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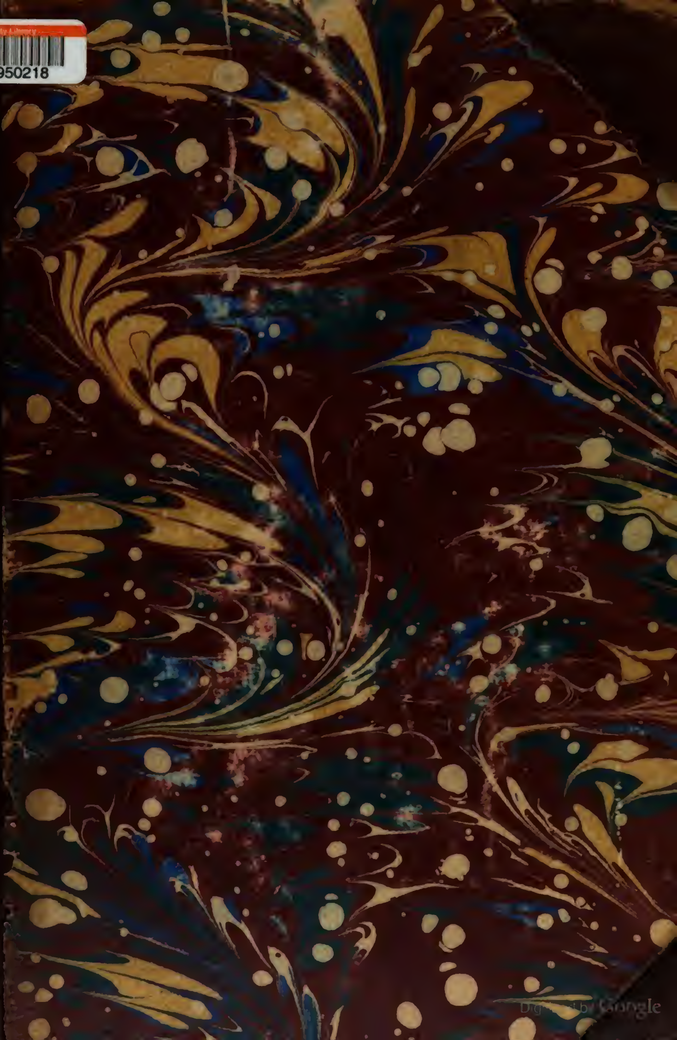

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ERRATA.

Page 471, line 1 of the second paragraph, *for* $\frac{1}{1.2\ldots x}$ *read* $\frac{(-1)^{x-1}}{1.2\ldots x}$

Page 472, line 18, *for* $(-1) \underset{-x}{C}$ *read* $(-1)^r \underset{-x}{C}$

Page 638, line 14, *for* $273^\circ(C)$ *read* $-273^\circ(C)$

Page 653, line 5, for u read u_0

Page 653, line 11, for u^0 read u

Page 573, line 3 from bottom, *for* 11 feet *read* 3 feet

Pages 573, 574, *dele* beds cxv. cxvi.

I. *On the Dynamical Theory of Diffraction.* By G. G. STOKES, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *November* 26, 1849.]

WHEN light is incident on a small aperture in a screen, the illumination at any point in front of the screen is determined, on the undulatory theory, in the following manner. The incident waves are conceived to be broken up on arriving at the aperture; each element of the aperture is considered as the centre of an elementary disturbance, which diverges spherically in all directions, with an intensity which does not vary rapidly from one direction to another in the neighbourhood of the normal to the primary wave; and the disturbance at any point is found by taking the aggregate of the disturbances due to all the secondary waves, the phase of vibration of each being retarded by a quantity corresponding to the distance from its centre to the point where the disturbance is sought. The square of the coefficient of vibration is then taken as a measure of the intensity of illumination. Let us consider for a moment the hypotheses on which this process rests. In the first place, it is no hypothesis that we may conceive the waves broken up on arriving at the aperture: it is a necessary consequence of the dynamical principle of the superposition of small motions; and if this principle be inapplicable to light, the undulatory theory is upset from its very foundations. The mathematical resolution of a wave, or any portion of a wave, into elementary disturbances must not be confounded with a physical breaking up of the wave, with which it has no more to do than the division of a rod of variable density into differential elements, for the purpose of finding its centre of gravity, has to do with breaking the rod in pieces. It *is* an hypothesis that we may find the disturbance in front of the aperture by merely taking the aggregate of the disturbances due to all the secondary waves, each secondary wave proceeding as if the screen were away; in other words, that the effect of the screen is *merely to stop* a certain portion of the incident light. This hypothesis, exceedingly probable *a priori*, when we are only concerned with points at no great distance from the normal to the primary wave, is confirmed by experiment, which shews that the same appearances are presented, with a given aperture, whatever be the nature of the screen in which the aperture is pierced, whether, for example, it consist of paper or of foil, whether a small aperture be divided by a hair or by a wire of equal thickness. It is an hypothesis, again, that the intensity in a secondary wave is nearly constant, at a given distance from the centre, in different directions very near the normal to the primary wave; but it seems to me almost impossible to conceive a mechanical theory which would not lead to this result. It is evident that the difference of phase of the various secondary waves which agitate a given point must be determined by the difference of their radii; and if it should afterwards be found necessary to add a constant to all the phases the results will not be at all affected.

Lastly, good reasons may be assigned why the intensity should be measured by the square of the coefficient of vibration; but it is not necessary here to enter into them.

In this way we are able to calculate the relative intensities at different points of a diffraction pattern. It may be regarded as established, that the coefficient of vibration in a secondary wave varies, in a given direction, inversely as the radius, and consequently, we are able to calculate the relative intensities at different distances from the aperture. To complete this part of the subject, it is requisite to know the absolute intensity. Now it has been shewn that the absolute intensity will be obtained by taking the reciprocal of the wave length for the quantity by which to multiply the product of a differential element of the area of the aperture, the reciprocal of the radius, and the circular function expressing the phase. It appears at the same time that the phase of vibration of each secondary wave must be accelerated by a quarter of an undulation. In the investigations alluded to, it is supposed that the law of disturbance in a secondary wave is the same in all directions; but this will not affect the result, provided the solution be restricted to the neighbourhood of the normal to the primary wave, to which indeed alone the reasoning is applicable; and the solution so restricted is sufficient to meet all ordinary cases of diffraction.

Now the object of the first part of the following paper is, to determine, on purely dynamical principles, the law of disturbance in a secondary wave, and that, not merely in the neighbourhood of the normal to the primary wave, but in all directions. The occurrence of the reciprocal of the radius in the coefficient, the acceleration of a quarter of an undulation, and the absolute value of the coefficient in the neighbourhood of the normal to the primary wave, will thus appear as particular results of the general formula.

Before attacking the problem dynamically, it is of course necessary to make some supposition respecting the nature of that medium, or *ether*, the vibrations of which constitute light, according to the theory of undulations. Now, if we adopt the theory of transverse vibrations—and certainly, if the simplicity of a theory which conducts us through a multitude of curious and complicated phenomena, like a thread through a labyrinth, be considered to carry the stamp of truth, the claims of the theory of transverse vibrations seem but little short of those of the theory of universal gravitation—if, I say, we adopt this theory, we are obliged to suppose the existence of a tangential force in the ether, called into play by the continuous sliding of one layer, or film, of the medium over another. In consequence of the existence of this force, the ether must behave, so far as regards the luminous vibrations, like an elastic solid. We have no occasion to speculate as to the cause of this tangential force, nor to assume either that the ether does, or that it does not, consist of distinct particles; nor are we directly called on to consider in what manner the ether behaves with respect to the motion of solid bodies, such as the earth and planets.

Accordingly, I have assumed, as applicable to the luminiferous ether in vacuum, the known equations of motion of an elastic medium, such as an elastic solid. These equations contain two arbitrary constants, depending upon the nature of the medium. The argument which Green has employed to shew that the luminiferous ether must be regarded as sensibly incompressible, in treating of the motions which constitute light*, appears to me of great force.

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* *Camb. Phil. Trans.* Vol. VII. p. 2.

The supposition of incompressibility reduces the two arbitrary constants to one; but as the equations are not thus rendered more manageable, I have retained them in their more general shape.

The first problem relating to an elastic medium of which the object that I had in view required the solution was, to determine the disturbance at any time, and at any point of an elastic medium, produced by a given initial disturbance which was confined to a finite portion of the medium. This problem was solved long ago by Poisson, in a memoir contained in the tenth volume of the memoirs of the Academy of Sciences. Poisson indeed employed equations of motion with but one arbitrary constant, which are what the general equations of motion become when a certain numerial relation is assumed to exist between the two constants which they involve. This relation was the consequence of a particular physical supposition which he adopted, but which has since been shewn to be untenable, inasmuch as it leads to results which are contradicted by experiment. Nevertheless nothing in Poisson's method depends for its success on the particular numerical relation assumed; and in fact, to save the constant writing of a radical, Poisson introduced a second constant, which made his equations identical with the general equations, so long as the particular relation supposed to exist between the two constants was not employed. I might accordingly have at once assumed Poisson's results. I have however begun at the beginning, and given a totally different solution of the problem, which will I hope be found somewhat simpler and more direct than Poisson's. The solution of this problem and the discussion of the result occupy the first two sections of the paper.

Having had occasion to solve the problem in all its generality, I have in one or two instances entered into details which have no immediate relation to light. I have also occasionally considered some points relating to the theory of light which have no immediate bearing on diffraction. It would occupy too much room to enumerate these points here, which will be found in their proper place. I will merely mention one very general theorem at which I have arrived by considering the physical interpretation of a certain step of analysis, though, properly speaking, this theorem is a digression from the main object of the paper. The theorem may be enunciated as follows.

If any material system in which the forces acting depend only on the positions of the particles be slightly disturbed from a position of equilibrium, and then left to itself, the part of the subsequent motion which depends on the initial displacements may be obtained from the part which depends on the initial velocities by replacing the arbitrary functions, or arbitrary constants, which express the initial velocities by those which express the corresponding initial displacements, and differentiating with respect to the time.

Particular cases of this general theorem occur so frequently in researches of this kind, that I think it not improbable that the theorem may be somewhere given in all its generality. I have not however met with a statement of it except in particular cases, and even then the subject was mentioned merely as a casual result of analysis.

In the third section of this paper, the problem solved in the second section is applied to the determination of the law of disturbance in a secondary wave of light. This determination forms the whole of the dynamical part of the theory of diffraction, at least when we confine ourselves to diffraction in vacuum, or, more generally, within a homogeneous singly refracting medium: the rest is a mere matter of integration; and whatever difficulties the solution of the problem may present for particular forms of aperture, they are purely mathematical.

In the investigation, the incident light is supposed to be plane-polarized, and the following results are arrived at. Each diffracted ray is plane-polarized, and the plane of polarization is determined by this law; *The plane of vibration of the diffracted ray is parallel to the direction of vibration of the incident ray.* The expression *plane of vibration* is here used to denote the plane passing through the ray and the direction of vibration. The direction of vibration in any diffracted ray being determined by the law above mentioned, the phase and coefficient of vibration at that part of a secondary wave are given by the formulæ of Art. 33.

The law just enunciated seems to lead to a crucial experiment for deciding between the two rival theories respecting the directions of vibration in plane-polarized light. Suppose the plane of polarization, and consequently the plane of vibration, of the incident light to be turned round through equal angles of say 5° at a time. Then, according to theory, the planes of vibration of the diffracted ray will not be distributed uniformly, but will be crowded towards the plane perpendicular to the plane of diffraction, or that which contains the incident and diffracted rays. The law and amount of the crowding will in fact be just the same as if the planes of vibration of the incident ray were represented in section on a plane perpendicular to that ray, and then projected on a plane perpendicular to the diffracted ray. Now experiment will enable us to decide whether the *planes of polarization* of the diffracted ray are crowded towards the plane of diffraction or towards the plane perpendicular to the plane of diffraction, and we shall accordingly be led to conclude, either that the direction of vibration is perpendicular, or that it is parallel to the plane of polarization.

In ordinary cases of diffraction, the light is insensible at such a small distance from the direction of the incident ray produced that the crowding indicated by theory is too small to be detected by experiment. It is only by means of a fine grating that we can obtain light of considerable intensity which has been diffracted at a large angle.

On mentioning to my friend, Professor Miller, the result at which I had arrived, and making some inquiries about the fineness, &c. of gratings, he urged me to perform the experiment myself, and kindly lent me for the purpose a fine glass grating, which he has in his possession. For the use of two graduated instruments employed in determining the positions of the planes of polarization of the incident and diffracted rays I am indebted to the kindness of my friend Professor O'Brien.

The description of the experiments, and the discussion of the results, occupies Part II. of this Paper. Since in a glass grating the diffraction takes place at the common surface of two different media, namely, air and glass, the theory of Part I. does not quite meet the case. Nevertheless it does not fail to point out whereabouts the plane of polarization of the diffracted ray ought to lie, according as we adopt one or other of the hypotheses respecting the direction of vibration. For theory assigns exact results on the two extreme suppositions, *first*, that the diffraction takes place before the light reaches the grooves; *secondly*, that it takes place after the light has passed between them; and these results are very different, according as we suppose the vibrations to be perpendicular or parallel to the plane of polarization. Most of the experiments were made on light which was diffracted in passing through the grating. The results appeared to be decisive in favour of Fresnel's hypothesis. In fact, theory shews that diffraction at a large angle is a powerful cause of crowding of the planes of vibration of the diffracted ray towards the perpendicular to the plane of diffraction, and experiment pointed out the existence of a powerful cause of crowding of the planes of polarization towards the plane of diffraction; for not only was the crowding in the contrary direction due to refraction overcome, but a considerable crowding was actually produced towards the plane of diffraction, especially when the grooved face of the glass plate was turned towards the incident light.

The experiments were no doubt rough, and are capable of being repeated with a good deal more accuracy by making some small changes in the apparatus and method of observing. Nevertheless the quantity with respect to which the two theories are at issue is so large that the experiments, such as they were, seem amply sufficient to shew which hypothesis is discarded by the phenomena.

The conclusive character of the experimental result with regard to the question at issue depends, I think, in a great measure on the simplicity of the law which forms the only result of theory that it is necessary to assume. This law in fact merely asserts that, whereas the direction of vibration in the diffracted ray cannot be parallel to the direction of vibration in the incident ray, being obliged to be perpendicular to the diffracted ray, it makes with it as small an angle as is consistent with the above restriction. This law seems only just to lie beyond the limits of the geometrical part of the theory of undulations. At the same time I may be permitted to add that, for my own part, I feel very great confidence in the equations of motion of the luminiferous ether in vacuum, and in that view of the nature of the ether which would lead to these equations, namely, that in the propagation of light, the ether, from whatever reason, behaves like an elastic solid. But when we consider the mutual action of the luminiferous ether and ponderable matter, a wide field, as it seems to me, is thrown open to conjecture. Thus, to take the most elementary of all the phenomena which relate to the action of transparent media on light, namely, the diminution of the velocity of propagation, this diminution seems capable of being accounted for on several different hypotheses. And if this elementary phenomenon leaves so much room for conjecture, much more may we form various hypotheses as to the state of things at the confines of two media, such as air and glass. Accordingly, conclusions in favour of either hypothesis which are derived from the comparison of theoretical and experimental results relating to the effects of reflection and refraction on the polarization of light, appear to me much more subject to doubt than those to which we are led by the experiments here described.

In commencing the theoretical investigation of diffraction, I naturally began with the simpler case of sound. As, however, the results which I have obtained for sound are of far less interest than those which relate to light, I have here omitted them, more especially as the paper has already swelled to a considerable size. I may, perhaps, on some future occasion bring them before the notice of this Society.

PART I.

THEORETICAL INVESTIGATION.

SECTION I.

PRELIMINARY ANALYSIS.

1. In what follows there will frequently be occasion to express a triple integration which has to be performed with respect to all space, or at least to all points of space for which the quantity to be integrated has a value different from zero. The conception of such an integration, regarded as a limiting summation, presents itself clearly and readily to the mind, without the consideration of co-ordinates of any kind. A system of co-ordinates forms merely the machinery by which the integration is to be effected in particular cases; and when the function to be integrated is arbitrary, and the nature of the problem does not point to one system rather than another, the employment of some particular system, and the analytical expression thereby of the function to be integrated, serves only to distract the attention by the introduction of a foreign element, and to burden the pages with a crowd of unnecessary symbols. Accordingly, in the case mentioned above, I shall merely take dV to represent an element of volume, and write over it the sign $\iiint$, to indicate that the integration to be performed is in fact triple. Integral signs will be used in this manner without limits expressed when the integration is to extend to all points of space for which the function to be integrated differs from zero.

There will frequently be occasion too to represent a double integration which has to be performed with reference to the surface of a sphere, of radius r, described round the point which is regarded as origin, or else a double integration which has to be performed with reference to all angular space. In this case the sign $\iint$ will be used, and dS will be taken to represent an element of the surface of the sphere, and $d\sigma$ to represent an elementary solid angle, measured by the corresponding element of the surface of a sphere described about its vertex with radius unity. Hence, if dV, dS, $d\sigma$ denote corresponding elements, $dS = r^2 d\sigma$, $dV = dr\,dS = r^2 dr\,d\sigma$. When the signs $\iiint$ and $\iint$, referring to differentials which are denoted by a single symbol, come together, or along with other integral signs, they will be separated by a dot, as for example $\iiint . \iint U\,dV\,d\sigma$.

2. As the operation denoted by $\dfrac{d^2}{dx^2} + \dfrac{d^2}{dy^2} + \dfrac{d^2}{dz^2}$ will be perpetually recurring in this paper, I shall denote it for shortness by ∇. This operation admits of having assigned to it a geometrical meaning which is independent of co-ordinates. For if P be the point (x, y, z), T a small space containing P, which will finally be supposed to vanish, dn an element of a normal drawn outwards at the surface of T, U the function which is the subject of the operation, and if ∇ be defined as the equivalent of $\dfrac{d^2}{dx^2} + \dfrac{d^2}{dy^2} + \dfrac{d^2}{dz^2}$, it is easy to prove that

$$\nabla U = \text{limit of } \frac{\iint \frac{dU}{dn}\, dS}{T}, \quad \ldots \ldots \ldots \quad (1)$$

the integration extending throughout the surface of T, of which dS is an element. In fact, if l, m, n be the direction-cosines of the normal, we shall have

$$\iint \frac{dU}{dn}\, dS = \iint \left(l \frac{dU}{dx} + m \frac{dU}{dy} + n \frac{dU}{dz} \right) dS$$

$$= \iint \frac{dU}{dx}\, dy\, dz + \iint \frac{dU}{dy}\, dz\, dx + \iint \frac{dU}{dz}\, dx\, dy. \quad \ldots \ldots \quad (2).$$

We have also, supposing the origin of co-ordinates to be at the point P, as we may without loss of generality,

$$\frac{dU}{dx} = \left(\frac{dU}{dx}\right) + \left(\frac{d^2U}{dx^2}\right) x + \left(\frac{d^2U}{dx\, dy}\right) y + \left(\frac{d^2U}{dx\, dz}\right) z + \text{terms of the 2nd order, \&c.} \quad (3),$$

where the parentheses denote that the differential coefficients which are enclosed in them have the values which belong to the point P. In the integral $\iint \frac{dU}{dx}\, dy\, dz$, each element must be taken positively or negatively, according as the normal which relates to it makes an acute or an obtuse angle with the positive direction of the axis of x. If we combine in pairs the elements of the integral which relate to opposite elements of the surface of T, we must write $\iint \left(\frac{dU_{\prime\prime}}{dx_{\prime\prime}} - \frac{dU_{\prime}}{dx_{\prime}} \right) dy\, dz$, where the single and double accents subscribed refer respectively to the first and second points in which the surface of T is cut by an indefinite straight line drawn parallel to the axis of x, and in the positive direction, through the point (o, y, z). We thus get by means of (3), omitting the terms of a higher order than the first, which vanish in the limit,

$$\iint \left(\frac{dU_{\prime\prime}}{dx_{\prime\prime}} - \frac{dU_{\prime}}{dx_{\prime}} \right) dy\, dz = \left(\frac{d^2U}{dx^2}\right) \iint (x_{\prime\prime} - x_{\prime})\, dy\, dz.$$

But $\iint (x_{\prime\prime} - x_{\prime})\, dy\, dz$ is simply the volume T. Treating in the same manner the two other integrals which appear on the right-hand side of equation (2), we get

$$\iint \frac{dU}{dn}\, dS = T \left\{ \left(\frac{d^2U}{dx^2}\right) + \left(\frac{d^2U}{dy^2}\right) + \left(\frac{d^2U}{dz^2}\right) \right\}, \text{ ultimately.}$$

Dividing by T and passing to the limit, and omitting the parentheses, which are now no longer necessary, we obtain the theorem enunciated.

If in equation (1) we take for T the elementary volume $r^2 \sin\theta\, dr\, d\theta\, d\phi$, or $r\, dr\, d\theta\, dx$, according as we wish to employ polar co-ordinates, or one of three rectangular co-ordinates combined with polar co-ordinates in the plane of the two others, we may at once form the expression for ∇U, and thus pass from rectangular co-ordinates to either of these systems without the trouble of the transformation of co-ordinates in the ordinary way.

3. Let f be a quantity which may be regarded as a function of the rectangular co-ordinates of a point of space, or simply, without the aid of co-ordinates, as having a given value for each point of space. It will be supposed that f vanishes outside a certain portion T of infinite space, and that within T it does not become infinite. It is required to determine a function U by the conditions that it shall satisfy the partial differential equation

$$\nabla U = f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4).$$

at all points of infinite space, that it shall nowhere become infinite, and that it shall vanish at an infinite distance.

These conditions are precisely those which have to be satisfied by the potential of a finite mass whose density is $-\dfrac{f}{4\pi}$; and we shall have accordingly, if O be the point for which the value of U is required, and r be the radius vector of any element drawn from O,

$$U = -\frac{1}{4\pi}\iiint \frac{f}{r}\,dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5).$$

In fact, it may be proved, just as in the theory of potentials, that the expression for U given by (5) does really satisfy (4) and the given conditions; and consequently, if $U + U'$ be the most general solution, U' must satisfy the equation $\nabla U' = 0$ at all points, must nowhere become infinite, and must vanish at an infinite distance. But this being the case it is easy to prove that U' cannot be different from zero.

The solution will still hold good in certain cases when f is infinite at some points, or when it is not confined to a finite space T, but only vanishes at an infinite distance. But such instances may be regarded as limiting cases of the problem restricted as above, and therefore need not be supposed to be excluded by those restrictions.

4. Let U be a quantity depending upon the time t, as well as upon the position of the point of space to which it relates, and satisfying the partial differential equation

$$\frac{d^2U}{dt^2} = a^2 \nabla U. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6).$$

It is required to determine U by the above equation and the conditions that when $t = 0$, U and $\dfrac{dU}{dt}$ shall have finite values given arbitrarily within a finite space T, and shall vanish outside T.

Let O be the point for which the value of U is sought, r the radius vector of any element drawn from O; $f(r)$, $F(r)$ the initial values of U, $\dfrac{dU}{dt}$. By this notation it is not meant that these values are functions of r alone, for they will depend likewise upon the two angles which determine the direction of r; but there will be no occasion to express analytically their dependance on those angles. The solution of the problem is

$$U = \frac{t}{4\pi}\iint F(at)\,d\sigma + \frac{1}{4\pi}\frac{d}{dt}\,t\iint f(at)\,d\sigma. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7).$$

See a memoir by Poisson *Mém. de l'Académie*, Tom. III. p. 130, or *Gregory's Examples*, p. 499.

5. Let δ be a function which has given finite values within a finite portion of space, and vanishes elsewhere; and let it be required to determine three functions ξ, η, ζ by the conditions

$$\frac{d\zeta}{dy} - \frac{d\eta}{dz} = \frac{d\xi}{dz} - \frac{d\zeta}{dx} = \frac{d\eta}{dx} - \frac{d\xi}{dy} = 0; \quad \dots\dots \quad (8).$$

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \delta. \quad \dots\dots\dots \quad (9).$$

The functions ξ, η, ζ are further supposed not to become infinite, and to vanish at an infinite distance. To save repetition, it will here be remarked, once for all, that the same supposition will be made in similar cases.

By virtue of equations (8), $\xi dx + \eta dy + \zeta dz$ is an exact differential $d\psi$, and (9) gives $\nabla \psi = \delta$. Hence we have by the formula (5)

$$\psi = -\frac{1}{4\pi} \iiint \frac{\delta}{r} \, dV. \quad \dots\dots\dots\dots \quad (10),$$

and ψ being known, ξ, η, ζ will be obtained by mere differentiation. To differentiate ψ with respect to x, it will be sufficient to differentiate δ under the integral sign. For draw OO' parallel to the axis of x, and equal to Δx, let P, P' be two points similarly situated with respect to O, O', respectively, and consider the part of ψ and that of $\psi + \Delta\psi$ due to equal elements of volume dV situated at P, P' respectively. For these two elements r has the same value, since $OP = O'P'$, and in passing from the first to the second δ is changed into $\delta + \Delta\delta$, and therefore the increment of ψ is simply $-\frac{\Delta\delta}{4\pi r} dV$. To get the complete increment of ψ we have only to perform the triple integration, an integration which is always real, even though r vanishes in the denominator, as may be readily seen on passing momentarily to polar co-ordinates. Dividing now by Δx and passing to the limit, we get

$$\xi = \frac{d\psi}{dx} = -\frac{1}{4\pi} \iiint \frac{d\delta}{dx} \frac{dV}{r}. \quad \dots\dots\dots \quad (11).$$

By employing temporarily rectangular co-ordinates in the triple integration, integrating by parts with respect to x, and observing that the quantity free from the integral sign vanishes at the limits, we get

$$\xi = -\frac{1}{4\pi} \iiint \frac{\delta}{r^2} \cos(rx) \, dV. \quad \dots\dots\dots \quad (12),$$

as might have been readily proved from (10), by referring O to a fixed origin, and then differentiating with respect to x. The expressions for η and ζ may be written down from symmetry.

6. Let ϖ', ϖ'', ϖ''' be three functions which have given finite values throughout a finite space and vanish elsewhere; it is required to determine three other functions ξ, η, ζ by the conditions

$$\frac{d\zeta}{dy} - \frac{d\eta}{dz} = 2\varpi', \quad \frac{d\xi}{dz} - \frac{d\zeta}{dx} = 2\varpi'', \quad \frac{d\eta}{dx} - \frac{d\xi}{dy} = 2\varpi''', \quad \dots\dots\dots \quad (13).$$

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0. \quad \dots\dots\dots\dots \quad (14).$$

It is to be observed that ϖ', ϖ'', ϖ''' are not independent. For differentiating equations (13) with respect to x, y, z, and adding, we get

$$\frac{d\varpi'}{dx}+\frac{d\varpi''}{dy}+\frac{d\varpi'''}{dz}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

Hence ϖ', ϖ'', ϖ''' must be supposed given arbitrarily only in so far as is consistent with the above equation.

Eliminating ζ from (14), and the second of equations (13), we get

$$\frac{d}{dx}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}\right)+\frac{d^2\xi}{dz^2}=2\frac{d\varpi''}{dz},$$

$$\text{or } \nabla\xi+\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)=2\frac{d\varpi''}{dz},$$

which becomes by the last of equations (13)

$$\nabla\xi=2\left(\frac{d\varpi''}{dz}-\frac{d\varpi'''}{dy}\right).$$

Consequently, by equation (5),

$$\xi=\frac{1}{2\pi}\iiint\left(\frac{d\varpi'''}{dy}-\frac{d\varpi''}{dz}\right)\frac{dV}{r}.$$

Transforming this equation in the same manner as (11), supposing x, y, z measured from O, and writing down the two equations found by symmetry, we have finally,

$$\left.\begin{aligned}\xi&=\frac{1}{2\pi}\iiint(y\varpi'''-z\varpi'')\frac{dV}{r^3},\\ \eta&=\frac{1}{2\pi}\iiint(z\varpi'-x\varpi''')\frac{dV}{r^3},\\ \zeta&=\frac{1}{2\pi}\iiint(x\varpi''-y\varpi')\frac{dV}{r^3}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

7. Let δ, ϖ', ϖ'', ϖ''' be as before; and let it be required to determine three functions ξ, η, ζ from the equations (9) and (13).

From the linearity of the equations it is evident that we have merely to add together the expressions obtained in the last two articles.

8. Let ξ_0, η_0, ζ_0 be three functions given arbitrarily within a finite space outside of which they are equal to zero: it is required to decompose these functions into two parts ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 such that $\xi_1 dx+\eta_1 dy+\zeta_1 dz$ may be an exact differential $d\psi_1$, and ξ_2, η_2, ζ_2 may satisfy (14).

Observing that $\xi_2=\xi_0-\xi_1$, η_2, $=\eta_0-\eta_1$, $\zeta_2=\zeta_0-\zeta_1$, expressing ξ_1, η_1, ζ_1, in terms of ψ_1, and substituting in (14), we get

$$\nabla\psi_1=\delta_0,$$

where δ_0 is what δ becomes when ξ_0, η_0, ζ_0 are written for ξ, η, ζ. The above equation gives

$$\psi_1=-\frac{1}{4\pi}\iiint\frac{\delta_0}{r}dV,$$

whence ξ_1, η_1, ζ_1, and consequently ξ_2, η_2, ζ_2, are known.

SECTION II.

PROPAGATION OF AN ARBITRARY DISTURBANCE IN AN ELASTIC MEDIUM.

9. THE equations of motion of a homogeneous uncrystallized elastic medium, such as an elastic solid, in which the disturbance is supposed to be very small, are well known. They contain two distinct arbitrary constants, which cannot be united in one without adopting some particular physical hypothesis. These equations may be obtained by supposing the medium to consist of ultimate molecules, but they by no means require the adoption of such a hypothesis, for the same equations are arrived at by regarding the medium as continuous.

Let x, y, z be the co-ordinates of any particle of the medium in its natural state; ξ, η, ζ the displacements of the same particle at the end of the time t, measured in the directions of the three axes respectively. Then the first of the equations may be put under the form

$$\frac{d^2\xi}{dt^2} = b^2\left(\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} + \frac{d^2\xi}{dz^2}\right) + (a^2 - b^2)\frac{d}{dx}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right),$$

where a^2, b^2, denote the two arbitrary constants. Put for shortness

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \delta, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17).$$

and as before represent by $\nabla\xi$ the quantity multiplied by b^2. According to this notation, the three equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= b^2\nabla\xi + (a^2 - b^2)\frac{d\delta}{dx}, \\ \frac{d^2\eta}{dt^2} &= b^2\nabla\eta + (a^2 - b^2)\frac{d\delta}{dy}, \\ \frac{d^2\zeta}{dt^2} &= b^2\nabla\zeta + (a^2 - b^2)\frac{d\delta}{dz}. \end{aligned}\right\}, \quad . \; . \; . \; . \; . \; . \; . \quad (18).$$

It is to be observed that δ denotes the dilatation of volume of the element situated at the point (x, y, z). In the limiting case in which the medium is regarded as absolutely incompressible δ vanishes; but in order that equations (18) may preserve their generality, we must suppose a at the same time to become infinite, and replace $a^2\delta$ by a new function of the co-ordinates. If we take $-p$ to denote this function, we must replace the last terms in these equations by $-\frac{dp}{dx}$, $-\frac{dp}{dy}$, $-\frac{dp}{dz}$, respectively, and we shall thus have a fourth unknown function, as well as a fourth equation, namely that obtained by replacing the second member of (17) by zero. But the retention of equations (18) in their present more general form does not exclude the supposition of incompressibility, since we may suppose a to become infinite in the end just as well as at first.

10. Suppose the medium to extend infinitely in all directions, and conceive a portion of it occupying the finite space T to receive any arbitrary small disturbance, and then to be left to itself, the whole of the medium outside the space T being initially at rest; and let it be required to determine the subsequent motion.

Differentiating equations (18) with respect to x, y, z, respectively, and adding, we get by virtue of (17)

$$\frac{d^2\delta}{dt^2} = a^2\nabla\delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Again, differentiating the third of equations (18) with respect to y, and the second with respect to z, and subtracting the latter of the two resulting equations from the former, and treating in a similar manner the first and third, and then the second and first of equations (18), we get

$$\frac{d^2\varpi'}{dt^2} = b^2\nabla\varpi', \qquad \frac{d^2\varpi''}{dt^2} = b^2\nabla\varpi'', \qquad \frac{d^2\varpi'''}{dt^2} = b^2\nabla\varpi''', \quad . \quad . \quad (20).$$

where ϖ', ϖ'', ϖ''' are the quantities defined by equations (13). These quantities express the rotations of the element of the medium situated at the point (x, y, z) about axes parallel to the three co-ordinate axes respectively.

Now the formula (7) enables us to express δ, ϖ', ϖ'', and ϖ''' in terms of their initial values and those of their differential coefficients with respect to t, which are supposed known; and these functions being known, we shall determine ξ, η, and ζ as in Art. 7. Our equations being thus completely integrated, nothing will remain but to simplify and discuss the formulæ obtained.

11. Let O be the point of space at which it is required to determine the disturbance, r the radius vector of any element drawn from O; and let the initial values of δ, $\frac{d\delta}{dt}$ be represented by $f(r)$, $F(r)$, respectively, with the same understanding as in Art 4. By the formula (7), we have

$$\delta = \frac{t}{4\pi}\iint F(at)\,d\sigma + \frac{1}{4\pi}\frac{d}{dt}\,t\iint f(at)\,d\sigma \quad . \quad . \quad . \quad . \quad (21).$$

The double integrals in this expression vanish except when a spherical surface described round O as centre, with a radius equal to at, cuts a portion of the space T. Hence, if O be situated outside the space T, and if r_1, r_2 be respectively the least and greatest values of the radius vector of any element of that space, there will be no dilatation at O until $at = r_1$. The dilatation will then commence, will last during an interval of time equal to $a^{-1}(r_2 - r_1)$, and will then cease for ever. The dilatation here spoken of is understood to be either positive or negative, a negative dilatation being the same thing as a condensation.

Hence a *wave of dilatation* will be propagated in all directions from the originally disturbed space T, with a velocity a. To find the portion of space occupied by the wave, we have evidently only got to conceive a spherical surface, of radius at, described about each point of the space T as centre. The space occupied by the assemblage of these surfaces is that in which the wave of dilatation is comprised. To find the limits of the wave, we need evidently only attend to those spheres which have their centres situated in the surface of the space T. When t is small, this system of spheres will have an exterior envelope of two sheets, the outer of these sheets being exterior, and the inner interior to the shell formed by the assemblage of the spheres. The outer sheet forms the outer limit to the portion of the medium in which the dilatation is different from zero. As t increases, the inner sheet contracts, and at last its opposite sides cross, and it changes its character from being exterior, with reference to the spheres, to interior. It then expands, and forms the inner boundary of the shell in which the wave of condensation is comprised. It is easy to shew geometrically that each envelope is propagated with a velocity a in a normal direction.

12. It appears in a similar manner from equations (20) that there is a similar wave, propagated with a velocity b, to which are confined the rotations ϖ', ϖ'', ϖ'''. This wave may be called for the sake of distinction, the *wave of distortion*, because in it the medium is not dilated nor condensed, but only distorted in a manner consistent with the preservation of a constant density. The condition of the stability of the medium requires that the ratio of b to a be not greater than that of $\sqrt{3}$ to 2[*].

13. If the initial disturbance be such that there is neither dilatation nor velocity of dilatation initially, there will be no wave of dilatation, but only a wave of distortion. If it be such that the expressions $\xi dx + \eta dy + \zeta dz$ and $\dfrac{d\xi}{dt}dx + \dfrac{d\eta}{dt}dy + \dfrac{d\zeta}{dt}dz$ are initially exact differentials, there will be no wave of distortion, but only a wave of dilatation. By making $b = 0$ we pass to the case of an elastic fluid, such as air. By supposing $a = \infty$ we pass to the case of an incompressible elastic solid. In this case we must have initially $\delta = 0$ and $\dfrac{d\delta}{dt} = 0$; but in order that the results obtained by at once putting $a = \infty$ may have the same degree of generality as those which would be obtained by retaining a as a finite quantity, which in the end is supposed to increase indefinitely, we must not suppose the initial disturbance confined to the space T, but only the initial rotations and the initial angular velocities. Consequently, outside T the expression $\xi dx + \eta dy + \zeta dz$ must be initially an exact differential $d\psi$, where ψ satisfies the equation $\nabla\psi = 0$ derived from (14), and the expression $\dfrac{d\xi}{dt}dx + \dfrac{d\eta}{dt}dy + \dfrac{d\zeta}{dt}dz$ must be initially an exact differential $d\psi_1$, where ψ_1 satisfies the equation $\nabla\psi_1 = 0$. So long as a is finite, it comes to the same thing whether we regard the medium as animated initially by certain velocities given arbitrarily throughout the space T, or as acted on by impulsive accelerating forces capable of producing those velocities; and the latter mode of conception is equally applicable to the case of an incompressible medium, for which a is infinite, although we cannot in that case conceive the initial velocities as given arbitrarily, but only arbitrarily in so far as is compatible with their satisfying the condition of incompressibility. It is not so easy to see what interpretation is to be given, in the case of an incompressible medium, to the initial displacements which are considered in the general case, in so far as these displacements involve dilatation or condensation. As no simplicity worth mentioning is gained by making a at once infinite, this constant will be retained in its present shape, more especially as the results arrived at will thus have greater generality.

14. The expressions for the disturbance of the medium at the end of the time t are linear functions of the initial displacements and initial velocities; and it appears from (21), and the corresponding equations which determine ϖ', ϖ'', and ϖ''', that the part of the disturbance which is due to the initial displacements may be obtained from the part which is due to the

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[*] See a memoir by Green, *On the reflexion and refraction of Light.* Camb. Phil. Trans. Vol. VII. p. 2. See also Camb. Phil. Trans. Vol. VIII. p. 319.

initial velocities by differentiating with respect to t, and replacing the arbitrary functions which represent the initial velocities by those which represent the initial displacements. The same result constantly presents itself in investigations of this nature: on considering its physical interpretation it will be found to be of extreme generality.

Let any material system whatsoever, in which the forces acting depend only on the positions of the particles, be slightly disturbed from a position of equilibrium, and then left to itself. In order to represent the most general initial disturbance, we must suppose small initial displacements and small initial velocities, the most general possible consistent with the connexion of the parts of the system, communicated to it. By the principle of the superposition of small motions, the subsequent disturbance will be compounded of the disturbance due to the initial velocities and that due to the initial displacements. It is immaterial for the truth of this statement whether the equilibrium be stable or unstable; only, in the latter case, it is to be observed that the time t which has elapsed since the disturbance must be sufficiently small to allow of our neglecting the square of the disturbance which exists at the end of that time. Still, as regards the purely mathematical question, for any previously assigned interval t, however great, it will be possible to find initial displacements and velocities so small that the disturbance at the end of the time t shall be as small as we please; and in this sense the principle of superposition, and the results which flow from it, will be equally true whether the equilibrium be stable or unstable.

Suppose now that no initial displacements were communicated to the system we are considering, but only initial velocities, and that the disturbance has been going on during the time t. Let $f(t)$ be the type of the disturbance at the end of the time t, where $f(t)$ may represent indifferently a displacement or a velocity, linear or angular, or in fact any quantity whereby the disturbance may be defined. In the case of a rigid body, or a finite number of rigid bodies, there will be a finite number of functions $f(t)$ by which the motion of the system will be defined: in the cases of a flexible string, a fluid, an elastic solid, &c., there will be an infinite number of such functions, or, in other words, the motion will have to be defined by functions which involve one or more independent variables besides the time. Let v_0 be in a similar manner the type of the initial velocities, and let τ be an increment of t, which in the end will be supposed to vanish. The disturbance at the end of the time $t + \tau$ will be represented by $f(t + \tau)$; but since by hypothesis the forces acting on the system do not depend explicitly on the time, this disturbance is the same as would exist at the end of the time t in consequence of the system of velocities v_0 communicated to the material system at the commencement of the time $-\tau$, the system being at that instant in its position of equilibrium. Suppose then the system of velocities v_0 communicated in this manner, and in addition suppose the system of velocities $-v_0$ communicated at the time 0. On account of the smallness of the motion, the disturbance produced by the system of velocities v_0 will be expressed by linear functions of these velocities; and consequently, if $f(t)$ represent the disturbance due to the system of velocities v_0, $-f(t)$ will represent the disturbance due to the system $-v_0$. Hence the disturbance at the end of the time t will be represented by $f(t + \tau) - f(t)$. Now we may evidently regard the state of the material system immediately after the communication of the system of velocities $-v_0$ as its initial state, and then seek the disturbance which would be produced by the initial disturbance. The velocities v_0 going on during the time τ will have produced by the end of that time a system of displacements represented by τv_0. By hypothesis, the system was in a position of equilibrium at the commencement of the time $-\tau$; and since the forces are supposed not to depend on the velocities, but only on the positions of the particles, the effective forces during the time τ vary from zero to small quantities of the order τ, and therefore the velocities generated by the end of the time $-\tau$ are small quantities of the order τ^2. Hence the velocities $-v_0$ communicated at the time 0 destroy the previously existing velocities, except so far as regards small quantities of the order τ^2, which vanish in the limit, and therefore we have nothing to consider but the system of displacements τv_0. Hence the disturbance produced by a system of initial displacements τv_0 is represented by $f(t+\tau) - f(t)$, ultimately; and therefore the disturbance produced by a system of initial displacements v_0 is represented by the limit of $\tau^{-1}\{f(t+\tau) - f(t)\}$, or by $f'(t)$. Hence, to get the disturbance due to the initial displacements from that due to the initial velocities, we have only to differentiate with respect to t, and to replace the arbitrary constants or arbitrary functions which express the initial velocities by those which express the corresponding initial displacements. Conversely, to get the disturbance due to the initial velocities from that due to the initial displacements, we have only to change the arbitrary constants or functions, and to integrate with respect to t, making the integral vanish with t if the disturbance is expressed by displacements, or correcting it so as to give the initial velocities when $t = 0$ if the disturbance is expressed by velocities.

The reader may easily, if he pleases, verify this theorem on some dynamical problem relating to small oscillations.

15. Let us proceed now to determine the general values of ξ, η, ζ in terms of their initial values, and those of their differential coefficients with respect to t. By the formulæ of Section I., ξ, η, ζ are linear functions of δ, ϖ', ϖ'', and ϖ''', and we may therefore first form the part which depends upon δ, and afterwards the part which depends upon ϖ', ϖ'', ϖ''', and then add the results together. Moreover, it will be unnecessary to retain the part of the expressions which depends upon initial displacements, since this can be supplied in the end by the theorem of the preceding article.

Omitting then for the present ϖ', ϖ'', ϖ''', as well as the second term in equations (21), we get from equations (10) and (21),

$$\psi = -\frac{t}{16\pi^2}\iiint\cdot\iint\frac{F(at)}{r}\,dV\,d\sigma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

To understand the nature of the integration indicated in this equation, let O be the point of space for which the value of ψ is sought; from O draw in an arbitrary direction OP equal to r, and from P draw, also in an arbitrary direction, PQ equal to at. Then $F(at)$ denotes the value of the function F, or the initial rate of dilatation, at the point Q of space, and we have first to perform a double integration referring to all such points as Q, P being fixed, and then a triple integration referring to all such points as P. To facilitate the transformation of the integral (22), conceive PQ produced to Q', let $PQ' = s$, let dV' be an element of volume, and replace the double integral $\iint F.\,d\sigma$ by the triple integral $h^{-1}\iiint F.\,s^{-2}\,dV'$, taken between the limits defined by the imparities $at < s < at + h$, which may be done, provided h be finally made to vanish. We shall thus have two triple integrations to perform, each of which we may conceive to extend to all space, provided we regard the quantity to be integrated as equal to zero when PQ', (or as it may now be denoted PQ, Q being a point taken generally,) lies beyond the limits at and $at + h$, as well as when the point Q falls outside the space T, to which the disturbance was originally confined. Now perform the first of the two triple integrations on the supposition that Q remains fixed while P is variable, instead of supposing P to remain fixed while Q is variable. We shall thus have F constant and r variable, instead of having F variable and r constant. This first triple integration must evidently extend throughout the spherical shell which has Q for centre and at, $at + h$ for radii of the interior and exterior surfaces. We get, on making h vanish,

$$\text{limit of } \frac{1}{h}\iiint\frac{dV}{r} = \iint\frac{dS}{r},$$

dS being an element of the surface of a sphere described with Q for centre and at for radius. Now if $OQ = r'$, the integral $\iint r^{-1}\,dS$, which expresses the potential of a spherical shell, of radius at and density unity, at a point situated at a distance r' from the centre, is equal to $4\pi at$ or $4\pi r'^{-1}a^2t^2$, according as $r' < > at$. Substituting in (22), and omitting the accents, which are now no longer necessary, we get

$$\psi = -\frac{1}{4\pi a}\iiint F.\,dV(r<at) - \frac{t}{4\pi}\iiint\frac{F}{r}.\,dV(r>at) \quad . \quad . \quad . \quad (23),$$

where the limits of integration are defined by the imparities written after the integrals, as will be done in similar cases.

16. Let u_0, v_0, w_0, be the initial velocities; then

$$F = \frac{du_0}{dx} + \frac{dv_0}{dy} + \frac{dw_0}{dz}.$$

Substituting in the first term of the right-hand member of equation (23), and integrating by parts, exactly as in Art. 5, we get

$$-\frac{1}{4\pi a}\iiint F.\,dV(r<at) = -\frac{1}{4\pi a}\,\Sigma\iint\{(u_0)_{\prime\prime} - (u_0)_{\prime}\}\,dy\,dz,$$

where the Σ denotes that we must take the sum of the expression written down and the two formed from it by passing from x to y and from y to z, and the single and double accents refer respectively to the first and second point in which the surface of a sphere having O for centre, and at for radius, is cut by an indefinite line drawn parallel to the axis of x, and in the positive direction, through the point (o, y, z). Treating the last term in equation (23) in the same way, and observing that the quantities once integrated vanish at an infinite distance, or, to speak more properly, at the limits of the space T, we get

$$-\frac{t}{4\pi}\iiint\frac{F}{r}\,dV(r>at) = \frac{1}{4\pi a}\,\Sigma\iint\{(u_0)_{\prime\prime} - (u_0)_{\prime}\}\,dy\,dz$$
$$-\frac{t}{4\pi}\iiint(u_0x + v_0y + w_0z)\frac{dV}{r^3}(r>at).$$

The double integrals arising from the transformation of the second member of equation (23) destroy one another, and we get finally

$$\psi = -\frac{t}{4\pi}\iiint(u_0 x + v_0 y + w_0 z)\frac{dV}{r^3}\,(r > at). \quad . \quad . \quad . \quad . \quad (24).$$

17. To obtain the part of the displacement ξ due to the initial velocity of dilatation, we have only to differentiate ψ with respect to x, and this will be effected by differentiating u_0, v_0, w_0 under the integral signs, as was shewn in Art. 5. Treating the resulting expression by integration by parts, as before, and putting l, m, n for the direction-cosines of the radius vector drawn to the point to which the accents refer, and ξ_1 for the part of ξ due to F, we get

$$\xi_1 = \frac{1}{4\pi a^2 t}\iint\{(lu_0 + mv_0 + nw_0)_{\prime\prime} - (lu_0 + mv_0 + nw_0)_{\prime}\}\,dy\,dz$$
$$+ \frac{t}{4\pi}\iiint\left(u_0\frac{d}{dx}\frac{x}{r^3} + v_0\frac{d}{dx}\frac{y}{r^3} + w_0\frac{d}{dx}\frac{z}{r^3}\right)dV\,(r > at).$$

Let q_0 be the initial velocity revolved along the radius vector, so that $q_0 = lu_0 + mv_0 + nw_0$, and let $(q_0)_{at}$ be the value of q_0 at a distance at from O; then

$$\iint\{(lu_0 + mv_0 + nw_0)_{\prime\prime} - (lu_0 + mv_0 + nw_0)_{\prime}\}\,dy\,dz = \iint l\,(q_0)_{at}\,dS = a^2t^2\iint l\,(q_0)_{at}\,d\sigma,$$

$$\text{and}\quad u_0\frac{d}{dx}\frac{x}{r^3} + v_0\frac{d}{dx}\frac{y}{r^3} + w_0\frac{d}{dx}\frac{z}{r^3} = \frac{u_0 - 3lq_0}{r^3}.$$

Substituting in the expression for ξ_1, we get finally

$$\xi_1 = \frac{t}{4\pi}\iint l\,(q_0)_{at}\,d\sigma + \frac{t}{4\pi}\iiint(u_0 - 3lq_0)\frac{dV}{r^3}\,(r > at). \quad . \quad . \quad . \quad . \quad (25).$$

18. Let us now form the part of ξ which depends on the initial rotations and angular velocities, and which may be denoted by ξ_2. The theorem of Art. 14 allows us to omit for the present the part due to the initial rotations, which may be supplied in the end. Let ω_0', ω_0'', ω_0''' be the initial angular velocities. Then ξ_2 is given in terms of ϖ'' and ϖ''' by the first of equations (16), and ϖ'', ϖ''' are given in terms of ω_0'', ω_0''' by the formula (7), in which however b must be put for a. We thus get

$$\xi_2 = \frac{t}{8\pi^2}\iiint\cdot\iint(y\,\omega_0''' - z\,\omega_0'')\frac{dV\,d\sigma}{r^3}.$$

The integrations in this expression are to be understood as in Art. 15, and ω_0'', ω_0''' are supposed to have the values which belong to the point Q, but PQ is now equal to bt instead of at. The quintuple integral may be transformed into a triple integral just as before. We get in the first place

$$\xi_2 = \frac{t}{8\pi^2}\left(\iiint\omega_0'''\,dV'\iint\frac{y\,d\sigma}{r^3} - \iiint\omega_0''\,dV'\iint\frac{z\,d\sigma}{r^3}\right). \quad . \quad . \quad . \quad . \quad (26).$$

The double integration in this expression refers to all angular space, considered as extending round Q; x, y, z are the co-ordinates, measured from O, of a point P situated at a distance bt from Q, and $r = OP$. If $dS = (bt)^2 d\sigma$, the expressions for the integrals

$$\iint xr^{-3}dS, \quad \iint yr^{-3}dS, \quad \iint zr^{-3}dS$$

may be at once written down by observing that these integrals express the components of the attraction of a spherical shell, of radius bt and density 1, having Q for centre, on a particle situated at O. Hence if x', y', z' be the co-ordinates of Q, measured from O, and $r' = OQ$, the integrals vanish when $r' < bt$, and are equal to

$$4\pi (bt)^2 x' r'^{-3}, \quad 4\pi (bt)^2 y' r'^{-3}, \quad 4\pi (bt)^2 z' r'^{-3},$$

respectively, when $r' > bt$. Hence we get from (26), omitting the accents, which are now no longer necessary, since we have done with the point P,

$$\xi_2 = \frac{t}{2\pi} \iiint (\omega_0''' y - \omega_0'' z) \frac{dV}{r^3} (r > bt). \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Now

$$2\omega_0' = \frac{dw_0}{dy} - \frac{dv_0}{dz}; \quad 2\omega_0'' = \frac{du_0}{dz} - \frac{dw_0}{dx}; \quad 2\omega_0''' = \frac{dv_0}{dx} - \frac{du_0}{dy}.$$

Substituting in (27), and adding and subtracting $x \dfrac{du_0}{dx}$ under the integral signs, we get

$$\xi_2 = \frac{t}{4\pi} \iiint \left\{ - \left(x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz} \right) u_0 + \left(x \frac{du_0}{dx} + y \frac{dv_0}{dx} + z \frac{dw_0}{dx} \right) \right\} \frac{dV}{r^3} (r > bt).$$

But $x \dfrac{d}{dx} + y \dfrac{d}{dy} + z \dfrac{d}{dz}$ is the same thing as $r \dfrac{d}{dr}$, and we get accordingly

$$\iiint \left(x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz} \right) u_0 \frac{dV}{r^3} (r > bt) = \int . \iint \frac{du_0}{dr} \, dr \, d\sigma \, (r > bt) = - \iint (u_0)_{bt} \, d\sigma.$$

The second part of ξ_2 is precisely the expression transformed in the preceding article, except that the sign is changed, and b put for a. Hence we have

$$\xi_2 = \frac{t}{4\pi} \iint (u_0 - lq_0)_{bt} \, d\sigma - \frac{t}{4\pi} \iiint (u_0 - 3lq_0) \frac{dV}{r^3} (r > bt). \quad . \quad . \quad . \quad (28).$$

19. Adding together the expressions for ξ_1 and ξ_2, we get for the disturbance due to the initial velocities

$$\xi = \frac{t}{4\pi} \iint l (q_0)_{at} \, d\sigma + \frac{t}{4\pi} \iint (u_0 - lq_0)_{bt} \, d\sigma + \frac{t}{4\pi} \iiint (3lq_0 - u_0) \frac{dV}{r^3} (bt < r < at). \quad (29).$$

The part of the disturbance due to the initial displacements may be obtained immediately by the theorem of Art. 14. Let ξ_0, η_0, ζ_0 be the initial displacements, ρ_0 the initial displacement resolved along a radius vector drawn from O. The last term in equation (29), it will be observed, involves t in two ways, for t enters as a coefficient, and likewise the limits depend upon t. To find the part of the differential coefficient which relates to the variation of the limits, we have only to replace dV by $r^2 dr d\sigma$, and treat the integral in the usual way. We get for the part of the disturbance due to the initial displacements

$$\xi = \frac{1}{4\pi} \iint \left\{ l \left(4\rho_0 + at \frac{d\rho_0}{dr} \right) - \xi_0 \right\}_{at} d\sigma + \frac{1}{4\pi} \iint \left\{ 2\xi_0 + bt \frac{d\xi_0}{dr} - l \left(4\rho_0 + bt \frac{d\rho_0}{dr} \right) \right\}_{bt} d\sigma$$

$$+ \frac{1}{4\pi} \iiint (3l\rho_0 - \xi_0) \frac{dV}{r^3} (bt < r < at). \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

It is to be recollected that in this and the preceding equation l denotes the cosine of the angle between the axis of x and an arbitrary radius vector drawn from O, whose direction varies from one element $d\sigma$ of angular space to another, and that the at or bt subscribed denotes that r is to be supposed equal to at or bt after differentiation. To obtain the whole displacement parallel to x which exists at the end of the time t at the point O, we have only to add together the second members of equations (29) and (30). The expressions for η and ζ may be written down from symmetry, or rather the axis of x may be supposed to be measured in the direction in which we wish to estimate the displacement.

20. The first of the double integrals in equations (29), (30) vanishes outside the limits of the wave of dilatation, the second vanishes outside the limits of the wave of distortion. The triple integrals vanish outside the outer limit of the wave of dilatation, and inside the inner limit of the wave of distortion, but have finite values within the two waves and between them. Hence a particle of the medium situated outside the space T does not begin to move till the wave of dilatation reaches it. Its motion then commences, and does not wholly cease till the wave of distortion has passed, after which the particle remains absolutely at rest.

21. If the initial disturbance be such that there is no wave of distortion, the quantities ϖ', ϖ'', ϖ''', ω', ω'', ω''' must be separately equal to zero, and the expression for ξ will be reduced to ξ_1, given by (25), and the expression thence derived which relates to the initial displacements. The triple integral in the expression for ξ_1 vanishes when the wave of dilatation has passed, and the same is the case with the corresponding integral which depends upon the initial displacements. Hence the medium returns to rest as soon as the wave of dilatation has passed; and since even in the general case each particle remains at rest until the wave of dilatation reaches it, it follows that when the initial disturbance is such that no wave of distortion is formed the disturbance at any time is confined to the wave of dilatation. The same conclusion might have been arrived at by transforming the triple integral.

22. When the initial motion is such that there is no wave of dilatation, as will be the case when there is initially neither dilatation nor velocity of dilatation, ξ will be reduced to ξ_2, given by (28), and the corresponding expression involving the initial displacements. By referring to the expression in Art. 17, from which the triple integral in equation (28) was derived, we get

$$\iiint (u_0 - 3lq_0)\frac{dV}{r^3} = \iiint \left(u_0 \frac{d}{dx}\frac{x}{r^3} + v_0 \frac{d}{dx}\frac{y}{r^3} + w_0 \frac{d}{dx}\frac{z}{r^3}\right) dV. \quad . \quad . \quad (31).$$

Now

$$\iiint u_0 \frac{d}{dx}\frac{x}{r^3} dV = \iiint u_0 \frac{d}{dx}\frac{x}{r^3} dx\,dy\,dz = \iint \left(\frac{u_0 x}{r^3}\right) dy\,dz - \iiint \frac{du_0}{dx}\frac{x}{r^3} dx\,dy\,dz,$$

the parentheses denoting that the quantity enclosed in them is to be taken between limits. By the condition of the absence of initial velocity of dilatation we have

$$-\iiint \frac{du_0}{dx}\frac{x}{r^3} dx\,dy\,dz = \iiint \left(\frac{dv_0}{dy} + \frac{dw_0}{dz}\right)\frac{x}{r^3} dx\,dy\,dz.$$

Substituting in the second member of equation (31), and writing down for the present only the terms involving v_0, we obtain

$$\iiint \left(\frac{dv_0}{dy}\,\frac{x}{r^3} + v_0\,\frac{d}{dx}\,\frac{y}{r^3}\right) dx\,dy\,dz,$$

which, since $\frac{d}{dx}\,\frac{y}{r^3} = \frac{d}{dy}\,\frac{x}{r^3}$, becomes

$$\iiint \frac{d}{dy}\,\frac{v_0 x}{r^3}\,dx\,dy\,dz \text{ or } \iint \left(\frac{v_0 x}{r^3}\right) dx\,dz.$$

Treating the terms involving w_0 in the same manner, and substituting in (31), we get

$$\iiint (u_0 - 3lq_0)\,\frac{dV}{r^3} = \iint \left(\frac{u_0 x}{r^3}\right) dy\,dz + \iint \left(\frac{v_0 x}{r^3}\right) dz\,dx + \iint \left(\frac{w_0 x}{r^3}\right) dx\,dy.$$

Now the integration is to extend from $r = bt$ to $r = \infty$. The quantities once integrated vanish at the second limit, and the first limit relates to the surface of a sphere described round O as centre with a radius equal to bt. Putting dS or $b^2t^2\,d\sigma$ for an element of the surface of this sphere, we obtain for the value of the second member of the last equation

$$-(bt)^{-2} \iint (lu_0 + mv_0 + nw_0)_{bt}\,l\,dS, \quad \text{or} - \iint l\,(q_0)_{bt}\,d\sigma;$$

and therefore the triple integral in equation (28) destroys the second part of the double integral in the same equation. Hence, writing down also the terms depending upon the initial displacements, we obtain for ξ the very simple expression

$$\xi = \frac{t}{4\pi} \iint (u_0)_{bt}\,d\sigma + \frac{1}{4\pi}\,\frac{d}{dt}\,t \iint (\xi_0)_{bt}\,d\sigma.$$

This expression might have been obtained at once by applying the formula (7) to the first of equations (18), which in this case take the form (6), since $\delta = 0$.

23. Let us return now to the general case, and consider especially the terms which alone are important at a great distance from the space to which the disturbance was originally confined; and, first, let us take the part of ξ which is due to the initial velocities, which is given by equation (29).

Let the three parts of the second member of this equation be denoted by ξ_a, ξ_b, ξ_c, respectively, and replace $d\sigma$ by $(at)^{-2}dS$ or $(bt)^{-2}dS$, as the case may be; then

$$\xi_a = \frac{1}{4\pi a^2 t} \iint l\,(q_0)_{at}\,dS. \quad \ldots\ldots\ldots \quad (32).$$

Let O_1 be a fixed point, taken within the space T, and regarded as the point of reference for all such points as O. Then when O is at such a distance from O_1 that the radius vector, drawn from O, of any element of T makes but a very small angle with OO_1, we may regard l as constant in the integration, and equal to the cosine of the angle between OO_1 and the direction in which we wish to estimate the displacement at O. Moreover the portion of the surface of a sphere having O for centre which lies within T will be ultimately a plane perpendicular to OO_1, and q_0 will be ultimately the initial velocity resolved in the direction OO_1. Hence we have ultimately

$$\xi_a = \frac{l}{4\pi a^2 t} \iint (q_0)_{at} dS,$$

where, for a given direction of O_1O, the integral receives the same series of values, as at increases through the value OO_1, whatever be the distance of O from O_1. Since the direction of the axis of x is arbitrary, and the component of the displacement in that direction is found by multiplying by l a quantity independent of the direction of the axes, it follows that the displacement itself is in the direction OO_1, or in the direction of a normal to the wave. For a given direction of O_1O, the law of disturbance is the same at one distance as at another, and the magnitude of the displacements varies inversely as at, the distance which the wave has travelled in the time t.

We get in a similar manner

$$\xi_b = \frac{1}{4\pi b^2 t} \iint (u_0 - lq_0)_{bt} dS, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

where l, and the direction of the resolved part, q_0, of the initial velocity are ultimately constant, and the surface of which dS is an element is ultimately plane. To find the resolved part of the displacement in the direction OO_1, we must suppose x measured in that direction, and therefore put $l = 1$, $q_0 = u_0$, which gives $\xi_b = 0$. Hence the displacement now considered takes place in a direction perpendicular to OO_1, or is *transversal*.

For a given direction of O_1O, the law of disturbance is constant, but the magnitude of the displacements varies inversely as bt, the distance to which the wave has been propagated. To find the displacement in any direction, OE, perpendicular to OO_1, we have only to take OE for the direction of the axis of x, and therefore put $l = 0$, and suppose u_0 to refer to this direction.

Consider, lastly, the displacement, ξ_c, expressed by the last term in equation (29). The form of the expression shews that ξ_c will be a small quantity of the order $\frac{t}{r^3}$ or $\frac{1}{r^2}$, since t is of the same order as r; for otherwise the space T would lie outside the limits of integration, and the triple integral would vanish. But ξ_a and ξ_b are of the order $\frac{1}{r}$, and therefore ξ_c may be neglected, except in the immediate neighbourhood of T.

To see more clearly the relative magnitudes of these quantities, let v be a velocity which may be used as a standard of comparison of the initial velocities, R the radius of a sphere whose volume is equal to that of the space T, and compare the displacements ξ_a, ξ_b, ξ_c which exist, though at different times, at the same point O, where $O_1O = r$. These displacements are comparable with

$$\frac{vR^3}{ar}, \quad \frac{vR^3}{br}, \quad \frac{vR^3 t}{r^3},$$

which are proportional to

$$\frac{1}{a}, \quad \frac{1}{b}, \quad \frac{R}{r} \cdot \frac{t}{r}.$$

But, in order that the triple integral in (29) may not wholly vanish, $\frac{t}{r}$ must lie between the limits $\frac{1}{a}$ and $\frac{1}{b}$, or at most lie a very little outside these limits, which it may do in consequence of the finite thickness of the two waves. Hence the quantity neglected in neglecting ξ_e is of the order $\frac{R}{r}$ compared with the quantities retained.

The important terms in the disturbance due to the initial displacements might be got from equation (30), but they may be deduced immediately from the corresponding terms in the disturbance due to the initial velocities by the theorem of Art. 14.

24. If we confine our attention to the terms which vary ultimately inversely as the distance, and which alone are sensible at a great distance from T, we shall be able, by means of the formulæ of the preceding article, to obtain a clear conception of the motion which takes place, and of its connexion with the initial disturbance.

From the fixed point O_1, draw in any direction the right line O_1O equal to r, r being so large that the angle subtended at O by any two elements of T is very small; and let it be required to consider the disturbance at O. Draw a plane P perpendicular to OO_1, and cutting OO_1 produced at a distance p from O_1. Let $-p_1$, $+p_2$ be the two extreme values of p for which the plane P cuts the space T. Conceive the displacements and velocities resolved in three rectangular directions, the first of these, to which ξ and u relate, being the direction OO_1. Let $f_u(p)$, $f_v(p)$, $f_w(p)$ be three functions of p defined by the equations

$$f_u(p) = \iint u_0\, dS,\ f_v(p) = \iint v_0\, dS,\ f_w(p) = \iint w_0\, dS, \quad . \quad . \quad (34)$$

and $f_\xi(p)$, $f_\eta(p)$, $f_\zeta(p)$ three other functions depending on the initial displacements as the first three do on the initial velocities, so that

$$f_\xi(p) = \iint \xi_0\, dS,\ f_\eta(p) = \iint \eta_0\, dS,\ f_\zeta(p) = \iint \zeta_0\, dS. \quad . \quad . \quad . \quad (35).$$

These functions, it will be observed, vanish when the variable lies outside of the limits $-p_1$ and $+p_2$. They depend upon the direction O_1O, so that in passing to another direction their values change, as well as the limits of the variable between which they differ from zero. It may be remarked however that in passing from any one direction to its opposite the functions receive the same values, as the variable decreases from $+p_1$ to $-p_2$, that they before received as the variable increased from $-p_1$ to $+p_2$, provided the directions in which the displacements are resolved, as well as the sides towards which the resolved parts are reckoned positive, are the same in the two cases.

The medium about O remains at rest until the end of the time $a^{-1}(r - p_1)$, when the wave of dilatation reaches O. During the passage of this wave, the displacements and velocities are given by the equations

$$\left.\begin{aligned} \xi &= \frac{1}{4\pi a r} f_u(at - r) + \frac{1}{4\pi r} f_\xi'(at - r), \quad \eta = 0, \quad \zeta = 0, \\ u &= \frac{1}{4\pi r} f_u'(at - r) + \frac{a}{4\pi r} f_\xi''(at - r), \quad v = 0, \quad w = 0. \end{aligned}\right\} \quad . \quad (36).$$

The first term in the right-hand member of the first of these equations is got from (32) by putting $l = 1$, introducing the function f_u, and replacing at in the denominator by r, which may be done, since at differs from r only by a small quantity depending upon the finite dimensions of the space T. The second term is derived from the first by the theorem of Art. 14, and u is of course got from ξ by differentiating with respect to t. Had t been retained in the denominator, the differentiation would have introduced terms of the order t^{-2}, and therefore of the order r^{-2}, but such terms are supposed to be neglected.

The wave of dilatation will have just passed over O at the end of the time $a^{-1}(r + p_2)$. The medium about O will then remain sensibly at rest in its position of equilibrium till the wave of distortion reaches it, that is, till the end of the time $b^{-1}(r - p_1)$. During the passage of this wave, the displacements and velocities will be given by the equations

$$\left.\begin{aligned} \xi = 0, \quad \eta &= \frac{1}{4\pi b r} f_v\,(bt - r) + \frac{1}{4\pi r} f_\eta'\,(bt - r), \\ \zeta &= \frac{1}{4\pi b r} f_w\,(bt - r) + \frac{1}{4\pi r} f_\zeta'\,(bt - r), \\ u = 0, \quad v &= \frac{1}{4\pi r} f_v'\,(bt - r) + \frac{b}{4\pi r} f_\eta''\,(bt - r), \\ w &= \frac{1}{4\pi r} f_w'\,(bt - r) + \frac{b}{4\pi r} f_\zeta''\,(bt - r). \end{aligned}\right\} \quad . \quad . \quad (37).$$

After the passage of the wave of distortion, which occupies an interval of time equal to $b^{-1}(p_1 + p_2)$, the medium will return absolutely to rest in its position of equilibrium.

25. A caution is here necessary with reference to the employment of equation (30). If we confine our attention to the important terms, we get

$$\xi = \frac{1}{4\pi a t} \iint l \left(\frac{d\rho_0}{dr}\right)_{at} dS + \frac{1}{4\pi b t} \iint \left\{\frac{d\xi_0}{dr} - l\,\frac{d\rho_0}{dr}\right\}_{bt} dS. \quad . \quad (38).$$

Now the initial displacements and velocities are supposed to have finite, but otherwise arbitrary, values within the space T, and to vanish outside. Consequently we cannot, without unwarrantably limiting of the generality of the problem, exclude from consideration the cases in which the initial displacements and velocities alter abruptly in passing across the surface of T. In particular, if we wish to determine the disturbance at the end of the time t due to the initial disturbance in a part only of the space throughout which the medium was originally disturbed, we are obliged to consider such abrupt variations; and this is precisely what occurs in treating the problem of diffraction. In applying equation (38) to such a case, we must consider the abrupt variation as a limiting case of a continuous, but rapid, variation, and we shall have to add to the double integrals found by taking for $\dfrac{d\rho_0}{dr}$ and $\dfrac{d\xi_0}{dr}$ the finite values which refer to the space T, certain single integrals referring to the perimeter of that portion of the plane P which lies within T. The easiest way of treating the integrals is, to reserve the differentiation with respect to t from which the differential coefficients just written have arisen until after the double integration, and we shall thus be led to the formulæ of the preceding article, where the correct values of the terms in question were obtained at once by the theorem of Art. 14.

26. It appears from Arts. 11 and 12, that in the wave of distortion the density of the medium is strictly the same as in equilibrium; but the result obtained in Art. 23, that the displacements in this wave are transversal, that is, perpendicular to the radius of the wave, is only approximate, the approximation depending upon the largeness of the radius, r, of the wave compared with the dimensions of the space T, or, which comes to the same, compared with the thickness of the wave. In fact, if it were strictly true that the displacement at O due to the original disturbance in each element of the space T was transversal, it is evident that the crossing at O of the various waves corresponding to the various elements of T under finite, though small angles, would prevent the whole displacement from being strictly perpendicular to the radius vector drawn to O from an arbitrarily chosen point, O_1, within T. But it is not mathematically true that the disturbance proceeding from even a single point O_1, when a disturbing force is supposed to act, or rather that part of the disturbance which is propagated with the velocity b, is perpendicular to OO_1, as will be seen more clearly in the next article. It is only so nearly perpendicular that it may be regarded as strictly so without sensible error. As the wave grows larger, the inclination of the direction of displacement to the wave's front decreases with great rapidity.

Thus the motion of a layer of the medium in the front of a wave may be compared with the tidal motion of the sea, or rather with what it would be if the earth were wholly covered by water. In both cases the density of the medium is unchanged, and there is a slight increase or decrease of thickness in the layer, which allows the motion along the surface to take place without change of density: in both cases the motion in a direction perpendicular to the surface is very small compared with the motion along the surface.

27. From the integral already obtained of the equations of motion, it will be easy to deduce the disturbance due to a given variable force acting in a given direction at a given point of the medium.

Let O_1 be the given point, T a space comprising O_1. Let the time t be divided into equal intervals τ; and at the beginning of the n^{th} interval let the velocity $\tau F(n\tau)$ be communicated, in the given direction, to that portion of the medium which occupies the space T. Conceive velocities communicated in this manner at the beginning of each interval, so that the disturbances produced by these several velocities are superposed. Let D be the density of the medium in equilibrium; and let $F(n\tau) = (DT)^{-1}f(n\tau)$, so that $\tau f(n\tau)$ is the momentum communicated at the beginning of the n^{th} interval. Now suppose the number of intervals τ indefinitely increased, and the volume T indefinitely diminished, and we shall pass in the limit to the case of a moving force which acts continuously.

The disturbance produced by given initial velocities is expressed, without approximation, by equation (29), that is, without any approximation depending on the largeness of the distance OO_1; for the square of the disturbance has been neglected all along. Let $OO_1 = r$; refer the displacement at O to the rectangular axes of x, y, z; let l, m, n be the direction-cosines of OO_1; l', m', n' those of the given force, and put for shortness k for the cosine of the angle between the direction of the force and the line OO_1 produced, so that

$$k = ll' + mm' + nn'.$$

Consider at present the first term of the right-hand side of (29). Since the radius vector drawn from O to any element of T ultimately coincides with OO_1, we may put l outside the integral signs, and replace $d\sigma$ by $r^{-2}dS$. Moreover, since this term vanishes except when at lies between the greatest and least values of the radius vector drawn from O to any element of T, we may replace t outside the integral signs by $a^{-1}r$. Conceive a series of spheres, with radii $a\tau$, $2a\tau \ldots na\tau, \ldots$ described round O, and let the n^{th} of these be the first which cuts T. Let S_1, $S_2 \ldots$ be the areas of the surfaces of the spheres, beginning with the n^{th}, which lie within T; then

$$\iint (q_0)_{at} dS = k\tau F(t - n\tau) S_1 + k\tau F\{t - (n+1)\tau\} S_2 + \ldots$$

But $F(t - n\tau)$, $F\{t - (n+1)\tau\} \ldots$ are ultimately equal to each other, and to

$$F\left(t - \frac{r}{a}\right), \text{ or } (DT)^{-1} f\left(t - \frac{r}{a}\right);$$

and $a\tau S_1 + a\tau S_2 + \ldots$ is ultimately equal to T. Hence we get, for the part of ξ which arises from the first of the double integrals, $\dfrac{lk}{4\pi Da^2 r} f\left(t - \dfrac{r}{a}\right)$. The second of the double integrals is to be treated in exactly the same way.

To find what the triple integral becomes, let us consider first only the impulse which was communicated at the beginning of the time $t - n\tau$, where $n\tau$ lies between the limits $a^{-1}r$ and $b^{-1}r$, and is not so nearly equal to one of these limits that any portion of the space T lies beyond the limits of integration. Then we must write $n\tau$ for t in the coefficient, and $3lq_0 - u_0$ becomes ultimately $(3lk - l')\tau F(t - n\tau)$, and, as well as r, is ultimately constant in the triple integration. Hence the triple integral ultimately becomes

$$\frac{(3lk - l')T}{4\pi r^3} n\tau \,.\, \tau F(t - n\tau),$$

and we have now to perform a summation with reference to different values of n, which in the limit becomes an integration. Putting $n\tau = t'$, we have ultimately

$$\tau = dt', \quad \Sigma n\tau \,.\, \tau F(t - n\tau) = \int_{\frac{r}{a}}^{\frac{r}{b}} t' F(t - t')\, dt'.$$

It is easily seen that the terms arising from the triple integral when it has to be extended over a part only of the space T vanish in the limit. Hence we have, collecting all the terms, and expressing $F(t)$ in terms of $f(t)$,

$$\xi = \frac{lk}{4\pi Da^2 r} f\left(t - \frac{r}{a}\right) + \frac{l' - lk}{4\pi Db^2 r} f\left(t - \frac{r}{b}\right) + \frac{3lk - l'}{4\pi Dr^3} \int_{\frac{r}{a}}^{\frac{r}{b}} t' f(t - t')\, dt'. \quad . \quad . \quad (39).$$

To get η and ζ, we have only to pass from l, l' to m, m' and then to n, n'. If we take OO_1 for the axis of x, and the plane passing through OO_1 and the direction of the force for the plane xz, and put a for the inclination of the direction of the force to OO_1 produced, we shall have

$$l = 1,\ m = 0,\ n = 0,\ l' = k = \cos a,\ m' = 0,\ n' = \sin a\,;$$

whence

$$\left.\begin{aligned} \xi &= \frac{\cos a}{4\pi D a^2 r} f\left(t - \frac{r}{a}\right) + \frac{\cos a}{2\pi D r^3} \int_{\frac{r}{a}}^{\frac{r}{b}} t' f(t - t')\, dt', \\ \eta &= 0, \\ \zeta &= \frac{\sin a}{4\pi D b^2 r} f\left(t - \frac{r}{b}\right) - \frac{\sin a}{4\pi D r^3} \int_{\frac{r}{a}}^{\frac{r}{b}} t' f(t - t')\, dt'. \end{aligned}\right\} \quad . \quad . \quad (40)$$

In the investigation, it has been supposed that the force began to act at the time 0, before which the fluid was at rest, so that $f(t) = 0$ when t is negative. But it is evident that exactly the same reasoning would have applied had the force begun to act at any past epoch, as remote as we please, so that we are not obliged to suppose $f(t)$ equal to zero when t is negative, and we may even suppose $f(t)$ periodic, so as to have finite values from $t = -\infty$ to $t = +\infty$.

By means of the formulæ (39), it would be very easy to write down the expressions for the disturbance due to a system of forces acting throughout any finite portion of the medium, the disturbing force varying in any given manner, both as to magnitude and direction, from one point of the medium to another, as well as from one instant of time to another.

The first term in ξ represents a disturbance which is propagated from O_1 with a velocity a. Since there is no corresponding term in η or ζ, the displacement, as far as relates to this disturbance, is strictly normal to the front of the wave. The first term in ζ represents a disturbance which is propagated from O_1 with a velocity b, and as far as relates to this disturbance the displacement takes place strictly in the front of the wave. The remaining terms in ξ and ζ represent a disturbance of the same kind as that which takes place in an incompressible fluid in consequence of the motion of solid bodies in it. If $f(t)$ represent a force which acts for a short time, and then ceases, $f(t - t')$ will differ from zero only between certain narrow limits of t, and the integral contained in the last terms of ξ and ζ will be of the order r, and therefore the terms themselves will be of the order r^{-2}, whereas the leading terms are of the order r^{-1}. Hence in this case the former terms will not be sensible beyond the immediate neighbourhood of O_1. The same will be true if $f(t)$ represent a periodic force, the mean value of which is zero. But if $f(t)$ represent a force always acting one way, as for example a constant force, the last terms in ξ and ζ will be of the same order, when r is as large as the first terms.

28. It has been remarked in the introduction that there is strong reason for believing that in the case of the luminiferous ether the ratio of a to b is extremely large, if not infinite. Consequently the first term in ξ, which relates to normal vibrations, will be insensible, if not absolutely evanescent. In fact, if the ratio of a to b were no greater than 100, the denominator in this term would be 10000 times as great as the denominator of the first term in ζ. Now the molecules of a solid or gas in the act of combustion are probably thrown into a state of violent vibration, and may be regarded, at least very approximately, as centres of disturbing forces. We may thus see why transversal vibrations should alone be produced,

unaccompanied by normal vibrations, or at least by any which are of sufficient magnitude to be sensible. If we could be sure that the ether was strictly incompressible, we should of course be justified in asserting that normal vibrations are impossible.

29. If we suppose $a = \infty$, and $f(t) = c \sin \dfrac{2\pi bt}{\lambda}$, we shall get from (40)

$$\left.\begin{aligned}
\xi &= \frac{c\lambda \cos a}{4\pi^2 D b^2 r^2} \cos \frac{2\pi}{\lambda}(bt - r) - \frac{c\lambda^2 \cos a}{4\pi^3 D b^2 r^3} \sin \frac{\pi r}{\lambda} \cos \frac{2\pi}{\lambda}\left(bt - \frac{r}{2}\right),\\
\eta &= 0,\\
\zeta &= \frac{c \sin a}{4\pi D b^2 r} \sin \frac{2\pi}{\lambda}(bt - r) - \frac{c\lambda \sin a}{8\pi^2 D b^2 r^2} \cos \frac{2\pi}{\lambda}(bt - r) + \frac{c\lambda^2 \sin a}{8\pi^3 D b^2 r^3} \sin \frac{\pi r}{\lambda} \cos \frac{2\pi}{\lambda}\left(bt - \frac{r}{2}\right);
\end{aligned}\right\} \quad (41)$$

and we see that the most important term in ξ is of the order $\dfrac{\lambda}{\pi r}$ compared with the leading term in ζ, which represents the transversal vibrations properly so called. Hence ξ, and the second and third terms in ζ, will be insensible, except at a distance from O, comparable with λ, and may be neglected; but the existence of terms of this nature, in the case of a spherical wave whose radius is not regarded as infinite, must be borne in mind, in order to understand in what manner transversal vibrations are compatible with the absence of dilatation or condensation.

30. The integration of equations (18) might have been effected somewhat differently by first decomposing the given functions ξ_0, η_0, ζ_0, and u_0, v_0, w_0 into two parts, as in Art. 8, and then treating each part separately. We should thus be led to consider separately that part of the initial disturbance which relates to a wave of dilatation and that part which relates to a wave of distortion. Either of these parts, taken separately, represents a disturbance which is not confined to the space T, but extends indefinitely around it. Outside T, the two disturbances are equal in magnitude and opposite in sign.

SECTION III.

DETERMINATION OF THE LAW OF THE DISTURBANCE IN A SECONDARY WAVE OF LIGHT.

31. Conceive a series of plane waves of plane-polarized light propagated in vacuum in a direction perpendicular to a fixed mathematical plane P. According to the undulatory theory of light, as commonly received, that is, including the doctrine of transverse vibrations, the light in the case above supposed consists in the vibrations of an elastic medium or *ether*, the vibrations being such that the ether moves in sheets, in a direction perpendicular to that of propagation, and the vibration of each particle being symmetrical with respect to the plane of polarization, and therefore rectilinear, and either parallel or perpendicular to that plane. In order to account for the propagation of such vibrations, it is necessary to suppose the existence of a tangential force, or tangential pressure, called into play by the continuous sliding of the

sheets one over another, and proportional to the amount of the displacement of sliding. There is no occasion to enter into any speculation as to the cause of this tangential force, nor to entertain the question whether the luminiferous ether consists of distinct molecules or is mathematically continuous, just as there is no occasion to speculate as to the cause of gravity in calculating the motions of the planets. But we are absolutely obliged to suppose the existence of such a force, unless we are prepared to throw overboard the theory of transversal vibrations, as usually received, notwithstanding the multitude of curious, and otherwise apparently inexplicable phenomena which that theory explains with the utmost simplicity. Consequently we are led to treat the ether as an elastic solid *so far as the motions which constitute light are concerned*. It does not at all follow that the ether is to be regarded as an elastic solid when large displacements are considered, such as we may conceive produced by the earth and planets, and solid bodies in general, moving through it. The mathematical theories of fluids and of elastic solids are founded, or at least may be founded, on the consideration of internal pressures. In the case of a fluid, these pressures are supposed normal to the common surface of the two portions whose mutual action is considered: this supposition forms in fact the mathematical definition of a fluid. In the case of an elastic solid, the pressures are in general oblique, and may even in certain directions be wholly tangential. The treatment of the question by means of pressures presupposes the absence of any sensible *direct* mutual action of two portions of the medium which are separated by a small but sensible interval. The state of constraint or of motion of any element affects the pressures in the surrounding medium, and in this way one element exerts an *indirect* action on another from which it is separated by a sensible interval.

Now the absence of prismatic colours in the stars, depending upon aberration, the absence of colour in the disappearance and reappearance of Jupiter's Satellites in the case of eclipses, and, still more, the absence of change of colour in the case of certain periodic stars, especially the star Algol, shew that the velocity of light of different colours is, if not mathematically, at least sensibly the same. According to the theory of undulations, this is equivalent to saying that in vacuum the velocity of propagation is independent of the length of the waves. Consequently the direct action of two elements of ether separated by a sensible interval must be sensibly if not mathematically equal to zero, or at least must be independent of the disturbance; for, were this not the case, the expression for the velocity of propagation would involve the length of a wave. An interval is here considered sensible which is comparable with the length of a wave. We are thus led to apply to the luminiferous ether in vacuum the ordinary equations of motion of an elastic solid, provided we are only considering those disturbances which constitute light.

Let us return now to the case supposed at the beginning of this section. According to the preceding explanation, we must regard the ether as an elastic solid, in which a series of rectilinear transversal vibrations is propagated in a direction perpendicular to the plane *P*. The disturbance at any distance in front of this plane is really produced by the disturbance continually transmitted across it; and, according to the general principle of the superposition of small motions, we have a perfect right to regard the disturbance in front as the aggregate of the elementary disturbances due to the disturbance continually transmitted across the several elements into which we may conceive the plane P divided. Let it then be required to determine the disturbance corresponding to an elementary portion only of this plane.

In practical cases of diffraction at an aperture, the breadth of the aperture is frequently sensible, though small, compared with the radius of the incident waves. But in determining the law of disturbance in a secondary wave we have nothing to do with an aperture; and in order that we should be at liberty to regard the incident waves as plane all that is necessary is, that the radius of the incident wave should be very large compared with the wave's length, a condition always fulfilled in experiment.

32. Let O_1 be any point in the plane P; and refer the medium to rectangular axes passing through O_1, x being measured in the direction of propagation of the incident light, and z in the direction of vibration. Let $f(bt - x)$ denote the displacement of the medium at any point behind the plane P, x of course being negative. Let the time t be divided into small intervals, each equal to τ, and consider separately the effect of the disturbance which is transmitted across the plane P during each separate interval. The disturbance transmitted during the interval τ which begins at the end of the time t' occupies a film of the medium, of thickness $b\tau$, and consists of a displacement $f(bt')$ and a velocity $bf'(bt')$. By the formulæ of Section II. we may find the effect, over the whole medium, of the disturbance which exists in so much only of the film as corresponds to an element dS of P adjacent to O_1. By doing the same for each interval τ, and then making the number of such intervals increase and the magnitude of each decrease indefinitely, we shall ultimately obtain the effect of the disturbance which is continually propagated across the element dS.

Let O be the point of the medium at which the disturbance is required; l, m, n the direction-cosines of O_1O measured from O_1, and therefore $-l$, $-m$, $-n$ those of OO_1 measured from O; and let $OO_1 = r$. Consider first the disturbance due to the velocity of the film. The displacements which express this disturbance are given without approximation by (29) and the two other equations which may be written down from symmetry. The first terms in these equations relate to normal vibrations, and on that account alone might be omitted in considering the diffraction of light. But, besides this, it is to be observed that t in the coefficient of these terms is to be replaced by $a^{-1}r$. Now there seems little doubt, as has been already remarked in the introduction, that in the case of the luminiferous ether a is incomparably greater than b, if not absolutely infinite*; so that the terms in question are insensible, if not absolutely evanescent. The third terms are insensible, except at a distance from O_1 comparable with λ, as has been already observed, and they may therefore be omitted if we suppose r very large compared with the length of a wave. Hence it will be sufficient to consider the second terms only. In the coefficient of these terms we must replace t by $b^{-1}r$; we must put $u_0 = 0$, $v_0 = 0$, $w_0 = bf'(bt - r)$, write $-l$, $-m$, $-n$ for l, m, n, and put $q_0 = -nw_0 = -nbf'(bt - r)$. The integral signs are to be omitted, since we want to get the disturbance which corresponds to an elementary portion only of the plane P.

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* I have explained at full my views on this subject in a paper *On the constitution of the luminiferous ether* printed in the 32nd volume of the *Philosophical Magazine*, p. 349.

It is to be observed that $d\sigma$ represents the elementary solid angle subtended at O by an element of the riband formed by that portion of the surface of a sphere described round O, with radius r, which lies between the plane yz and the parallel plane whose abscissa is $b\tau$. To find the aggregate disturbance at O corresponding to a small portion, S, of the plane P lying about O_1, we must describe spheres with radii $\ldots r - 2b\tau, r - b\tau, r, r + b\tau, r + 2b\tau \ldots$, describing as many as cut S. These spheres cut S into ribands, which are ultimately equal to the corresponding ribands which lie on the spheres. For, conceive a plane drawn through OO_1 perpendicular to the plane yz. The intersections of this plane by two consecutive spheres and the two parallel planes form a quadrilateral, which is ultimately a rhombus; so that the breadths of corresponding ribands on a sphere and on the plane are equal, and their lengths are also equal, and therefore their areas are equal. Hence we must replace $d\sigma$ by $r^{-2}\,dS$, and we get accordingly

$$\xi = -\frac{l\,n\,dS}{4\pi r} f'(bt - r), \quad \eta = -\frac{m\,n\,dS}{4\pi r} f'(bt - r), \quad \zeta = \frac{(1 - n^2)\,dS}{4\pi r} f'(bt - r). \quad . \quad . \quad (42)$$

Since $l\xi + m\eta + n\zeta = 0$, the displacement takes place in a plane through O perpendicular to O_1O. Again, since $\xi : \eta :: l : m$, it takes place in a plane through O_1O and the axis of z. Hence it takes place along a line drawn in the plane last mentioned perpendicular to OO_1. The direction of displacement being known, it remains only to determine the magnitude. Let ζ_1 be the displacement, and ϕ the angle between O_1O and the axis of z, so that $n = \cos\phi$. Then $\zeta_1 \sin\phi$ will be the displacement in the direction of z, and equating this to ζ in (42) we get

$$\zeta_1 = \frac{dS}{4\pi r} \sin\phi\, f'(bt - r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The part of the disturbance due to the successive displacements of the films may be got in the same way from (30) and the two other equations of the same system. The only terms which it will be necessary to retain in these equations are those which involve the differential coefficients of ξ_0, η_0, ζ_0, and ρ_0 in the second of the double integrals. We must put as before r for bt, and write $r^{-2}dS$ for $d\sigma$. Moreover we have for the incident vibrations

$$\xi = 0, \quad \eta = 0, \quad \zeta = f(bt' - x), \quad \rho = -nf(bt' - x).$$

To find the values of the differential coefficients which have to be used in (30) and the two other equations of that system, we must differentiate on the supposition that ξ, η, ζ, ρ are functions of r in consequence of being functions of x, and after differentiation we must put $x = 0$, $t' = t - \frac{r}{b}$. Since $\frac{d}{dr} = -l\frac{d}{dx}$, we get

$$\left(\frac{d\zeta_0}{dr}\right)_{bt} = lf'(bt - r), \quad \left(\frac{d\rho_0}{dr}\right)_{bt} = -lnf'(bt - r),$$

whence we get, remembering that the signs of l, m, n in (30) have to be changed,

$$\xi = -\frac{l^2 n\,dS}{4\pi r} f'(bt - r), \quad \eta = -\frac{lmn\,dS}{4\pi r} f'(bt - r), \quad \zeta = \frac{l(1 - n^2)\,dS}{4\pi r} f'(bt - r)$$

The displacement represented by these equations takes place along the same line as before; and if we put ζ_2 for the displacement, and write $\cos\theta$ for l, we get

$$\zeta_2 = \frac{dS}{4\pi r}\cos\theta\sin\phi f'(bt - r). \quad . \quad . \quad . \quad . \qquad (44).$$

33. By combining the partial results obtained in the preceding article, we arrive at the following theorem.

Let $\xi = 0$, $\eta = 0$, $\zeta = f(bt - x)$ be the displacements corresponding to the incident light; let O_1 be any point in the plane P, dS an element of that plane adjacent to O_1; and consider the disturbance due to that portion only of the incident disturbance which passes continually across dS. Let O be any point in the medium situated at a distance from the point O_1 which is large in comparison with the length of a wave; let $O_1O = r$, and let this line make angles θ with the direction of propagation of the incident light, on the axis of x, and ϕ with the direction of vibration, on the axis of z. Then the displacement at O will take place in a direction perpendicular to O_1O, and lying in the plane zO_1O; and if ζ' be the displacement at O, reckoned positive in the direction nearest to that in which the incident vibrations are reckoned positive,

$$\zeta' = \frac{dS}{4\pi r}(1 + \cos\theta)\sin\phi f'(bt - r)^*. \quad . \quad . \quad . \qquad (45).$$

In particular, if

$$f(bt - x) = c\sin\frac{2\pi}{\lambda}(bt - x),$$

we shall have

$$\zeta' = \frac{c\,dS}{2\lambda r}(1 + \cos\theta)\sin\phi\cos\frac{2\pi}{\lambda}(bt - r). \quad . \quad . \quad . \qquad (46).$$

34. On finding by means of this formula the aggregate disturbance at O due to all the elements of the plane P, O being supposed to be situated at a great distance from P, we ought to arrive at the same result as if the waves had not been broken up.

To verify this, let fall from O the perpendicular OO' on the plane P, and let $OO' = p$, or $= -p$, according as O is situated in front of the plane P or behind it. Through O' draw $O'x'$, $O'y'$, parallel to O_1x, O_1y, and let $O'O_1 = r'$, $O_1O'y' = \omega$. Then $dS = r'dr'd\omega = rdrd\omega$, since $r^2 = p^2 + r'^2$, and p is constant. Let $\zeta' = s\sin\phi$. The displacement ζ' takes place in the plane zO_1O, and perpendicular to O_1O; and resolving it along and perpendicular to O_1z, we get for resolved parts $s\sin^2\phi$, $s\sin\phi\cos\phi$, of which the latter is estimated in the direction OM, where M is the projection of O_1 on $O'y'$. Let $MOO' = \chi$, χ being reckoned positive when M falls on that side of O' on which y' is reckoned positive; then, resolving the displacement along OM parallel to $O'x'$, $O'y'$, we get for resolved parts $-s\sin\phi\cos\phi\cos\chi$, $s\sin\phi\cos\phi\sin\chi$. Hence we get for the displacements ξ, η, ζ at O

$$\xi = -s\sin\phi\cos\phi\cos\chi, \quad \eta = s\sin\phi\cos\phi\sin\chi, \quad \zeta = s\sin^2\phi.$$

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[*] The corresponding expression which I have obtained for sound differs from this only in having $\cos\theta$ in place of $\sin\phi$, provided we suppose b to be the velocity of propagation of sound, and ζ' to represent a displacement in the direction O_1O.

Now produce $O'O_1$ to O_2, and refer O_1x, O_1y, O_1z, O_1O_2, O_1O to a sphere described round O_1 with radius unity. Then zO_2O forms a spherical triangle, right angled at O_2, and $zO_2 = \frac{\pi}{2} - \omega$, $O_2O = \frac{\pi}{2} + \theta$, $Oz = \phi$, $OzO_2 = \frac{\pi}{2} + \chi$, whence we get from spherical trigonometry,

$$\cos\phi = -\sin\theta\sin\omega, \quad \sin\phi\cos\chi = \cos\theta, \quad \sin\phi\sin\chi = \cos\theta\tan\chi = \sin\theta\cos\omega.$$

We have therefore

$$\xi = s\sin\theta\cos\theta\sin\omega, \quad \eta = -s\sin^2\theta\sin\omega\cos\omega, \quad \zeta = s\,(1 - \sin^2\theta\sin^2\omega).$$

To find the aggregate disturbance at O, we must put for s its value, and perform the double integrations, the limits of ω being 0 and 2π, and those of r being $\sqrt{p^2}$ and ∞. The positive and negative parts of the integrals which give ξ and η will evidently destroy each other, and we need therefore only consider ζ. Putting for s its value, and expressing θ in terms of r, we get

$$\zeta = \frac{c}{2\lambda}\iint (r+p)\,(r^2\cos^2\omega + p^2\sin^2\omega)\cos\frac{2\pi}{\lambda}(bt-r)\,\frac{dr\,d\omega}{r^3} \quad . \quad . \quad (47)$$

Let us first conceive the integration performed over a large area A surrounding O', which we may afterwards suppose to increase indefinitely. Perform the integration with respect to r first, put for shortness $F(r)$ for the coefficient of the cosine under the integral signs, and let R, a function of ω, be the superior limit of r. We get by integration by parts

$$\int F(r)\cos\frac{2\pi}{\lambda}(bt-r)\,dr = -\frac{\lambda}{2\pi}F(r)\sin\frac{2\pi}{\lambda}(bt-r) + \left(\frac{\lambda}{2\pi}\right)^2 F'(r)\cos\frac{2\pi}{\lambda}(bt-r) + \ldots$$

Now the terms after the first must be neglected for consistency's sake, because the formula (46) is not exact, but only approximate, the approximation depending on the neglect of terms which are of the order λ compared with those retained. The first term, taken between limits, gives

$$\frac{\lambda}{2\pi}F(\pm p)\sin\frac{2\pi}{\lambda}(bt \mp p) - \frac{\lambda}{2\pi}F(R)\sin\frac{2\pi}{\lambda}(bt-R),$$

where the upper or lower sign has to be taken according as O lies in front of the plane P or behind it. We thus get from (47)

$$\zeta = \frac{c}{2}(1 \pm 1)\sin\frac{2\pi}{\lambda}(bt \mp p) - \frac{c}{4\pi}\int_0^{2\pi} F(R)\sin\frac{2\pi}{\lambda}(bt-R)\,d\omega.$$

When R becomes infinite, $F(R)$ reduces itself to $\cos^2\omega$, and the last term in ζ becomes

$$-\frac{c}{4\pi}\int_0^{2\pi}\cos^2\omega\sin\frac{2\pi}{\lambda}(bt-R)\,d\omega.$$

Suppose that no finite portion of the perimeter of A is a circular arc with O' for centre, and let this perimeter be conceived to expand indefinitely, remaining similar to itself. Then, for any finite interval, however small, in the integration with respect to ω, the function $\sin\frac{2\pi}{\lambda}(bt-R)$ will change sign an infinite number of times, having a mean value which is ultimately zero, and the limit of the above expression will be rigorously zero. Hence we get in the limit

$$\zeta = c \sin \frac{2\pi}{\lambda}(bt - p), \text{ or } = 0,$$

according as p is positive or negative. Hence the disturbance continually transmitted across the plane P produces the same disturbance in front of that plane as if the wave had not been broken up, and does not produce any back wave, which is what it was required to verify.

It may be objected that the supposition that the perimeter of A is free from circular arcs having O' for centre is an arbitrary restriction. The reply to this objection is, that we have no right to assume that the disturbance at O which corresponds to an area A approaches in all cases to a limit as A expands, remaining similar to itself. All we have a right to assert *a priori* is, that *if* it approaches a limit that limit must be the disturbance which would exist if the wave had not been broken up.

It is hardly necessary to observe that the more general formula (45) might have been treated in precisely the same way as (46).

35. In the third Volume of the *Cambridge Mathematical Journal*, p. 46, will be found a short paper by Mr. Archibald Smith, of which the object is to determine the intensity in a secondary wave of light. In this paper the author supposes the intensity at a given distance the same in all directions, and assumes the coefficient of vibration to vary, in a given direction, inversely as the radius of the secondary wave. The intensity is determined on the principle that when an infinite plane wave is conceived to be broken up, the aggregate effect of the secondary waves must be the same as that of the primary wave. In the investigation, the difference of direction of the vibrations corresponding to the various secondary waves which agitate a given point is not taken into account, and moreover a term which appears under the form $\cos \infty$ is assumed to vanish. The correctness of the result arrived at by the latter assumption may be shewn by considerations similar to those which have just been developed. If we suppose the distance from the primary wave of the point which is agitated by the secondary waves to be large in comparison with λ, it is only those secondary waves which reach the point in question in a direction nearly coinciding with the normal to the primary wave that produce a sensible effect, since the others neutralize each other at that point by interference. Hence the result will be true for a direction nearly coinciding with the normal to the primary wave, independently of the truth of the assumption that the disturbance in a secondary wave is equal in all directions, and notwithstanding the neglect of the mutual inclination of the directions of the disturbances corresponding to the various secondary waves. Accordingly, when the direction considered is nearly that of the normal to the primary wave, $\cos\theta$ and $\sin\phi$ in (46) are each nearly equal to 1, so that the coefficient of the circular function becomes $c\,dS\,(\lambda r)^{-1}$, nearly, and in passing from the primary to the secondary waves it is necessary to accelerate the phase by a quarter of an undulation. This agrees with Mr. Smith's results.

The same subject has been treated by Professor Kelland in a memoir *On the Theoretical*

Investigation of the Absolute Intensity of Interfering Light, printed in the fifteenth Volume of the *Transactions of the Royal Society of Edinburgh*, p. 315. In this memoir the author investigates the case of a series of plane waves which passes through a parallelogram in front of a lens, and is received on a screen at the focus of the lens, as well as several other particular cases. By equating the total illumination on the screen to the area of the aperture multiplied by the illumination of the incident light, the author arrives in all cases at the conclusion that in the coefficient of vibration of a secondary wave the elementary area dS must be divided by λr. In consequence of the employment of intensities, not displacements, the necessity for the acceleration of the phase by a quarter of an undulation does not appear from this investigation.

In the investigations of Mr. Smith and Professor Kelland, as well as in the verification of the formula (46) given in the last article, we are only concerned with that part of a secondary wave which lies near the normal to the primary. The correctness of this formula for all directions must rest on the dynamical theory.

36. In any given case of diffraction, the intensity of the illumination at a given point will depend mainly on the mode of interference of the secondary waves. If however the incident light be polarized, and the plane of polarization be altered, every thing else remaining the same, the mode of interference will not be changed, and the coefficient of vibration will vary as $\sin\phi$, so that the intensity will vary between limits which are as 1 to $\cos^2\theta$. If common light of the same intensity be used, the intensity of the diffracted light at the given point will be proportional to $\frac{1}{2}(1+\cos^2\theta)$.

PART II.

EXPERIMENTS ON THE ROTATION OF THE PLANE OF POLARIZATION OF DIFFRACTED LIGHT.

SECTION I.

DESCRIPTION OF THE EXPERIMENTS.

If a plane passing through a ray of plane-polarized light, and containing the direction of vibration, be called the *plane of vibration*, the law obtained in the preceding section for the nature of the polarization of diffracted light, when the incident light is plane-polarized, may be expressed by saying, that any diffracted ray is plane-polarized, and the plane of vibration of the diffracted ray is parallel to the direction of vibration of the incident ray. Let the angle between the incident ray produced and the diffracted ray be called the *angle of diffraction*, and the plane containing these two rays the *plane of diffraction ;* let α_i, α_d be the angles which the planes of vibration of the incident and diffracted rays respectively make with planes drawn through those rays perpendicular to the plane of diffraction, and θ the angle of diffraction. Then we easily get by a spherical triangle

$$\tan\alpha_d = \cos\theta\tan\alpha_i.$$

If then the plane of vibration of the incident ray be made to turn round with a uniform velocity, the plane of vibration of the diffracted ray will turn round with a variable velocity, the law connecting these velocities being the same as that which connects the sun's motions in right ascension and longitude, or the motions of the two axes of a Hook's joint. The angle of diffraction answers to the obliquity of the ecliptic in the one case, or the supplement of the angle between the axes in the other. If we suppose a series of equidifferent values given to α_i, such as 0°, 5°, 10°,...355°, the planes of vibration of the diffracted ray will not be distributed uniformly, but will be crowded towards the plane perpendicular to the plane of diffraction, according to the law expressed by the above equation.

Now the angles which the *planes of polarization* of the incident and diffracted rays, (if the diffracted ray prove to be really plane-polarized,) make with planes perpendicular to the plane of diffraction can be measured by means of a pair of graduated instruments furnished with Nicol's prisms. Suppose the plane of polarization of the incident light to be inclined at the angles 0°, 5°, 10°..., successively to the perpendicular to the plane of diffraction; then the readings of the instrument which is used as the analyzer will shew whether the planes of polarization of the diffracted ray are crowded towards the plane of diffraction or towards the plane perpendicular to the plane of diffraction. If ϖ, α be the azimuths of the planes of polarization of the incident and diffracted rays, both measured from planes perpendicular to the plane of diffraction, we should expect to find these angles connected by the equation $\tan\alpha = \sec\theta\tan\varpi$ in the former event, and $\tan\alpha = \cos\theta\tan\varpi$ in the latter. If the law and amount of the crowding agree with theory as well as could reasonably be expected, some allowance being made for the influence of modifying causes, (such as the direct action of the edge of the diffracting body,) whose exact effect cannot be calculated, then we shall be led to conclude that the vibrations in plane-polarized light are perpendicular or parallel to the plane of polarization, according as the crowding takes place towards or from the plane of diffraction.

In all ordinary cases of diffraction, the light becomes insensible at such a small angle from the direction of the incident ray produced that the crowding indicated by theory is too small to be sensible in experiment, except perhaps in the mean of a very great number of observations. It is only by means of a fine grating that we can obtain strong light which has been diffracted at a large angle. I doubt whether a grating properly so called, that is, one consisting of actual wires, or threads of silk, has ever been made which would be fine enough for the purpose. The experiments about to be described have accordingly been performed with the glass grating already mentioned, which consisted of a glass plate on which parallel and equi distant lines had been ruled with a diamond at the rate of about 1300 to an inch.

Although the law enunciated at the beginning of this section has been obtained for diffraction in vacuum, there is little doubt that the same law would apply to diffraction within a homogeneous uncrystallized medium, at least to the degree of accuracy that we employ when we speak of the refractive index of a substance, neglecting the dispersion. This is rendered probable by the simplicity of the law itself, which merely asserts that the vibrations in the diffracted light are rectilinear, and agree in direction with the vibrations in the incident light as nearly as is consistent with the necessary condition of being perpendicular to the diffracted ray. Moreover, when dispersion is neglected, the same equations of motion of the luminiferous ether are obtained, on mechanical theories, for singly refracting media as for vacuum; and if these equations be assumed to be correct, the law under consideration, which is deduced from the equations of motion, will continue to hold good. In the case of a glass grating however the diffraction takes place neither in air nor in glass, but at the confines of the two media, and thus theory fails to assign exact values to a. Nevertheless it does not fail to assign limits within which, or at least not far beyond which, a must reasonably be supposed to lie; and as the values comprised within these limits are very different according as one or other of the two rival theories respecting the direction of vibration is adopted, experiments with a glass grating may be nearly as satisfactory, so far as regards pointing to one or other of the two theories, as experiments would be which were made with a true grating.

The glass grating was mounted for me by Prof. Miller in a small frame fixed on a board which rested on three screws, by means of which the plane of the plate and the direction of the grooves could be rendered perpendicular to the plane of a table on which the whole rested.

The graduated instruments lent to me by Prof. O'Brien consisted of small graduated brass circles, mounted on brass stands, so that when they stood on a horizontal table the planes of the circles were vertical, and the zeros of graduation vertically over the centres. The circles were pierced at the centre to admit doubly refracting prisms, which were fixed in brass collars which could be turned round within the circles, the axes of motion being perpendicular to the planes of the circles, and passing through their centres. In one of the instruments, which I used for the polarizer, the circle was graduated to degrees from 0° to 360°, and the collar carried simply a pointer. To stop the second pencil, I attached a wooden collar to the brass collar, and inserted in it a Nicol's prism, which was turned till the more refracted pencil was extinguished. In a few of the latest experiments the Nicol's prism was dispensed with, and the more refracted pencil stopped by a screen with a hole which allowed the less refracted pencil to pass. In the other instrument, which I used for the analyzer, the brass collar carried a vernier reading to 5′. In this instrument the doubly refracting prism admitted of being removed, and I accordingly removed it, and substituted a Nicol's prism, which was attached by a wooden collar. The Nicol's prism was usually inserted into the collar at random, and the index error was afterwards determined from the observations themselves.

The light employed in all the experiments was the sun light reflected from a mirror placed at the distance of a few feet from the polarizer. On account of the rotation of the earth, the mirror required re-adjustment every three or four minutes. The continual change in the direction of the incident light was one of the chief sources of difficulty in the experiments and inaccuracy in the results; but lamplight would, I fear, be too weak to be of much avail in these experiments.

The polarizer, the grating, and the analyzer stood on the same table, the grating a few inches from the polarizer, and the analyzer about a foot from the grating. The plane of diffraction was assumed to be parallel to the table, which was nearly the case; but the change in the direction of the incident light produced continual small changes in the position of this plane. In most experiments the grating was placed perpendicular to the incident light, by making the light reflected from the surface go back into the hole of the polarizer. The angle of diffraction was measured at the conclusion of each experiment by means of a protractor, lent to me for the purpose by Prof. Miller. The grating was removed, and the protractor placed with its centre as nearly as might be under the former position of the bright spot formed on the grating by the incident light. The protractor had a pair of opposite verniers moveable by a rack; and the directions of the incident and diffracted light were measured by means of sights attached to the verniers. The angle of diffraction in the different experiments ranged from about 20° to 60°.

The deviation of the less refracted pencil in the doubly refracting prism of the polarizer, though small, was very sensible, and was a great source both of difficulty and of error. To understand this, let AB be a ray incident at B on a slip of the surface of the plate contained between two consecutive grooves, BC a diffracted ray. On account of the interference of the light coming from the different parts of the slip, if a small pencil whose axis is AB be incident on the slip, the diffracted light will not be sensible except in a direction BC, determined by the condition that $AB + BC$ shall be a minimum, A and C being supposed fixed. Hence AB, BC must make equal angles with the slip, regarded as a line, the acute angles lying towards opposite ends of the slip, and therefore C must lie in the surface of a cone formed by the revolution of the produced part of AB about the slip. If AB represent the pencil coming through the polarizer, it will describe a cone of small angle as the pointer moves round, and therefore both the position of the vertex and the magnitude of the vertical angle of the cone which is the locus of C will change. Hence the sheet of the cone may sometimes fall above or below the eye-hole of the analyzer. In such a case it is necessary either to be content to miss one or more observations, corresponding to certain readings of the polarizer, or else to alter a little the direction of the incident light, or, by means of the screws, to turn the grating through a small angle round a horizontal axis. The deviation of the light which passed through the polarizer, and the small changes in the direction of the incident light, I regard as the chief causes of error in my experiments. In repeating the experiments so as to get accurate results, these causes of error would have to be avoided.

At first I took for granted that the instrument-maker had inserted the doubly refracting prism in the polarizer in such a manner that the plane of polarization of the less refracted pencil was either vertical or horizontal, (the instrument being supposed to stand on a horizontal table,) when the pointer stood at 0°, having reason to know that it was not inserted at random; and having determined which, by an exceedingly rough trial, I concluded it was vertical. Meeting afterwards with some results which were irreconcilable with this supposition, I was led to make an actual measurement, and found that the plane of polarization was vertical when the pointer stood at 25°. Consequently 25° is to be regarded as the index error of the polarizer, to be subtracted from the reading of the pointer. The circumstance just mentioned accounts for the apparently odd selection of values of ϖ in the earlier experiments, the results of which are given in the tables at the end of this section.

On viewing a luminous point or line through the grating, the central colourless image was seen accompanied by side spectra, namely, the spectra which Fraunhofer called *Spectra of the second class*. After a little, these spectra overlapped in such a manner that the individual spectra could no longer be distinguished, and nothing was to be seen but two tails of light, which extended, one on each side, nearly 90° from the central image. On viewing the flame of a spirit lamp through the grating, the individual spectra of the second class could be seen, where, with sun-light, nothing could be perceived but a tail of light. The tails themselves were not white, but exhibited very broad impure spectra; about two such could be made out on each side. These spectra are what were called *spectra of the first class* by Fraunhofer, who shewed that their breadth depended on the smaller of the two quantities, the breadth of a groove, and the breadth of the polished interval between two consecutive grooves. In the grating, the breadth of the grooves was much smaller than the breadth of the intervals between[*].

In the experiments, the diffracted light observed belonged to a bright, though not always the brightest, part of a spectrum of the first class. The compound nature of the light was easily put in evidence by placing a screen with a vertical slit between the grating and the eye, and then viewing the slit through a prism with its edge vertical[†]. A spectrum was then seen which consisted of bright bands separated by dark intervals, strongly resembling the appearance presented when a pure spectrum is viewed through a pinhole, or narrow slit, which is half covered by a plate of mica, placed on the side at which the blue is seen. At a considerable angle of diffraction as many as 15 or 20 bands might be counted.

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[*] On viewing the grating under a microscope, the grooves were easily seen to be much narrower than the intervals between; their breadth was too small to be measured. On looking at the flame of a spirit lamp through the grating, I counted sixteen images on one side, then several images were too faint to be seen, and further still the images again appeared, though they were fainter than before. I estimated the direction of zero illumination to be situated about the eighteenth image. If we take this estimation as correct, it follows from the theory of these gratings that the breadth of a groove was the eighteenth part of the interval between any point of one groove and the corresponding point of its consecutive, an interval which in the case of the present grating was equal to the 1-1300th part of an inch. Hence the breadth of a groove was equal to the 1-23400th part of an inch.

[†] To separate the different spectra, Fraunhofer used a small prism with an angle of about 20°, fixed with its edge horizontal in front of the eye-piece of the telescope through which, in his experiments, the spectra were viewed.

In the first experiment the grating was placed with its plane perpendicular to the light which passed through the polarizer, the grooved face being turned from the polarizer. The light observed was that which was diffracted at emergence from the glass. It is only when the eye is placed close to the grating, or when, if the eye be placed a few inches off, the whole of the grating is illuminated, that a large portion of a tail of light can be seen at once. When only a small portion of the grating is illuminated, and the eye is placed at the distance of several inches, as was the case in the experiments, it is only a small portion of a tail which can enter the pupil. The appearance presented is that of a bright spot on the grooved face of the glass. The angle of diffraction in the first experiment was large, $57^\circ\ 5'$ by measurement. Besides the principal image, or bright spot, a row of images were seen to the left: the regularly transmitted light lay to the right, right and left being estimated with reference to the position of the observer. These images were due to internal diffraction and reflection, as will be better understood further on. They were separated by small angles, depending on the thickness of the glass, but sufficient to allow of one image being observed by itself. The observations were confined to the principal or right-hand image.

In the portion of a spectrum of the first class which was observed there was a predominance of red light. In most positions of the pointer of the polarizer the diffracted light did not wholly vanish on turning round the analyser, but only passed through a minimum. In passing through the minimum the light rapidly changed colour, being blue at the minimum. This shews that the different colours were polarized in different planes, or perhaps not strictly plane-polarized. Nevertheless, as the intensity of the light at the minimum was evidently very small compared with its intensity at the maximum, and the change of colour was rapid, it is allowable to speak in an approximate way of the plane of polarization of the diffracted light, just as it is allowable to speak of the refractive index of a substance, although there is really a different refractive index for each different kind of light. It was accordingly the angular position of the plane which was the best representative of a plane of polarization that I sought to determine in this and the subsequent experiments.

In the first experiment the plane of polarization of the diffracted light was determined by six observations for each angle at which the pointer of the polarizer was set. This took a good deal of time, and increased the errors depending on changes in the direction of the light. Accordingly, in a second experiment, I determined the plane of polarization by single observations only, setting the pointer of the polarizer at smaller intervals than before. Both these experiments gave for result that the planes of polarization of the diffracted light were distributed very nearly uniformly. This result already points very decidedly to one of the two hypotheses respecting the direction of vibration. For according to theory the effect of diffraction alone would be, greatly to crowd the planes either in one direction or in the other. It seems very likely that the effect of oblique emergence alone should be to crowd the planes in the manner of refraction, that is, towards the perpendicular to the plane of diffraction. If then we adopt Fresnel's hypothesis, the two effects will be opposed, and may very well be supposed wholly or nearly to neutralize each other. But if we adopt the other hypothesis we shall be obliged to suppose that in the oblique emergence from the glass, or in something else, there exists a powerful cause of crowding towards the plane of diffraction, that is, *in the manner of reflection*, sufficient to neutralize the great crowding in the contrary direction produced by diffraction, which certainly seems almost incredible.

The nearly uniform distribution of the planes of polarization of the diffracted light shews that the two streams of light, polarized in and perpendicular to the plane of diffraction respectively, into which the incident light may be conceived to be decomposed, were diffracted at emergence from the glass in very nearly the same proportion. This result appeared to offer some degree of vague analogy with the depolarization of light produced by such substances as white paper. This analogy, if borne out in other cases, might seem to throw some doubt on the conclusiveness of the experiments with reference to the decision of the question as to the direction of the vibrations of plane-polarized light. For the deviation of the light from its regular course might seem due rather to a sort of scattering than to regular diffraction, though certainly the fact that the observed light was very nearly plane-polarized does not at all harmonize with such a view. Accordingly, I was anxious to obtain a case of diffraction in which the planes of polarization of the diffracted light should be decidedly crowded one way or other. Now, according to the explanation above given, the approximate uniformity of distribution of the planes of polarization in the first two experiments was due to the antagonistic effects of diffraction, (according to Fresnel's hypothesis respecting the direction of vibration,) and of oblique emergence from the glass, or irregular refraction, that is, refraction produced wholly by diffraction. If this explanation be correct, a very marked crowding towards the plane of diffraction ought to be produced by diffraction at reflection, since diffraction alone and reflection alone would crowd the planes in the same manner.

To put this anticipation to the test of experiment, I placed the grating with its plane perpendicular to the incident light, and the grooved face towards the polarizer, and observed the light which was diffracted at reflection. Since in this case there would be no crowding of the planes of polarization in the regularly reflected light, any crowding which might be observed would be due either to diffraction directly, or to the irregular reflection due to diffraction, or, far more probably, to a combination of the two.

The experiments indicated indeed a marked crowding towards the plane of diffraction, but the light was so strong at the minimum, for most positions of the pointer of the polarizer, that the observations were very uncertain, and it was evidently only a rough approximation to regard the diffracted light as plane-polarized. The reason of this was evident on consideration. Of the light incident on the grating, a portion is regularly reflected, forming the central image of the system of spectra produced by diffraction at reflection, a portion is diffracted externally at such an angle as to enter the eye, a small portion is scattered, and the greater part enters the glass. Of the light which enters the glass, a portion is diffracted internally at such an angle that after regular reflection and refraction it enters the eye, a portion diffracted at other angles, but the greater part falls perpendicularly on the second surface. A portion of this is reflected to the first surface, and of the light so reflected a portion is diffracted at emergence at such an angle as to enter the eye. Thus there are three principal images, each formed by light which has been once diffracted and once reflected, the externally diffracted light being considered as both diffracted and reflected, namely, one has been diffracted internally, and then regularly reflected and refracted, a second in which the light has been regularly refracted and reflected, and then diffracted at emergence, and a third in which the light has been diffracted externally. Any other light which enters the eye must have been at least twice diffracted, or once diffracted and at least three times reflected, and therefore will be comparatively weak, except perhaps when the angle of incidence, or else the angle of diffraction, is very large. Now when the grating is perpendicular to the incident light the second and third of the principal images are necessarily superposed; and as they might be expected to be very differently polarized, it was likely enough that the light arising from the mixture of the two should prove to be very imperfectly polarized.

To separate these images, I placed a narrow vertical slit in front of the grating, between it and the polarizer, and inclined the grating by turning it round a vertical axis so that the normal fell between the polarizer and the analyzer. As soon as the grating was inclined, the image which had been previously observed separated into two, and at a certain inclination the three principal images were seen equidistant. The middle image, which was the second of those above described, was evidently the brightest of the three. The three images were found to be nearly if not perfectly plane-polarized, but polarized in different planes. The third image, and perhaps also the first, did not wholly vanish at the minimum. This might have been due to some subordinate image which then appeared, but it was more probably due to a real defect of polarization.

The planes of polarization of the side images, especially the first, were greatly crowded towards the plane of diffraction, or, which is the same, the plane of incidence. Those of the middle image were decidedly crowded in the same direction, though much less so than those of the side images. The light of the first and second images underwent one regular refraction and one regular reflection besides the diffraction and the accompanying irregular refraction. The crowding of the planes of polarization in one direction or the other produced by the regular refraction and the regular reflection can readily be calculated from the known formulæ*, and thus the crowding due to diffraction and the accompanying irregular refraction can be deduced from the observed result.

The crowding of the planes of polarization of the third image is due solely to diffraction and the accompanying irregular reflection. The crowding in one direction or the contrary, according as one or other hypothesis respecting the direction of vibrations is adopted, is readily calculated from the dynamical theory, and thus is obtained the crowding which is left to be attributed to the irregular reflection. In the absence of an exact theory little or no use can be made of the result in the way of confirming either hypothesis; but it is sufficient to destroy the vague analogy which might have been formed between the effects of diffraction and of irregular scattering.

The crowding of the planes of polarization of the middle image, after the observations had

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[*] It is here supposed that the regularly reflected or refracted light which forms the central colourless image belonging to a system of spectra is affected as to its polarization in the same way as if the surface were free from grooves.

been reduced in the manner which will be explained in the next section, appeared somewhat greater than was to have been expected from the first two experiments. This led me to suspect that the crowding in the manner of reflection produced by diffraction accompanying the passage of light from air, across the grooved surface, into the glass plate, might be greater than the crowding had proved to be which was produced by diffraction accompanying the passage from glass, across the grooved surface, into air. I accordingly placed the grating with its plane perpendicular to the incident light, and the grooved face *towards* the polarizer, and placed the analyzer so as to receive the light which was diffracted in passing across the first surface, and then regularly refracted at the second. I soon found that the planes of polarization were very decidedly crowded towards the plane of diffraction, and that, notwithstanding the crowding in the contrary direction which must have been produced by the regular refraction at the second surface of the plate, and the crowding, likewise in the contrary direction, which might naturally be expected to result from the irregular refraction at the first surface, considered apart from diffraction. This result seemed to remove all doubt respecting the hypothesis as to the direction of vibration to which the experiments pointed as the true one.

On account of the decisive character of the result just mentioned, I took several sets of observations on light diffracted in this manner at different angles. I also made two more careful experiments of the same nature as the first two. The result now obtained was, that there was a very sensible crowding towards the plane of diffraction when the grooved face was turned from the polarizer, although there was evidently a marked difference between the two cases, the crowding being much less than when the grooved face was turned towards the polarizer. Even the first two experiments, now that I was aware of the index error of the polarizer, appeared to indicate a small crowding in the same direction.

Before giving the numerical results of the experiments, it may be as well to mention what was observed respecting the defect of polarization. I would here remark that an investigation of the precise nature of the diffracted light was beside the main object of my experiments, and only a few observations were taken which belong to such an investigation. In what follows, ϖ denotes the inclination of the plane of polarization of the light incident on the grating to a vertical plane passing through the ray, that is, to a plane perpendicular to the plane of diffraction. It is given by the reading of the pointer of the polarizer corrected for the index error 25°, and is measured positive in the direction of revolution of the hands of a watch placed with its back towards the incident light.

Whether the diffraction accompanied reflection or refraction, external or internal, the diffracted light was perfectly plane-polarized when ϖ had any one of the values 0°, 90°, 180°, or 270°. The defect of polarization was greatest about 45° from any of the above positions. When the diffracted light observed was red or reddish, on analyzation a blue light was seen at or near the minimum; when the diffracted light was blue or blueish, a red light was seen at or near the minimum. When the angle of diffraction was moderately small, such as 15° or 20°, the defect of polarization was small or insensible; when the angle of diffraction was large, such as 50° or 60°, the defect of polarization was considerable. For equal angles of diffraction, the defect of polarization was much greater when the grooved face was turned towards the polarizer than when it was turned in the contrary direction. By the term *angle of diffraction*, as applied to the case in which the grooved face was turned towards the polarizer, is to be understood the angle measured in air, from which the angle of diffraction within the glass may be calculated from a knowledge of the refractive index.

The grating being placed perpendicularly to the incident light, with the grooved face towards the polarizer, the light diffracted at a considerable angle, ($59^\circ 52'$ by measurement,) to the left of the regularly transmitted light was nearly white. When the pointer of the polarizer stood at 70°, so that $\varpi = +45^\circ$, on turning the Nicol's prism of the analyzer in the positive direction through the position of minimum illumination, the light became in succession greenish yellow, blue, plum colour, nearly red. When ϖ was equal to -45°, the same appearance was presented on reversing the direction of rotation. Since the colours appeared in the order blue, red, when $\varpi = +45^\circ$, and in the order red, blue, when $\varpi = -45^\circ$, the analyzer being in both cases supposed to turn in the direction of the hands of a watch, the deficiency of colour took place in the order red, blue, when $\varpi = +45^\circ$, and in the order blue, red, when $\varpi = -45^\circ$. Hence the planes of polarization, or approximate polarization, of the blue were more crowded towards the plane of diffraction than those of the red.

On placing a narrow slit so as to allow a small portion only of the diffracted light to pass, and decomposing the light by a prism, in the manner already described, so as to get a spectrum consisting of bright bands with dark intervals, and then analyzing this spectrum with a Nicol's prism, it was found that at a moderate angle of diffraction all the colours were sensibly plane-polarized, though the planes of polarization did not quite coincide. At a large angle of diffraction the bright part of the spectrum did not quite disappear on turning round the Nicol's prism, while the red and blue ends, probably on account of their less intensity, appeared to be still perfectly plane-polarized, though not quite in the same plane. On treating in the same manner the diffracted light produced when the grooved face of the glass plate was turned from the polarizer, all the colours appeared to be sensibly plane-polarized. In the former case the light of the brightest part of the spectrum was made to disappear, or nearly so, by using a thin plate of mica in combination with the Nicol's prism, which shews that the defect of plane polarization was due to a slight elliptic polarization.

The numerical results of the experiments on the rotation of the plane of polarization are contained in the following table. In this table ϖ is the reading of the polarizer corrected for the index error 25°. A reading such as 340° is entered indifferently in the column headed "ϖ" as $+315^\circ$ or -45°, that is, $340^\circ - 25^\circ$ or $-(360^\circ - 340^\circ) - 25^\circ$. α is the reading of the analyzer, determined by one or more observations. The analyzer was graduated only from -90° to $+90^\circ$, and any reading such as -20° is entered indifferently as -20°, $+160^\circ$, or $+340^\circ$, being entered in such a manner as to avoid breaking the sequence of the numbers. On account of the light left at the minimum, the determination of α was very uncertain when the angle of diffraction was large, except when ϖ had very nearly one of the values 0°, 90°, 180°, or 270°. In the most favourable circumstances the mean error in the determination of α was about a quarter of a degree. In some of the experiments a red glass was used to assist in rendering the observations more definite. This had the advantage of stopping all rays except the red, but the disadvantage of considerably diminishing the intensity of the light. The minutes in the given value of θ, the angle of diffraction, cannot be trusted; in fact, during any experiment θ was liable to changes to at least that extent in consequence of the changes in the direction of the light. The same remark applies to i, the angle of incidence, in experiments 11 and 12. In these experiments the three principal images already described were observed separately. The angle of diffraction is measured from the direction of the regularly reflected ray, so that i is the angle of incidence, and $i + \theta$ the angle of reflection, or, in the case of the images which suffered one internal reflection, the angle of emergence.

The eleven experiments which are not found in the following tables consist of five on diffraction by reflection, which did not appear worth giving on account of the superposition of different images; one on diffraction by refraction, to which the same remark applies, the grating having been placed at a considerable distance from the polarizer, so that the spot illuminated was too large to allow of the separate observation of different images; one on diffraction by reflection, in which the grating was placed perpendicularly to the incident light, with the grooved face turned from the polarizer, but the errors of observation, though much smaller than the whole quantity to be observed, were so large on account of the large angle of diffraction, (about 75°,) with which the observations were attempted, that the details are not worth giving; one on diffraction by refraction, in which the different observations were so inconsistent that the experiment seemed not worth reducing; one which was only just begun; and two qualitative experiments, the results of which have been already given. I mention this that I may not appear to have been biassed by any particular theory in selecting the experiments of which the numerical results are given.

The following remarks relate to the particular experiments:

No. 1. In this experiment each value of α was determined by six observations, of which the mean error[*] ranged from about 15′ to 55′. So far the experiment was very satisfactory, but it was vitiated by changes in the direction of the light, sufficient care not having been taken in the adjustment of the mirror.

No. 2. α determined by single observations.

No. 13. α determined by two observations at least, of which the mean error ranged from about 10′ to nearly 1°, but was usually decidedly less than 1°. At and about the octants, that is to say, when ϖ was nearly equal to 45°, or an odd multiple of 45°, the light was but very imperfectly polarized in one plane.

No. 14. α determined by two observations. Marked in note book as "a very satisfactory experiment." The mean of the mean errors was only 11′.

No. 15. α determined by three observations at least. The light was very imperfectly polarized, except near the standard points, that is to say when ϖ was equal to 0° or 90°, or a multiple of 90°. This rendered the observations very uncertain. About the octants the mean error in a set of observations taken one immediately after another amounted to near 2°.

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[*] The difference between each individual observation and the mean of the six is regarded as the error of that observation, and the mean of these differences taken positively is what is here called the *mean error*. When two observations only are taken, the mean error is the same thing as the semi-difference between the observations. Since, for a given position of the pointer of the polarizer, the readings of the analyzer were usually taken one immediately after another, the mean error furnishes no criterion by which to judge of the errors produced by the small changes in the direction of the light incident on the grating, but only of those which arise from the vagueness of the object observed. The reader will be much better able to judge of the amount of probable error from all causes after examining the reduction of the experiments given in the next section.

No. 17. α determined by two observations. The light was very imperfectly polarized, except near the standard points. Yet the observations agreed very fairly with one another. The mean of the mean errors was 25′, and the greatest of them not quite 1°.

No. 18. α determined by two observations, which, generally speaking, agreed well with one another. For $\varpi = -90^\circ$ and $\varpi = +22.5^\circ$ the light observed was rather scattered than regularly diffracted, the sheet of the cone of illumination having fallen above or below the hole of the analyzer.

No. 21. α determined by two observations at least. In this experiment the polarizer was covered with red glass.

No. 22. α determined by two observations. Marked in note book as "a very satisfactory experiment, though the light was not perfectly polarized."

No. 23. α determined by two observations at least. The hole in a screen placed between the polarizer and the grating was covered with red glass. This appears to have been a good experiment.

No. 11. α determined by two observations, which agreed well with one another. In the table, $\alpha(1)$, $\alpha(2)$, $\alpha(3)$ refer respectively to the first, second, and third of the three principal images already mentioned. In this experiment the polarizer was reversed, that face being turned towards the mirror which in the other experiments was turned towards the grating, which is the reason why α and ϖ increase together, although the light observed suffered one reflection. The same index error as before, namely 25°, is supposed to belong to the polarizer in its reversed position.

No. 12. α determined by three observations. The largeness of the angle of diffraction rendered the determination of α very uncertain.

TABLE I.

Experiment, No. 1.

Grooved face *from* Polarizer.

θ = 57°5′.

ϖ	α
−115°	− 76°41′
− 92½°	− 52°56′
− 70°	
− 47½°	− 6°52′
− 25°	+ 14°51′
− 2½°	+ 37°51
+ 20°	+ 61° 5
+ 42½°	+ 82°54
+ 65°	+106°46

Experiment, No. 2.

Grooved face *from* Polarizer.

θ = 50°23′.

ϖ	α
−105°	− 80°
− 95°	− 70°25′
− 85°	− 61°15′
− 75°	− 51°30′
− 65°	− 41°10′
− 55°	− 29°15′
− 45°	− 20° 5′
− 35°	− 9°55′
− 25°	+ 0°20′
− 15°	+ 10°15′
− 5°	+ 20°20′
+ 5°	+ 30°55′
+ 15°	+ 40°55′
+ 25°	+ 50°45′
+ 35°	+ 61°45′
+ 45°	+ 70°55′
+ 55°	+ 82°15′

Experiment, No. 13.

Grooved face *towards* Polarizer.

θ = 39°50′.

ϖ	α
− 60°	− 6°5′
− 50°	+ 4°53′
− 40°	+ 15°52′

No. 13, continued.

ϖ	α
− 30°	+ 25°
− 20°	+ 33°25′
− 10°	+ 46° 5′
0°	+ 56°35′
+ 10°	+ 67°50′
+ 20°	+ 76°58′
+ 30°	+ 87°55′
+ 40°	+ 99°27′
+ 50°	+108°30′
+ 60°	+120°35′
+ 70°	+129° 2′
+ 80°	+137°42′
+ 90°	+146°57′

Experiment, No. 14.

Grooved face *from* Polarizer.

θ = 29°57′

ϖ	α
− 50°	+ 22°25′
− 40°	+ 31°15′
− 30°	+ 41°40′
− 20°	+ 51°55′
− 10°	+ 62°37′
0°	+ 71°10′
+ 10°	+ 81°47′
+ 20°	+ 93°47′
+ 30°	+103°10′
+ 40°	+113°15′
+ 50°	+122°42′
+ 60°	+132°42′
+ 70°	+143°
+ 80°	+152°47′
+ 90°	+161°57′
+100°	+171°52′
+110°	+182°52′
+120°	+191°47′
+130°	+202°12′
+140°	+211°42′

Experiment, No. 15.

Grooved face *towards* Polarizer.

θ = 59°52′.

ϖ	α
0°	− 68°10′
− 10°	− 81°

No. 15, continued.

ϖ	α
− 20°	− 92°23′
− 30°	−115°55′
− 40°	−124°25′
− 50°	−133°41′
− 60°	−140°29′
− 70°	−148°18′
− 80°	−152°50′
− 90°	−158°30′

Experiment, No. 17.

Grooved face *towards* Polarizer.

θ = 50°45′.

ϖ	α
− 90°	+ 77°15′
− 80°	+ 85°30′
− 70°	+ 93°12′
− 60°	+101°15′
− 50°	+109°47′
− 40°	+117°12′
− 30°	+129°57′

Experiment, No. 18.

Grooved face *towards* Polarizer.

θ = 21°39′.

ϖ	α
− 90°	−103°23′
− 45°	− 59°53′
0°	− 12°58′
+ 45°	+ 33°37′
+ 90°	+ 77°27′
+135°	+120° 2′
+180°	+167°57′
+225°	+214°10′

Experiment, No. 21.

Grooved face *towards* Polarizer.

Red glass used.

θ = 28°26′.

ϖ	α
− 90°	− 29°
− 75°	− 16° 2′
− 60°	− 2°12′

No. 21, continued.

ϖ	α
− 45°	+ 12°35′
− 30°	+ 27°52′
− 15°	+ 44°47′
0°	+ 61°40′
+ 15°	+ 78°25′
+ 30°	+ 92°18′
+ 45°	+107°25′
+ 60°	+122°30′
+ 75°	+137°
+ 90°	+151°32′

Experiment, No. 22.

Grooved face *from* Polarizer.

θ = 55°38′.

ϖ	α
−180°	−187° 2′
−165°	−170°37′
−150°	−154°30′
−135°	−140°25′
−120°	−124°45′
−105°	−110°40′
− 90°	− 96°55′
− 75°	− 83°32′
− 60°	− 69° 7′
− 45°	− 54°50′
− 30°	− 38°55′
− 15°	− 22°50′

Experiment, No. 23.

Grooved face *towards* Polarizer.

Red glass used.

θ = 54°53′.

ϖ	α
0°	− 6°30′
+ 15°	+ 11° 5′
+ 30°	+ 27°55′
+ 45°	+ 42°30′
+ 60°	+ 58°22′
+ 75°	+ 71° 5′
+ 90°	+ 83°22′
+105°	+ 96°12′
+120°	+108°30′
+135°	+122°45′

Experiment, No. 11.

$i = 14°50'$; $\theta = 22°30'$.

ϖ	α (1)	α (2)	α (3)
−105°	−113°35′	−117°50′	
− 85°	−103° 5′	−101°	−102°20′
− 65°	− 90°	− 83° 5′	− 89°
− 45°	− 78°40′	− 63°55′	− 74°50′
− 25°	− 58°50′	− 44°	− 53°19′
− 5°	− 25°5′	− 21°10′	− 23°10′
+ 15°	+ 13°15′	+ 1°25′	+ 7°55′
+ 35°	+ 38°35′	+ 24° 5′	+ 32°
+ 55°	+ 53°50′	+ 43°10′	+ 51°30′

Experiment, No. 12.

$i = 9°1'$; $\theta = 53°39'$.

ϖ	α (1)	α (2)	α (3)
− 25°	+ 5°35′	− 32°	− 13°45′
− 45°	+ 15°	− 9°40′	+ 2°
− 90°	+ 26°15′	+ 26°15′	+ 26°15′
−135°	+ 34°30′	+ 65°	+ 51°15′

SECTION II.

DISCUSSION OF THE NUMERICAL RESULTS OF THE EXPERIMENTS, WITH REFERENCE TO THEORY.

According to the known formulæ which express the laws of the rotation of the plane of polarization of plane-polarized light which has undergone reflection or refraction at the surface of a transparent uncrystallized medium, if ϖ, α' be the azimuths of the planes of polarization of the incident and reflected or refracted light, both measured from planes perpendicular to the plane of incidence, they are connected by the equation

$$\tan \alpha' = m \tan \varpi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where m is constant, if the position of the surface and the directions of the rays be given, but is a different constant in the two cases of reflection and refraction. According to the theory developed in this paper, the same law obtains in the case of diffraction in air, or even within an uncrystallized medium, but m has a value distinct from the two former. It seems then extremely likely that the same law should hold good in the case of that combination of diffraction with reflection or refraction which exists when the diffraction takes place at the common surface of two transparent uncrystallized media, such as air and glass. If this be true, it is evident that by combining all the observations belonging to one experiment in such a manner as to get the value of m which best suits that experiment, we shall obtain the crowding of the planes of polarization better than we could from the direct observations, and we shall moreover be able in this way easily to compare the results of different experiments. It seems reasonable then to try in the first instance whether the formula (48) will represent the observations with sufficient accuracy.

In applying this formula to any experiment, there are two unknown quantities to be determined, namely, m, and the index error of the analyzer. Let ϵ be this index error, so that $\alpha = \alpha' + \epsilon$. The regular way to determine ϵ and m would no doubt be to assume an approximate value ϵ_1 of ϵ, put $\epsilon = \epsilon_1 + \Delta\epsilon_1$, where $\Delta\epsilon_1$ is the small error of ϵ_1, form a series of equations of which the type is

$$\tan(\alpha - \epsilon_1) - \sec^2(\alpha - \epsilon_1)\,\Delta\epsilon_1 = m \tan \varpi,$$

and then combine the equations so as to get the most probable values of $\Delta\epsilon_1$ and m. But such a refinement would be wholly unnecessary in the case of the present experiments, which are confessedly but rough. Moreover ϵ can be determined with accuracy, except so far as relates to errors produced by changes in the direction of the light, by means of the observations taken at the standard points, the light being in such cases perfectly polarized. By accuracy is here meant such accuracy as experiments of this sort admit of, where a set of observations giving a mean error of a quarter of a degree would be considered accurate. Besides, whenever the values of ϖ selected for observation are symmetrically taken with respect to one of the standard points, a small error in ϵ would introduce no sensible error into the value of m which would result from the experiment, although it might make the formula appear in fault when the only fault lay in the index error.

Accordingly I have determined the index error of the analyzer in a way which will be most easily explained by an example. Suppose the values of α to have been determined by experiment corresponding to the following values of ϖ, -15^0, 0^0, $+15^0$, ... $+75^0$, $+90^0$, $+105^0$. The value of α for $\varpi = 0^0$, and the mean of the values for $\varpi = -15^0$ and $\varpi = +15^0$, furnish two values of ϵ; and the value of α for $\varpi = +90^0$, and the mean of the values for $\varpi = +75^0$ and $\varpi = +105^0$, furnish two values of $\epsilon + 90^0$. The mean of the four values of ϵ thus determined is likely to be more nearly correct than any of them. In some few experiments no two values of ϖ were symmetrically taken with respect to the standard points. In such cases I have considered it sufficient to take proportional parts for a small interval. Thus if α_1, α_2 be the readings of the analyzer for $\varpi = -10^0$, $\varpi = +5^0$, assuming $\alpha_1 = \epsilon - 10^0 - 2x$, $\alpha_2 = \epsilon + 5^0 + x$, we get $3x = \alpha_2 - \alpha_1 - 15^0$, whence ϵ, which is equal to $\alpha_2 - 5^0 - x$, is known. The index error of the analyzer having been thus determined, it remains to get the most probable value of m from a series of equations of the form (48). For facility of numerical calculation it is better to put this equation under the form

$$\log m = \log \tan \alpha' - \log \tan \varpi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

where it is to be understood that the signs of α and ϖ are to be changed if these angles should lie between 0 and -90^0, or their supplements taken if they should lie between $+90^0$ and $+180^0$. Now the mean of the values of $\log m$ determined by the several observations belonging to one experiment is not at all the most probable value. For the error in $\log \tan \alpha'$ produced by a small given error in α' increases indefinitely as α' approaches indefinitely to 0^0 or 90^0, so that in this way of combining the observations an infinite weight would be attributed to those which were taken infinitely close to the standard points, although such observations are of no use for the direct determination of $\log m$, their use being to determine ϵ. Let $\alpha' + \Delta\alpha'$ be the true angle of which α' is the approximate value, α' being deduced from the observed angle α corrected for the assumed index error ϵ. Then, neglecting $(\Delta\alpha')^2$, we get for the true equation which ought to replace (49),

$$\log m = \log \tan \alpha' + \frac{2M\Delta\alpha'}{\sin 2\alpha'} - \log \tan \varpi,$$

M being the modulus of the common system of logarithms. Since the effect of the error $\Delta\alpha'$ is increased by the division by $\sin 2\alpha'$, a quantity which may become very small, in combining the equations such as (49) I have first multiplied the several equations by $\sin 2\alpha'$, or the sine of $2(\alpha - \epsilon)$ taken positively, and then added together the equations so formed, and determined $\log m$ from the resulting equation. Perhaps it would have been better to have used for multiplier $\sin^2 2\alpha'$, which is what would have been given by the rule of least squares, if the several observations be supposed equally liable to error; but on the other hand the use of $\sin 2\alpha'$ for multiplier instead of $\sin^2 2\alpha'$ has the effect of diminishing the comparative weight of the observations taken about the octants, where, in consequence of the defect of polarization, the observations were more uncertain.

The following table contains the result of the reduction of the experiments in the way just explained. The value of ϵ used in the reduction, and the resulting value of $\log m$, are written down in each case. The first column belonging to each experiment gives the value of $\alpha' - \varpi$

calculated from (49) with the assumed value of log m, and is put down for the sake of comparison with the value of $\alpha' - \varpi$ deduced from the difference, $\alpha - \varpi$, of the observed angles α, ϖ, corrected for the assumed index error ϵ. The upper and lower angles belonging to any pair found in the second and third columns refer respectively to the upper and lower signs in the first column, and in the column headed "ϖ". In the table, the experiments are arranged in classes, according to their nature, and those belonging to the same class are arranged according to the values of θ. The first three experiments in the table relate to diffraction at refraction, in which the grooved face of the grating was turned from the polarizer, the next six to diffraction at refraction, in which the grooved face was turned towards the polarizer, and the last two to the experiments in which the grating was a little inclined, and the three principal images were observed. The result of Experiment No 1, is here given separately, on account of the different values of ϖ there employed.

Experiment No 1. $\theta = 57^\circ 5'$; assumed index error $\epsilon = +0^\circ 5'$.

ϖ	$\alpha' - \varpi$
-115°	$-1^\circ 46'$
$-92\frac{1}{2}^\circ$	$-0^\circ 31'$
-70°	
$-47\frac{1}{2}^\circ$	$+0^\circ 33'$
-25°	$-0^\circ 14'$
$-2\frac{1}{2}^\circ$	$+0^\circ 16'$
$+20^\circ$	$+1^\circ$
$+42\frac{1}{2}^\circ$	$+0^\circ 19'$
$+65^\circ$	$+1^\circ 41'$

The values of α' for $\varpi = -115^\circ$ and $\varpi = +65^\circ$ ought to differ by 180°, whereas they differ by $3^\circ 27'$ more. This angle is so large compared with the angles $\alpha' - \varpi$ given just above, that it seems best to reject the experiment. The experiment is sufficient however to shew that the crowding of the planes of polarization, be it in what direction it may, is very small. On combining all the observations belonging to this experiment in the manner already described, a small positive value of log m, namely $+ .002$, appeared to result. This value, if exact, would indicate an extremely small crowding in the manner of reflection.

TABLE II.

ϖ	Experiment, No. 14. $\theta = 29^\circ 57'$; $\epsilon = +72^\circ 23'$; $\log m = +.009$. $\alpha' - \varpi$ (in tenths of 1°).			Experiment, No. 2. $\theta = 50^\circ 23'$; $\epsilon = +24^\circ 12'$; $\log m = +.010$. $\alpha' - \varpi$			Experiment, No. 22. $\theta = 55^\circ 38'$; $\epsilon = -7^\circ 27'$; $\log m = +.035$. $\alpha' - \varpi$			Experiment, No. 18. $\theta = 21^\circ 39'$; $\epsilon = -12^\circ 44'$; $\log m = +.029$. $\alpha' - \varpi$			Experiment, No. 21. $\theta = 28^\circ 26'$; $\epsilon = +60^\circ 49'$; $\log m = +.039$. $\alpha' - \varpi$		
	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.
0°	0	−12	−12							0	−2	−2	0	+8	+8
±5°				±1	+6 0	+5 +1									
±10°	±2	−6 +2	−8 +4												
±15°				±3	+6 0	+3 +3	±12	−8	+4				±13	+26 −10	+13 +3
±20°	±4	+14 −5	−10 −1												
±25°				±5	+4 0	−1 +5									
±30°	±5	+8 −7	+3 −2				±20	−19	+1				±23	+15 −29	−8 −6
±35°				±6	? −2	? +4									
±40°	±6	+9 −11	+3 −5												
±45°				±7	+4 −4	−3 +3	±23	−28	−5	±19	+13 −21	−6 −2	±26	+16 −32	−10 −6
±50°	±6	+3 0	−3 +6												
±55°				±6	+19 +4	+13 +10									
±60°	±5	+3	−2				±20	−21	−1				±22	+17 −30	−5 −8
±65°				±5	−15	−10									
±70°	±4	+6	+2												
±75°				±3	−18	−15	±11	−15	−4				±12	+12 −16	0 −4
±80°	±2	+4	+2												
±85°				±1	−15	−14									
±90°	0	−4	−4				0	+1	+1	0	+2 −6	+2 −6	0	+7 +2	+7 +2
±95°				∓1	−7	−8									
±100°	∓2	−5	−3												
±105°				∓3	−3	−6	∓11	+13	+2						
±110°	∓4	+5	+9												
±115°				∓5											
±120°	∓5	−6	−1				∓20	+22	+2						
±125°				∓6											
±130°	∓6	−2	+4												
±135°				∓7			∓23	+16	−7	∓19	−22 +19	−3 0			
±140°	∓6	−7	−1												
±145°				∓6											
±150°	∓5						∓20	+25	+5						
±155°				∓5											
±160°	∓4														
±165°				∓3			∓12	+14	+2						
±170°	∓2														
±175°				∓1											
180°	0						0	0	0	0	+7	+7			

TABLE II, *continued.*

	Experiment, No. 13. $\theta = 39^\circ 50'$; $\epsilon = +56^\circ 50'$; $\log m = +.034$. $a' - \varpi$			Experiment, No. 17. $\theta = 50^\circ 45'$; $\epsilon = +167^\circ 15'$; $\log m = +.122$. $a' - \varpi$			Experiment, No. 23. $\theta = 54^\circ 53'$; $\epsilon = -7^\circ 27'$; $\log m = +.082$. $a' - \varpi$			Experiment, No. 15. $\theta = 59^\circ 52'$; $\epsilon = -68^\circ 15'$; $\log m = +.225$. $a' - \varpi$		
ϖ	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.
0°	0	− 2	− 2	0			0	+ 2	+ 2	0	+ 1	+ 1
± 5°												
± 10°	± 8	+ 10 − 7	+ 2 + 1	± 31						± 65	− 27	+38
± 15°							± 29	+ 27	− 2			
± 20°	± 15	+ 1 − 34	−14 −19	± 57						±114	− 41	+73
± 25°												
± 30°	± 20	+ 11 − 18	− 9 + 2	± 74	− 73	+ 1	± 43	+ 46	+ 3	±141	− 177	−36
± 35°												
± 40°	± 22	+ 26 − 10	+ 4 −12	± 80	− 100	−20				±146	− 162	−16
± 45°							± 54	+ 42	−12			
± 50°	± 22	+ 17 − 19	− 5 + 3	± 76	− 75	+ 1				±134	− 154	−20
± 55°												
± 60°	± 19	+ 37 − 9	+18 −10	± 64	− 60	+ 4	± 44	+ 50	+ 6	±110	− 122	−12
± 65°												
± 70°	± 14	+ 22	+ 8	± 46	− 40	+ 6				± 78	− 100	−22
± 75°							± 25	+ 35	+10			
± 80°	± 7	+ 9	+ 2	± 24	− 17	+ 7				± 40	− 46	− 6
± 85°												
± 90°	0	+ 1	+ 1	0	0	0	0	0	0	0	− 2	− 2
± 95°												
±100°												
±105°							∓ 25	− 31	− 6			
±110°												
±115°												
±120°							∓ 44	− 48	− 4			
±125°												
±130°												
±135°							∓ 54	− 56	− 2			
±140°												
±145°												
±150°							∓ 43					
±155°												
±160°												
±165°							∓ 29					
±170°												
±175°												
180°							0					

TABLE II, *continued.*

	Experiment, No. 11. $i = 14^\circ 50'$; $\theta = 22^\circ 30'$; $\epsilon = -15^\circ 30'$.									Experiment, No. 12. $i = 9^\circ 1'$; $\theta = 53^\circ 39'$; $\epsilon = -63^\circ 45'$.								
	First Image.			Second Image.			Third Image.			First Image.			Second Image.			Third Image.		
	$\log m = +.289.$			$\log m = +.061.$			$\log m = +.209.$			$\log m = +.756.$			$\log m = +.122.$			$\log m = +.366.$		
	$a' - \varpi$			$a' - \varpi$			$a' - \varpi$			$a' - \varpi$			$a' - \varpi$			$a' - \varpi$		
ϖ	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.	calc.	obs.	diff.
0°																		
± 5°	± 46	− 46	0	± 7	− 7	0	± 30	− 27	+ 3									
± 10°																		
± 15°	±125	+ 137	+12	± 21	+ 19	− 2	± 84	+ 84	0									
± 20°																		
± 25°	±172	− 183	−11	± 32	− 35	− 3	±120	− 128	− 8	±444	+ 443	− 1	± 67	+ 67	0	±223	+ 250	+27
± 30°																		
± 35°	±187	+ 191	+ 4	± 39	+ 46	+ 7	±136	+ 125	−11									
± 40°																		
± 45°	±178	− 182	− 4	± 40	− 34	+ 6	±133	− 143	−10	±351	+ 337	−14	± 79	+ 91	+12	±217	+ 217	+ 0
± 50°																		
± 55°	±152	+ 143	− 9	± 37	+ 37	0	±116	+ 120	+ 4									
± 60°																		
± 65°	±115	− 95	+10	± 29	− 26	+ 3	± 89	− 85	+ 4									
± 70°																		
± 75°																		
± 80°																		
± 85°	± 24	− 26	− 2	± 7	− 5	+ 2	± 19	− 18	+ 1									
± 90°										0	0	0	0	0	0	0	0	0
± 95°																		
±100°																		
±105°	∓ 71	+ 69	− 2	∓ 19	+ 27	+ 8												
±110°																		
±115°																		
±120°																		
±125°																		
±130°																		
±135°										∓351	− 367	−16	∓ 79	− 62	+17	∓217	− 200	+17
±140°																		
±145°																		
±150°																		
±155°																		
±160°																		
±165°																		
±170°																		
±175°																		
180°																		

The reader will please to remark that in order to follow the observed values of $\alpha' - \varpi$ beginning with $\varpi = 0°$, and going in the positive direction, it will be necessary to begin at the top of the table and go downwards, taking the upper number in each bracket. In order to go in the negative direction from $\varpi = 0°$, it would be necessary to begin at the top and go downwards, taking the under number in each bracket. A nearly constant error appearing in the table of differences would indicate merely that the value of ϵ used in the reduction was slightly erroneous. A slight error in ϵ, it is to be remembered, produces no sensible error in $\log m$, whenever the observations are balanced with respect to one of the standard points.

In the first two experiments entered in the table, the crowding of the planes of polarization is so small that it is masked by errors of observation, and it is only by combining all the observations that a slight crowding towards the plane of diffraction can be made out. In all the other experiments, however, a glance at the numbers in the second column is sufficient to shew in what direction the crowding takes place. From an inspection of the numbers found in the columns headed "diff." it seems pretty evident that if the formula (49) be not exact the error cannot be made out without more accurate observations. In the case of experiment No. 15, the errors are unusually large, and moreover appear to follow something of a regular law. In this experiment the observations were extremely uncertain on account of the large angle of diffraction and the great defect of polarization of the light observed, but besides this there appears to have been some confusion in the entry of the values of ϖ. This confusion affecting one or two angles, or else some unrecorded change of adjustment, was probably the cause of the apparent break in the second column between the third and fourth numbers. Since the value of $\log m$ is deduced from all the observations combined, there seems no occasion to reject the experiment, since even a large error affecting one angle would not produce a large error in the value of $\log m$ resulting from the whole series. In the entry of experiment No. 12 the signs of ϖ have been changed, to allow for the reversion produced by reflection. This change of sign was unnecessary in No. 11, because in that experiment the polarizer was actually reversed. The results of experiment No. 12 would be best satisfied by using slightly different values of the index error of the analyzer for the three images, adding to the assumed index error about $-1\frac{1}{2}°$, $+1\frac{1}{2}°$, $+2°$, for the first, second, and third images respectively. The largest error in the third columns, 2.7°, is for $\varpi = +25°$, third image. The three readings by which α was determined in this case were $-15°$, $-13°30'$, $-12°$? Hence the error $+2.7°$, even if no part of it were due to an index error, would hardly be too large to be attributed to errors of observation.

Since the formula (49), even if it be not strictly true, represents the experiments with sufficient accuracy, we may consider the value of $\log m$ which results from the combination of all the observations belonging to one experiment as itself the result of direct observation, and proceed to discuss its magnitude. Let us consider first the experiments on diffraction at refraction, in which the light was incident perpendicularly on the grating.

Although the theory of this paper does not meet the case in which diffraction takes place at the confines of air and glass, it leads to a definite result on each of the three following suppositions:

First, that the diffraction takes place in air, before the light reaches the glass:

Second, that the diffraction takes place in glass, after the light has entered the first surface perpendicularly:

Third, that the diffraction takes place in air, after the light has passed perpendicularly through the plate.

On the first supposition let α_1, α_2, α be the azimuths of the plane of polarization of the light after diffraction, after the first refraction, and after the second refraction respectively, and θ' the angle of refraction corresponding to the angle of incidence θ, so that $\sin\theta = \mu \sin\theta'$, μ being the refractive index of the plate: and first, let us suppose the vibrations of plane-polarized light to be perpendicular to the plane of polarization. Then by the theory of this paper we have $\tan\alpha_1 = \sec\theta \tan\varpi$, and by the known formula applying to refraction we have $\tan\alpha_2 = \cos(\theta - \theta')\tan\alpha_1$, $\tan\alpha = \cos(\theta - \theta')\tan\alpha_2$, whence $\tan\alpha = m \tan\varpi$, where

$$m = \sec\theta \cos^2(\theta - \theta').$$

On the second supposition, if α_1 be the azimuth after diffraction at an angle θ' within the glass, we have $\tan\alpha_1 = \sec\theta' \tan\varpi$, $\tan\alpha = \cos(\theta - \theta')\tan\alpha_1$, whence $\tan\alpha = m\tan\varpi$, where

$$m = \sec\theta' \cos(\theta - \theta').$$

On the third supposition we have $\tan\alpha = m \tan\varpi$, where

$$m = \sec\theta.$$

If we suppose the vibrations parallel to the plane of polarization, we shall obtain the same formulæ except that $\cos\theta$, $\cos\theta'$ will come in place of $\sec\theta$, $\sec\theta'$, the factor $\cos(\theta - \theta')$ being unaltered.

Theory would lead us to expect to find the value of $\log m$ deduced from observations in which the grooved face was turned from the polarizer lying between the values obtained on the second and third of the suppositions respecting the place of diffraction, or at most not much differing from one of these limits. Similarly, we should expect from theory to find the value of $\log m$ deduced from observations in which the grooved face was turned towards the polarizer lying between the values obtained on the first and second suppositions, or at most not lying far beyond one of these values.

The following table contains the values of $\log m$ calculated from theory on each of the hypotheses respecting the direction of vibration, and on each of the three suppositions respecting the place of diffraction. The numerals refer to these suppositions. The table extends from $\theta = 0$ to $\theta = 90^\circ$, at intervals of 5°. When $\theta = 0$, $m = 1$, and $\log m = 0$, in all cases. In calculating the table, I have supposed $\mu = 1.52$, or rather equal to the number, (1.5206,) whose common logarithm is .182. This table is followed by another containing the values of $\log m$ deduced from experiment.

TABLE III. Values of log m from theory, μ being supposed equal to 1.5206.

θ	Vibrations supposed perpendicular to the plane of polarization.			Vibrations supposed parallel to the plane of polarization.		
	I	II	III	I	II	III
5°	+.001	+.001	+ .002	− .002	−.001	− .002
10°	+.005	+.002	+ .007	− .008	−.004	− .007
15°	+.011	+.004	+ .015	− .019	−.008	− .015
20°	+.020	+.008	+ .027	− .034	−.015	− .027
25°	+.032	+.012	+ .043	− .053	−.023	− .043
30°	+.047	+.017	+ .062	− .078	−.033	− .062
35°	+.065	+.022	+ .087	− .109	−.044	− .087
40°	+.086	+.028	+ .116	− .146	−.058	− .116
45°	+.111	+.033	+ .150	− .190	−.073	− .150
50°	+.139	+.037	+ .192	− .244	−.090	− .192
55°	+.173	+.040	+ .241	− .310	−.109	− .241
60°	+.214	+.040	+ .301	− .388	−.129	− .301
65°	+.262	+.039	+ .374	− .486	−.151	− .374
70°	+.324	+.034	+ .466	− .608	−.175	− .466
75°	+.408	+.022	+ .587	− .766	−.202	− .587
80°	+.533	+.005	+ .760	− .987	−.231	− .760
85°	+.773	−.022	+1.060	−1.347	−.265	−1.060
90°	+ ∞	−.059	+ ∞	− ∞	−.305	− ∞

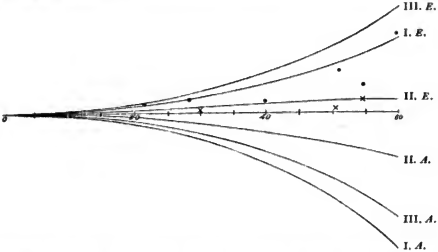
TABLE IV. Values of log m from observation.

Nature of experiment.	No.	θ	log m
Diffraction at refraction. Incidence perpendicular. Grooved face of glass plate turned *from* the incident light.	14	29°57′	+.009
	2	50°23′	+.010
	22	54°38′	+.035
Diffraction at refraction. Incidence perpendicular. Grooved face of glass plate turned *towards* the incident light.	18	21°39′	+.029
	21	28°26′	+.039
	13	39°50′	+.034
	17	50°45′	+.122
	23	54°53′	+.082
	15	59°52′	+.225

A comparison of the two tables will leave no reasonable doubt that the experiments are decisive in favour of Fresnel's hypothesis, if the theory be considered well founded. In considering the conclusiveness of the experiments, it is to be remembered that on either the first or second supposition respecting the place of diffraction, (and the third certainly cannot apply to the case in which the grooved face is turned towards the incident light,) the planes of polarization of the diffracted light are crowded by refraction towards the perpendicular to the plane of diffraction, and therefore the observed crowding towards the plane of diffraction does not represent the whole effect of the cause, be it what it may, of crowding in that direction.

If β be the value of $\alpha' - \varpi$ for $\varpi = 45^\circ$, $\beta = 1^\circ$ when $\log m = .015$, nearly; and when $\log m$ is not large, β is nearly proportional to $\log m$. In this case β is nearly the maximum value of $\alpha' - \varpi$. Hence the greatest value of $\alpha' - \varpi$, expressed in degrees, may be obtained approximately from Table IV, and, within the range of observation, from Table III, by regarding the decimals as integers and dividing by 15. Thus, for $\log m = -.388$ the real maximum is $24^\circ.8$, and the approximate rule gives $25^\circ.9$, so that this rule is abundantly sufficient to allow us to judge of the magnitude of the quantity by which the two theories differ. For $\theta = 60^\circ$, the two columns in Table III headed "I", as well as those headed "III", differ by .602, and those headed "II", differ by .169, so that the values assigned to β by the two theories differ by about 40° or 11°, according as we suppose the diffraction to take place in air or in glass. For $\theta = 40^\circ$, the corresponding differences are 15° and 6°, nearly. These differences, even those which belong to diffraction within the glass plate, are large compared with the errors of observation; for the probable cause of the large errors in experiment No 15, has been already mentioned.

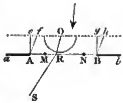
In the following figure the abscissæ of the curves represent the angle of diffraction, and the ordinates the values of $\log m$ calculated from theory. The numerals refer to the three suppositions respecting the place of diffraction, and the letters E, A, (the first vowels in the words *perpendicular* and *parallel*,) to the two hypotheses respecting the direction of vibration. The dots represent the results of the experiments in which the grooved face of the glass plate was turned towards the polarizer, and the crosses those of the experiments in which it was turned in the contrary direction.



The smallness of $\log m$ in experiment No. 23, to which the 5th dot belongs, is probably due in part to the use of the red glass, since, as has been already remarked, the planes of polarization of the blue were more crowded towards the plane of diffraction than those of the red. On this account the dot ought to be slightly raised to make this experiment comparable with its neighbours. On the other hand it will be seen by referring to Table II, that No. 23 was a much better experiment than No. 15, which is represented by the 6th dot, and apparently also better than No. 17, which is represented by the 4th dot. No. 21, represented by the 2nd dot, seems to have been decidedly better than No. 13, which is represented by the 3rd. Nos. 14 and 22, represented by the 1st and 3rd crosses respectively, were probably much better, especially the latter of them, than No. 2, which is represented by the 2nd cross. Now, bearing in mind the character of the experiments, conceive two curves drawn with a free hand, both starting from the origin, where they touch the axis, and passing, the one among the dots, and the other among the crosses. The former of these would apparently lie a little below the curve marked I. *E*, and the latter a very little below the curve II. *E*.

Hence the observations are very nearly represented by adopting Fresnel's hypothesis respecting the direction of vibration, and, whether the grooved face be turned towards or from the incident light, suposing the wave broken up *before* it reaches the grooves.

I think a physical reason may be assigned why the supposition of the wave's being broken up before it reaches the grooves should be a better representation of the actual state of things than the supposition of its being broken up after it has passed between them. Till it reaches the grooves, the wave is regularly propagated, and, according to what has been already remarked in the introduction, we have a perfect right to conceive it broken up at any distance we please in front of the grooves. Let the figure represent a section of the grooves, &c., by the plane of diffraction. Let *aA*, *Bb* be sections of two consecutive grooves, *AB* being the polished interval. Let *eh* be the plane at which a wave incident in the direction represented by the arrow is conceived to be broken up. Let *O* be any point in *eh*, and from *O* draw *ORS* in the direction of a ray proceeding regularly from *O* and entering the eye; so that *OR*, *RS* are inclined to the normal at angles θ, θ', or θ', θ, according as the light is passing from air into glass or from glass into air. The latter case is represented in the figure. Of a secondary wave diverging spherically from *O*, which is only partly represented in the figure, those rays which are situated between the limits *OA*, *OB*, and are not inclined at a small angle to either of these limiting directions, may be regarded as regularly refracted across *AB*. In a direction inclined at a small angle only to *OA* or *OB*, it would be necessary to take account of the diffraction at the edge *A* or *B*. Let γ be a small angle such that if *OR* be inclined to *OA* and *OB* at angles greater than γ the ray *OR* may be regarded as regularly refracted, and draw *Ae*, *Bg* inclined at angles γ to *OR*, and *Af*, *Bh* inclined at angles $-\gamma$. Then, in finding the illumination in the direction *RS*, all the secondary waves except those which come from points situated in portions such as *ef*, *gh* of the plane *eh* may be regarded as regularly refracted, or else completely stopped, those which come from points in *fg* and similar portions being regularly refracted, and those which come from points to the left of *e*, between *e* and the point which bears to *a* the same relation that *h* bears to *b*, as well as those which come from similar portions of the plane *eh*, being completely



stopped. Now the whole of the aperture *AB* is not effective in producing illumination in the direction *RS*. For let *C* be the centre of *AB*, and through *C* draw a plane perpendicular to *RS*, and then draw a pair of parallel planes each at a distance $\frac{1}{2}\lambda$ from the former plane, cutting *AB* in M_1, N_1, another pair at a distance λ, and cutting *AB* in M_2, N_2, and so on as long as the points of section fall between *A* and *B*. Let *M*, *N* be the last points of section. Then the vibrations proceeding from *MN* in the direction *RS* neutralize each other by interference, so that the effective portions of the aperture are reduced to *AM*, *NB*. Now the distance between the feet of the perpendiculars let fall from *A*, *M* on *RS* may have any value from 0 to $\frac{1}{2}\lambda$, and for the angle of diffraction actually employed *AM* was equal to about twice that distance on the average, or rather less. Hence *AM* may be regarded as ranging from 0 to λ; and since for the brightest part of a band forming that portion of a spectrum of the first class which belongs to light of given refrangibility *AM* has just half its greatest value, we may suppose $AM = \frac{1}{2}\lambda$. But if the distance between the planes *eh*, *ab* be a small multiple of λ, and γ be small, *ef* will be small compared with λ, and therefore compared with *AM*. Hence the breadth of the portions of the plane *eh*, such as *ef*, for which we are not at liberty to regard the light as first diffracted and then regularly refracted, is small compared with the breadth of the portions of the aperture, such as *AM*, which are really effective; and therefore, so far as regards the main part of the illumination, we are at liberty to make the supposition just mentioned. But we must not suppose the wave to be first regularly refracted and then diffracted, because the regular refraction presupposes the continuity of the wave.

The above reasoning is not given as perfectly satisfactory, nor could we on the strength of it venture to predict with confidence the result; but the result having been obtained experimentally, the explanation which has just been given seems a plausible way of accounting for it. According to this view of the subject, the result is probably not strictly exact, but only a very near approximation to the fact. For, if we suppose the distance between the planes *eh*, *ab* to be only a small multiple of λ, we cannot apply the regular law of refraction, except as a near approximation. Moreover, the dynamical theory of diffraction points to the existence of terms which, though small, would not be wholly insensible at the distance of the plane *ab*. Lastly, when the radius of a secondary wave which passes the edge *A* or *B* is only a small multiple of λ, we cannot regard γ as exceedingly small.

Let us consider now the results of experiments Nos. 11 and 12. In diffraction at refraction, the amount of crowding with respect to which the theory leaves us in doubt vanishes along with $\mu - 1$; and although this amount is far from insensible in the actual experiments, it is still not sufficiently large to prevent the results from being decisive in favour of one of the two hypotheses respecting the direction of vibration. Thus the curves marked "*A*" in the first figure are well separated from those marked "*E*", and if μ were to approach indefinitely to 1, the curves I. *A* and II. *A* would approach indefinitely to III. *A*, and I. *E*, and II. *E* to III. *E*. In diffraction at reflection, however, the case is quite different, and in the absence of a precise theory little can be made of the experiments, except that they tend to confirm the law expressed by the equation (49). In the case of the first and second images the diffraction accompanied refraction, and so far the experiments were of the same nature as those which have been just discussed, but the angle of incidence was not equal to zero, and in that respect they differ.

Let i', ρ be the angles of refraction corresponding to the angles of incidence, i, $i+\theta$. Then in the case of the first image the tangent of the azimuth of the plane of polarization is multiplied by $\cos(i+\theta-\rho)\sec(i+\theta+\rho)$ in consequence of reflection, and by $\cos(i+\theta-\rho)$ in consequence of refraction; and in the case of the second image by $\cos(i-i')$ in consequence of refraction, and by $\cos(i-i')\sec(i+i')$ in consequence of reflection. Hence if m' be the factor corresponding to diffraction and the accompanying refraction, m the factor got from observation, and regarded as correct, we have

$$\text{for 1st image, } \log m' = \log m + \log\cos(i+\theta+\rho) - 2\log\cos(i+\theta-\rho),$$

$$\text{for 2nd image, } \log m' = \log m + \log\cos(i+i') - 2\log\cos(i-i').$$

In the case of the first image, m' relates to diffraction at refraction from air into glass, where i is the angle of incidence in air, and $\rho - i'$ the angle of diffraction in glass. In the case of the second image, m' relates to diffraction from glass into air, where i' is the angle of incidence in glass, and θ the angle of diffraction in air.

In experiment No. 11, 1st image, we have from Table II, $\log m = +.289$; for the 2nd image $\log m = +.061$. In this experiment $i = 14°50'$, $\theta = 22°30'$, whence $i' = 9°41'$, $\rho = 23°30'$. We thus get

$$\text{for 1st image, } \log m' = +.289 - .286 = +.003,$$

$$\text{for 2nd image, } \log m' = +.061 - .037 = +.024.$$

The positive values of $\log m'$ which result from these experiments, notwithstanding the refraction which accompanied the diffraction, bear out the results of the experiments already discussed, and confirm the hypothesis of Fresnel. It may be remarked that $\log m'$ comes out larger for the second image, in which diffraction accompanied refraction from air into glass, than for the first image, in which diffraction accompanied refraction from glass into air. This also agrees with the experiments just referred to.

In experiment No. 12, the light which entered the eye came in a direction not much different from that in which light regularly reflected would have been perfectly polarized. Since in regularly reflected light the amount of crowding of the planes of polarization changes rapidly about the polarizing angle, it is probable that small errors in μ, i, and θ would produce large errors in m. Hence little can be made of this experiment beyond confirming the formula (49).

I will here mention an experiment of Fraunhofer's, which, when the whole theory is made out, will doubtless be found to have a most intimate connexion with those here described. In this experiment the light observed was reflected from the grooved face of a glass-grating; the reflection from the second surface was stopped by black varnish. In Fraunhofer's notation ϵ is the interval from one groove to the corresponding point of its consecutive, and is measured in parts of a French inch, σ is the angle of incidence, τ the inclination of the light observed to the plane of the grating, $(E\tau)$ the value of τ for the fixed line E, and the numerals mark the order of the spectrum, reckoned from the axis, or central colourless image, the order being reckoned positive on the side of the acute angle made by the regularly reflected light with the plane of the grating. The following is a translation of Fraunhofer's description of the experiment.

"It is very remarkable that, under a certain angle of incidence, a part of a spectrum arising from reflection consists of *perfectly polarized light.* This angle of incidence is very different for the different spectra, and even very sensibly different for the different colours of one and the same spectrum. With the glass-grating $\epsilon = 0.0001223$ there is polarized: $(E\tau)^{(+\mathrm{I})}$, that is, the *green* part of this first spectrum, when $\sigma = 49°$; $(E\tau)^{(+\mathrm{II})}$, or the green part in the second spectrum lying on the same side of the axis, when $\sigma = 40°$; lastly, $(E\tau)^{(-\mathrm{I})}$, or the green part of the first spectrum lying on the opposite side of the axis, when $\sigma = 69°$. When $(E\tau)^{(+\mathrm{I})}$ is polarized perfectly, the remaining colours of this spectrum are still but imperfectly polarized. This is less the case with $(E\tau)^{(+\mathrm{II})}$, and σ can be sensibly changed while this colour still remains polarized. $(E\tau)^{(-\mathrm{I})}$ is under no angle of incidence so completely polarized (so ganz vollständig polarisirt) as $(E\tau)^{(+\mathrm{I})}$. With a grating in which ϵ is greater than in that here spoken of, the angle of incidence would have to be quite different in order that the above-mentioned spectra should be polarized*."

If we suppose σ_ν a function of ν such that $\sigma_{-1} = 69$, $\sigma_{+1} = 49$, $\sigma_{+2} = 40$, we get by interpolation $\sigma_0 = 58.33$; so that if we suppose the central colourless image, which arises from light reflected according to the regular law, to have been polarized at the polarizing angle for light reflected at a surface free from grooves, we get $\mu = \tan 58° 40' = 1.64$, from which it appears that the grating was made of flint glass. The inclination of E in the spectrum of the order ν to the plane of the grating may be calculated from the formula $\cos\tau = \sin\sigma + \frac{\nu\lambda}{\epsilon}$†, given by Fraunhofer, and obtained from the theory of interference; and $\theta = 90° - \tau - \sigma$, where θ is the angle of diffraction. We thus get for green light polarized by reflection and the accompanying diffraction,

order of spectrum	σ	θ	$\sigma + \theta$
− 1	69°	− 18°13′	50°47′
0	58°40′	0	58°40′
+ 1	49°	+ 17°1′	66°1′
+ 2	40°	+ 33°52′	73°52′.

If we suppose the formula (49) to hold good in this case, m becomes infinite for the angles of incidence σ and the corresponding angles of reflection $\sigma + \theta$ contained in the preceding table.

Another observation of Fraunhofer's described in the same paper deserves to be mentioned in connexion with the present investigation, because at first sight it might seem to invalidate the conclusions which have been built on the results of the experiments. On examining the spectra produced by refraction in another glass-grating on which the light was incident perpendicularly, Fraunhofer found that the spectra on one side of the axis were more than twice as bright as those on the other§. To account for this phenomenon, he supposed that in ruling the grating the diamond had had such a position with respect to the plate that one side of each groove was sharp, the other less defined. This view was confirmed by finding that a glass plate covered with a thin coat of grease, and purposely ruled in such a manner, gave similar results.

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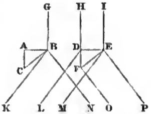
* Gilbert's Annalen der Physik, B. xiv. (1823) S. 364.

† In Fraunhofer's notation the wave length is denoted by ψ.

§ Ibid. p. 353.

Now with reference to the present investigation the question might naturally be asked, If such material changes in intensity are capable of being produced by such slight modifications in the diffracting edge, how is it possible to build any certain conclusions on an investigation in which the nature of the diffracting edge is not taken into account?

To facilitate the explanation of the apparent cause of the above-mentioned want of symmetry, suppose the diffraction produced by a wire grating in which the section of each wire is a right-angled triangle, with one side of the right angle parallel to the plane of the grating, and perpendicular to the incident light, and the equal acute angles all turned the same way. The triangles *ABC*, *DEF* in the figure represent sections of two consecutive wires, and *GB*, *HD*, *IE* represent incident rays, or normals to the incident waves, which are supposed plane. Let $BE = \epsilon$, and $BD : DE :: n : 1 - n$. Draw *BK*, *DL*, *EM* parallel to one another in the direction of the spectrum of the order ν on the one side of the axis, so that $\nu\lambda$ is the retardation of the ray *EM* relatively to *BK*, and therefore $\sin\theta = \frac{\nu\lambda}{\epsilon}$, θ being the angle of diffraction, or the inclination of *BK* to *GB* produced. Draw *BN*, *FO*, *EP* at an inclination θ on the other side of the axis, and let $\angle DBF = \alpha$. Then the retardation of *DL* relatively to *BK* is equal to $n\nu\lambda$, or $n\epsilon\sin\theta$, and that of *BN* relatively to *FO* is equal to $n\epsilon\sin\theta + n\epsilon\tan\alpha\cos\theta - n\epsilon\tan\alpha$, so that if we denote these retardations by $R_1, R_2, R_1 = n\epsilon\sin\theta$, $R_2 = n\epsilon\sin\theta - n\epsilon\tan\alpha \text{ versin }\theta$. Let p_1, p_2 be the greatest integers contained in the quotients of R_1, R_2 divided by λ, and let $R_1 = p_1\lambda + r_1$, $R_2 = p_2\lambda + r_2$. Then the relative intensities of the two spectra of the order $+\nu$ and $-\nu$ depend on r_1, r_2: in fact, we find for the ratio of intensities, on the theory of interference, $\sin^2\frac{\pi r_1}{\lambda} : \sin^2\frac{\pi r_2}{\lambda}$. Now this ratio may have any value, and we may even have a bright spectrum on one side of the axis answering to an evanescent spectrum on the other side. It appears then in the highest degree probable that the want of symmetry of illumination in Fraunhofer's experiment was due to *a different mode of interference* on opposite sides of the axis. But this has nothing whatsoever to do with the nature of the polarization of the incident light, and consequently does not in the slightest degree affect the ratio of the intensities, or rather the ratio of the coefficients of vibration, of the two streams of light *belonging to the same spectrum* corresponding to the two streams of oppositely polarized light into which we may conceive the incident light decomposed, and consequently does not affect the law of the rotation of the plane of polarization of the diffracted light.



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P. S. Since the above was written, Professor Miller has determined for me the refractive index of the glass plate by means of the polarizing angle. Four observations, made by candle-light, of which the mean error was only 1′½, gave for the double angle 113° 20′, whence $\mu = \tan 56^\circ 40' = 1.52043$, which agrees almost exactly with the value I had assumed. In two of these observations the light was reflected at the ruled, and in two at the plane surface. The accordance of the results bears out the supposition made in Part II, that the light belonging to the central colourless image, which is reflected or refracted according to the regular laws, is also affected as to its polarization in the same manner as if the surface were free from grooves. The refractive index of the plate being now known for certain, the experiments described in this paper render it probable that the crowding of the planes of polarization which actually takes place is rather less than that which results from theory on the supposition, (which is in a great measure empirical,) that the diffraction takes place before the light reaches the grooves. The difference is however so small that more numerous and more accurate experiments would be required before we could affirm with confidence that such is actually the case.

When a stream of light is incident obliquely on an aperture, it is sometimes necessary to conceive each wave broken up as its elements arrive in succession at the plane of the aperture. In applying the formula (46) to such a case, it will be sufficient to substitute for dS the projection of an element of the aperture on the wave's front, θ being measured as before from the normal to the wave, which no longer coincides with the normal to the plane of the aperture.

Before concluding, it will be right to say a few words respecting M. Cauchy's dynamical investigation of the problem of diffraction, if it be only to shew that I have not been anticipated in the results which I here lay before the Society. This investigation is referred to in Moigno's *Repertoire d'Optique moderne*, p. 190, and will be found in the fifteenth Volume of the *Comptes Rendus*, where two short memoirs of M. Cauchy's on the subject are printed, the first of which begins at p. 605, and the second at p. 670. The first contains the analysis which M. Cauchy had some years before applied to the problem. This solution he afterwards, as it appears, saw reason to abandon, or at least greatly to restrict; and he has himself stated, (p. 675), that it is only applicable when certain conditions are fulfilled, and when moreover the nature of the medium is such that normal and transversal vibrations are propagated with equal velocity. This latter condition, as Green has shewn, is incompatible with the stability of the medium. In the second memoir M. Cauchy has explained the principles of a new solution of the problem which he had obtained, without giving any of the analysis. The principal result, it would appear, at which he has arrived is, that light incident on an aperture in a screen is capable of being reflected, so to speak, *by the aperture itself* (p. 675); and he proposes seeking, by the use of very black screens, for these new rays which are reflected and diffracted. But it follows from reasoning similar to that of Art. 34, or even from the general formula (45) or (46), that such rays would be wholly insensible in all ordinary cases of diffraction, even were the screen to reflect absolutely no light. The only way apparently of rendering them sensible would be, to construct a grating of actual threads, so fine as to allow of observations at a large angle of diffraction. Such a grating I believe has never been made; and even if it could be made it would apparently be very difficult, if not impossible, to separate the effect to be investigated from the effect of reflection at the threads of the grating.

II. *Criticism of Aristotle's Account of Induction.* *By* W. WHEWELL, D.D.

[Read *February* 11, 1850.]

THE Cambridge Philosophical Society has willingly admitted among its proceedings not only contributions to science, but also to the philosophy of science; and it is to be presumed that this willingness will not be less if the speculations concerning the philosophy of science which are offered to the Society involve a reference to ancient authors. Induction, the process by which general truths are collected from particular examples, is one main point in such philosophy: and the comparison of the views of Induction entertained by ancient and modern writers has already attracted much notice. I do not intend now to go into this subject at any length; but there is a cardinal passage on the subject in Aristotle's *Analytics*, (*Analyt. Prior.* II. 25) which I wish to explain and discuss. I will first translate it, making such emendations as are requisite to render it intelligible and consistent, of which I shall afterwards give an account.

I will number the sentences of this chapter of Aristotle in order that I may afterwards be able to refer to them readily.

§ 1 "We must now proceed to observe that we have to examine not only syllogisms according to the aforesaid *figures*,—syllogisms logical and demonstrative,—but also rhetorical syllogisms,—and, speaking generally, any kind of proof by which belief is influenced, following any method.

§ 2 "All belief arises either from Syllogism or from Induction: [we must now therefore treat of Induction.]

§ 3 "Induction, and the Inductive Syllogism, is when by means of one extreme term we infer the other extreme term to be true of the middle term.

§ 4 "Thus if A, C, be the extremes, and B the mean, we have to shew, by means of C, that A is true of B.

§ 5 "Thus let A be *long-lived*; B, *that which has no gall-bladder*; and C, particular long-lived animals, as *elephant*, *horse*, *mule*.

§ 6 "Then every C is A, for all the animals above named are long-lived.

§ 7 "Also every C is B, for all those animals are destitute of gall-bladder.

§ 8 "If then B and C are convertible, and the mean (B) does not extend further than extreme (C), it necessarily follows that every B is A.

§ 9 "For it was shewn before, that, if any two things be true of the same, and if either of them be convertible with the extreme, the other of the things predicated is true of the convertible (extreme).

§ 10 "But we must conceive that C consists of a collection of all the particular cases; for Induction is applied to all the cases.

§ 11 "But such a syllogism is an inference of a first truth and immediate proposition.

§ 12 "For when there is a mean term, there is demonstrative syllogism through the mean; but when there is not a mean, there is proof by Induction.

§ 13 "And in a certain way, Induction is contrary to Syllogism; for Syllogism proves, by the middle term, that the extreme is true of the third thing: but Induction proves, by means of the third thing, that the extreme is true of the mean.

§ 14 "And Syllogism concluding by means of a middle term is prior by nature and more usual to us; but the proof by Induction is more luminous."

I think that the chapter, thus interpreted, is quite coherent and intelligible; although at first there seems to be some confusion, from the author sometimes saying that Induction is a kind of Syllogism, and at other times that it is not. The amount of the doctrine is this.

When we collect a general proposition by Induction from particular cases, as for instance, that all animals destitute of gall-bladder, (*acholous*) are long-lived, (if this proposition were true, of which hereafter) we may express the process in the form of a Syllogism, if we will agree to make a collection of particular cases our middle term, and assume that the proposition in which the second extreme term occurs is convertible. Thus the known propositions are

Elephant, horse, mule, &c., are long-lived.
Elephant, horse, mule, &c., are *acholous*.

But if we suppose that the latter proposition is convertible, we shall have these propositions,

Elephant, horse, mule, &c., are long-lived.
All acholous animals are elephant, horse, mule, &c.,

from whence we infer, quite rigorously as to *form*,

All acholous animals are long-lived.

This mode of putting the Inductive inference shews both the strong and the weak point of the illustration of Induction by means of Syllogism. The strong point is this, that we make the inference perfect as to form, by including an indefinite collection of particular cases, elephant, horse, mule, &c., in a single term, *C*. The Syllogism then is

All *C* are long-lived.
All acholous animals are *C*.
Therefore all acholous animals are long-lived.

The weak point of this illustration is, that, at least in some instances, when the number of actual cases is necessarily indefinite, the representation of them as a single thing involves an unauthorized step. In order to give the reasoning which really passes in the mind, we must say

Elephant, horse, &c., are long-lived.
All acholous animals are *as* elephant, horse, &c.,
Therefore all acholous animals are long-lived.

This "*as*" must be introduced in order that the "all *C*" of the first proposition may be justified by the "*C*" of the second.

This step is, I say, necessarily unauthorized, where the number of particular cases is indefinite; as in the instance before us, the species of acholous animals. We do not know how many such species there are, yet we wish to be able to assert that *all* acholous animals are long-lived. In the proof of such a proposition, put in a syllogistic form, there must necessarily be a logical defect; and the above discussion shews that this defect is the substitution of the proposition, "All acholous animals are *as* elephant, &c.," for the converse of the experimentally proved proposition, "elephant, &c., are acholous."

In instances in which the number of particular cases is limited, the necessary existence of a logical flaw in the syllogistic translation of the process is not so evident. But in truth, such a flaw exists in all cases of Induction *proper:* (for Induction by *mere enumeration* can hardly be called *Induction.*) I will, however, consider for a moment the instance of a celebrated proposition which has often been taken as an example of Induction, and in which the number of particular cases is, or at least is at present supposed to be, limited. Kepler's laws, for instance the law that the planets describe ellipses, may be regarded as examples of Induction. The law was inferred, we will suppose, from an examination of the orbits of Mars, Earth, Venus. And the syllogistic illustration which Aristotle gives, will, with the necessary addition to it, stand thus,

Mars, Earth, Venus describe ellipses.
Mars, Earth, Venus are planets.

Assuming the convertibility of this last proposition, *and its universality,* (which is the necessary addition in order to make Aristotle's syllogism valid) we say

All the planets are as Mars, Earth, Venus.

Whence it follows that all the planets describe ellipses.

If, instead of this assumed universality, the astronomer had made a real enumeration, and had established the fact of each particular, he would be able to say

Saturn, Jupiter, Mars, Earth, Venus, Mercury, describe ellipses.
Saturn, Jupiter, Mars, Earth, Venus, Mercury are all the planets.

And he would obviously be entitled to convert the second proposition, and then to conclude that

All the planets describe ellipses.

But then, if this were given as an illustration of Induction by means of syllogism, we should have to remark, in the first place, that the conclusion that "all the planets describe ellipses," adds nothing to the major proposition, that "S., J., M., E., V., m., do so." It is merely the same proposition expressed in other words, so long as S., J., M., E., V., m., are supposed to be all the planets. And in the next place we have to make a remark which is more important; that the minor, in such an example, must generally be either a very precarious truth, or, as appears in this case, a transitory error. For that the planets known at any time are *all* the planets, must always be a doubtful assertion, liable to be overthrown to-night by an astronomical observation. And the assertion, as received in Kepler's time, has been overthrown. For Saturn, Jupiter, Mars, Earth, Venus, Mercury, are not all the planets. Not only have several new ones been discovered at intervals, as Uranus, Ceres, Juno, Pallas, Vesta, but we have new ones discovered every day; and any conclusion depending upon this premiss that $A, B, C, D, E, F, G, H,$ to Z are all the planets, is likely to be falsified in a few years by the discovery of A', B', C', &c. If therefore, this were the syllogistic analysis of Induction, Kepler's discovery rested upon a false proposition; and even if the analysis were now made conformable to our present knowledge, that induction, analysed as above, would still involve a proposition which to-morrow may shew to be false. But yet no one, I suppose, doubts that Kepler's discovery was really a discovery —the establishment of a scientific truth on solid grounds; or, that it is a scientific truth for us, notwithstanding that we are constantly discovering new planets. Therefore the syllogistic analysis of it now discussed (namely, that which introduces simple enumeration as a step) is not the right analysis, and does not represent the grounds of the Inductive Truth, that all the planets describe ellipses.

It may be said that all the planets discovered since Kepler's time conform to his law, and thus confirm his discovery. This we grant: but they only *confirm* the discovery, they do not make it; they are not its ground work. It was a discovery before these new cases were known; it was an inductive truth without them. Still, an objector might urge, if any one of these new planets had contradicted the law, it would have overturned the discovery. But this is too boldly said. A discovery which is so precise, so complex (in the phenomena which it explains) so supported by innumerable observations extending through space and time, is not so easily overturned. If we find that Uranus, or that Encke's comet, deviates from Kepler's and Newton's laws, we do not infer that these laws must be false; we say that there must be some disturbing cause in these cases. We seek, and we find these disturbing causes: in the case of Uranus, a new planet; in the case of Encke's comet, a resisting medium. Even in this case therefore, though the number of particulars is limited, the Induction was not made by a simple enumeration of all the particulars. It was made from a few cases, and when the law was discerned to be true in these, it was extended to all; the conversion and assumed universality of the proposition that "these are planets," giving us the proposition which we need for the syllogistic exhibition of Induction, "all the planets are as these."

I venture to say further, that it is plain, that Aristotle did not regard Induction as the result of simple enumeration. This is plain, in the first place, from his example. Any proposition with regard to a special class of animals, cannot be proved by simple enumeration: for the number of particular cases, that is, of animal species in the class, is indefinite at any period of zoological discovery, and must be regarded as infinite. In the next place, Aristotle says (§ 10 of the above extract) "We must conceive that *C* consists of a collection of all the particular cases; for induction is applied to all the cases." We must *conceive* (*νοεῖν*) that *C* in the major, consists of all the cases, in order that the conclusion may be true of all the cases; but we cannot *observe* all the cases. But the evident proof that Aristotle does not contemplate in this chapter an Induction by simple enumeration, is the contrast in which he places Induction and Syllogism. For Induction by simple enumeration stands in no contrast to Syllogism. The Syllogism of such Induction is quite logical and conclusive. But Induction from a comparatively small number of particular cases to a general law, does stand in opposition to Syllogism. It gives us a truth,—a truth which, as Aristotle says, (§ 14) is more luminous than a truth proved syllogistically, though Syllogism may be *more natural and usual*. It gives us (§ 11) immediate propositions, obtained directly from observation, and not by a chain of reasoning: "first truths," the principles from which syllogistic reasonings may be deduced. The Syllogism proves by means of a middle term (§ 13) that the extreme is true of a third thing: thus (*acholous* being the middle term)

Acholous animals are long-lived:
All elephants are acholous animals:
Therefore all elephants are long-lived.

But Induction proves by means of a third thing (namely, particular cases) that the extreme is true of the mean; thus (*acholous*, still being the middle term)

Elephants are long-lived:
Elephants are acholous animals:
Therefore acholous animals are long-lived.

It may be objected, such reasoning as this is quite inconclusive: and the answer is, that this is precisely what we, and as I believe, Aristotle, are here pointing out. Induction *is* inconclusive *as reasoning*. It is not reasoning: it is another way of getting at truth. As we have seen, no reasoning can prove such an inductive truth as this, that all planets describe ellipses. It is *known* from observation, but it is not *demonstrated*. Nevertheless, no one doubts its universal truth, (except, as aforesaid, when disturbing causes intervene). And thence, Induction is, as Aristotle says, opposed to syllogistic reasoning, and yet is a means of discovering truth: not only so, but a means of discovering primary truths, immediately derived from observation.

I have elsewhere taught that all Induction involves a *Conception* of the mind applied to facts. It may be asked whether this applies in such a case as that given by Aristotle. And I reply, that Aristotle's instance is a very instructive example of what I mean. The Conception which is applied to the facts in order to make the induction possible is the want of the gall-bladder;—and Aristotle supplies us with a special term for this conception; *achŏlous**. But, it may be said, that the animals observed, the elephant, horse, mule, &c., are acholous, is a mere Fact of observation, not a Conception. I reply that it is a *Selected* Fact, a fact selected and compared in several cases, which is what we mean by a *Conception*. That there is needed for such selection and comparison a certain activity of the mind, is evident; but this also may become more clear by dwelling a little further on the subject. Suppose that Aristotle, having a desire to know what class of animals are long-lived, had dissected for that purpose many animals; elephants, horses, cows, sheep, goats, deer and the like. How many resemblances, how many differences, must he have observed in their anatomy! He was very likely long in fixing upon any one resemblance which was common to all the long-lived. Probably he tried several other characters, before he tried the presence and absence of the gall-bladder:—perhaps, trying such characters, he found them succeed for a few cases, and then fail in others, so that he had to reject them as useless for his purpose. All the while, the absence of the gall-bladder in the long-lived animals was a fact: but it was of no use to him, because he had not selected it and drawn it forth from the mass of other facts. He was looking for a mean term to connect his first extreme, *long-lived*, with his second, the special cases. He sought this middle term in the entrails of the many animals which he used as extremes: it *was* there, but he could not find it. The fact existed, but it was of no use for the purpose of Induction, because it did not become a special Conception in his mind. He considered the animals in various points of view, it may be, as ruminant, as horned, as hoofed, and the contrary; but not as *acholous* and the contrary. When he looked at animals in that point of view,—when he took up that character as the ground of distinction, he forthwith imagined that he found a separation of long-lived and short-lived animals. When that Fact became a Conception, he obtained an inductive truth, or, at any rate, an inductive proposition.

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* This term occurs in other parts of Aristotle. See the additional Note.

He obtained an inductive proposition by applying the Conception *acholous* to his observation of animals. This Conception divided them into two classes; and these classes were, he fancied, long-lived and short-lived respectively. That it was the Conception, and not the Fact which enabled him to obtain his inductive proposition, is further plain from this, that the supposed Fact is not a fact. Acholous animals are not longer-lived than others. The presence or absence of the gall-bladder is no character of longevity. It is true, that in one familiar class of animals, the herbivorous kind, there is a sort of first seeming of the truth of Aristotle's asserted rule: for the horse and mule which have not the gall-bladder are longer-lived than the cow, sheep, and goat, which have it. But if we pursue the investigation further, the rule soon fails. The deer-tribe that want the gall-bladder are not longer-lived than the other ruminating animals which have it. And as a conspicuous evidence of the falsity of the rule, man and the elephant are perhaps, for their size, the longest-lived animals, and of these, man has, and the elephant has not, the organ in question. The inductive proposition, then, is false; but what we have mainly to consider is, where the fallacy enters, according to Aristotle's analysis of Induction into Syllogism. For the two premisses are still true; that elephants, &c., are long-lived; and that elephants, &c., are acholous. And it is plain that the fallacy comes in with that conversion and generalization of the latter proposition, which we have noted as necessary to Aristotle's illustration of Induction. When we say "All acholous animals are as elephants, &c.," that is, as those in their biological conditions, we say what is not true. Aristotle's condition (§ 8), is not complied with, that the middle term shall not extend beyond the extreme. For the character *acholous* does extend beyond the elephant and the animals biologically resembling it; it extends to deer, &c., which are not like elephants and horses, in the point in question. And thus, we see that the assumed conversion and generalization of the minor proposition, is the seat of the fallacy of false Inductions, as it is the seat of the peculiar logical character of true inductions.

As true Inductive Propositions cannot be logically demonstrated by syllogistic rules, so they cannot be discovered by any rule. There is no formula for the discovery of inductive truth. It is caught by a peculiar sagacity, or power of divination, for which no precepts can be given. But from what has been said, we see that this sagacity shews itself in the discovery of propositions which are both *true*, and *convertible* in the sense above explained. Both these steps may be difficult. The former is often very laborious: and when the labour has been expended, and a true proposition obtained, it may turn out useless, because the proposition is not convertible. It was a matter of great labour to Kepler to prove (from calculation of observations) that Mars moves elliptically. Before he proved this, he had tried to prove many similar propositions:—that Mars moved according to the "bisection of the eccentricity,"—according to the "vicarious hypothesis,"—according to the "physical hypothesis,"—and the like; but none of these was found to be exactly true. The proposition that Mars moves elliptically was proved to be true. But still, there was the question, Is it convertible? Do all the planets move as Mars moves? This was proved, (suppose,) to be true, for the Earth and Venus. But still the question remains, Do all the planets move as Mars, Earth, Venus, do? The inductive generalizing impulse boldly answers, Yes, to this question; though the rules of Syllogism do not authorize the answer, and though there remain untried cases. The inductive philosopher tries the cases as fast as they occur, in order to confirm his previous conviction; but if he had to wait for belief and conviction till he had tried every case, he never could have belief or conviction of such a proposition at all. He is prepared to modify or add to his inductive truth according as new cases and new observations instruct him; but he does not fear that new cases or new observations will overturn an inductive proposition established by exact comparison of many complex and various phenomena.

Aristotle's example offers somewhat similar reflections. He had to establish a proposition concerning long-lived animals, which should be true, and should be susceptible of generalized conversion. To prove that the elephant, horse, and mule are destitute of gall-bladder required, at least, the labour of anatomizing those animals in the seat of that organ. But this labour was not enough; for he would find those animals to agree in many other things besides in being acholous. He must have selected that character somewhat at a venture. And the guess was wrong, as a little more labour would have shewn him; if for instance, he had dissected deer: for they are acholous, and yet short-lived. A trial of this kind would have shewn him that the extreme term, *acholous*, did extend beyond the mean, namely, animals such as elephant, horse, mule; and therefore, that the conversion was not allowable, and that the Induction was untenable. In truth, there is no relation between bile and longevity[*], and this example given by Aristotle of generalization from induction is an unfortunate one.

In discussing this passage of Aristotle, I have made two alterations in the text, one of which is necessary on account of the fact; the other on account of the sense. In the received text, the particular examples of long-lived animals given are *man*, horse, and mule (*ἐφ' ᾧ δὲ Γ, τὸ καθέκαστον μακρόβιον, οἷον ἄνθρωπος, καὶ ἵππος, καὶ ἡμίονος*). And it is afterwards said that all these are *acholous*: (*ἀλλὰ καὶ τὸ* B, *τὸ μὴ ἔχον χολὴν, παντὶ ὑπάρχει τῷ* Γ.) But man *has* a gall-bladder: and the fact was well known in Aristotle's time, for instance, to Hippocrates; so that it is not likely that Aristotle would have made the mistake which the text contains. But at any rate, it is a mistake; if not of the transcriber, of Aristotle; and it is impossible to reason about the passage, without correcting the mistake. The substitution of *ἔλεφας* for *ἄνθρωπος* makes the reasoning coherent; but of course, any other acholous long-lived animal would do so equally well.

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[*] Mr Owen, to whom I am indebted for the physiological part of this criticism, tells me, "All mammalia have bile, the carnivora in greater proportion than the herbivora: the gall-bladder is a comparatively unimportant accessory to the biliary apparatus; adjusting it to certain modifications of stomach and intestine: there is no relation between natural longevity and bile. Neither has the presence or absence of the gall-bladder any connexion with age. Man and the elephant are perhaps for their size the longest lived animals, and the latest at coming to maturity: one has the gall bladder, and the other not."

The other emendation which I have made is in § 6. In the received text § 6 and 7 stand thus:

6. Then every *C* is *A*, for *every acholous animal is long-lived*
(*τῷ δὴ Γ ὅλῳ ὑπάρχει τὸ Α, πᾶν γὰρ τὸ ἄχολον μακρόβιον*)

7. Also every *C* is *B*, for all *C* is destitute of bile.

Whence it may be inferred, says Aristotle, under certain conditions, that every *B* is *A* (*τὸ* Α *τῷ* Β *ὑπάρχειν*) that is, that *every acholous animal is long-lived.* But this conclusion is, according to the common reading, identical with the major premiss; so that the passage is manifestly corrupt. I correct it by substituting for *ἄχολον*, Γ; and thus reading *πᾶν γὰρ τὸ* Γ *μακρόβιον* "for every *C* is long-lived": just as in the parallel sentence, 7, we have *ἀλλὰ καὶ τὸ* Β, *τὸ μὴ ἔχον χολὴν, παντὶ ὑπάρχει τῷ* Γ. In this way the reasoning becomes quite clear. The corrupt substitution of *ἄχολον* for Γ may have been made in various ways; which I need not suggest. As my business is with the sense of the passage, and as it makes no sense without the change, and very good sense with it, I cannot hesitate to make the emendation. And these emendations being made, Aristotle's view of the nature and force of Induction becomes, I think, perfectly clear and very instructive.

W. WHEWELL.

Additional Note.

I take the liberty of adding to this memoir the following remarks, for which I am indebted to Mr Edleston, Fellow of Trinity College.

Several of the earlier editions of Aristotle have *γ* instead of *ἄχολον* in the passage referred to in the above paper: ex. gr.

(1) The edition printed at Basle, 1539 (after Erasmus): "*τὸ γ*."

(2) Basil (Erasmus) 1550. "*τὸ γ*."

(3) Burana's Latin version, Venet. 1552, has "omne enim *C* longævum."

(4) Sylburg. Francf. 1587 "*τὸ γ*" is printed in brackets thus: "[*τὸ γ*] *τὸ ἄχολον*."

(5) So also in Casaubon's edition, 1590.

(6) Casaub. 1605 "*τὸ γ*," (though the Latin version has "vacans bile;") not "[*τὸ γ*] *τὸ ἄχολον*," as the edition of 1590.

(7) In the edition printed Aurel. Allobr. 1607, "[*τὸ γ*] *τὸ ἄχολον*," as in (4) and (5).

(8) Du Val's editions, Paris, 1619, 1629, 1654 "*τὸ γ*," though in Pacius's translation in the adjacent column we find "vacans bile."

(9) In the critical notes to Waitz's edition of the *Organon* (Lips. 1844) it is stated that "post *ἄχολον* del. *γ*. *n*," implying apparently, that in the MS. marked *n*, the letter *γ*, which had been originally written after *ἄχολον*, had been erased.

The following passages throw light upon the question whether *ἄνθρωπος* ought or ought not to be retained in the passage discussed in the memoir.

(A) Aristot. *De Animalibus Histor.* II. 15, 9 (Bekk.), *τῶν μὲν ζῳοτόκων καὶ τετραπόδων ἔλαφος οὐκ ἔχει* [*χολὴν*] *οὐδὲ πρόξ, ἔτι δὲ ἵππος, ὀρεύς, ὄνος, φώκη καὶ τῶν ὑῶν ἔνιοι.......* *Ἔχει δὲ καὶ ὁ ἐλέφας τὸ ἧπαρ ἄχολον μέν, κ.τ.λ.*

(B) Conf. Ib. I. 17, 10, 11. (In the beginning of chap. 16, he says that the external *μόρια* of man are *γνώριμα*, "*τὰ δ᾽ ἐντὸς τοὐναντίον. Ἄγνωστα γάρ ἐστι μάλιστα τὰ τῶν ἀνθρώπων, ὥστε δεῖ πρὸς τὰ τῶν ἄλλων μόρια ζῴων ἀνάγοντας σκοπεῖν,*"...)

(C) Id. *De Part Animal.* IV. 2, 2. *τὰ μὲν γὰρ ὅλως οὐκ ἔχει χολήν, οἷον ἵππος καὶ ὀρεὺς καὶ ὄνος καὶ ἔλαφος καὶ πρόξ.......Ἐν δὲ τοῖς γένεσι τοῖς αὐτοῖς τὰ μὲν ἔχειν φαίνεται, τὰ δ᾽ οὐκ ἔχειν, οἷον ἐν τῷ τῶν μυῶν. Τούτων δ᾽ ἐστὶ καὶ ὁ ἄνθρωπος· ἔνιοι μὲν γὰρ φαίνονται ἔχοντες χολὴν ἐπὶ τοῦ ἥπατος, ἔνιοι δ᾽ οὐκ ἔχοντες. Διὸ καὶ γίνεται ἀμφισβήτησις περὶ ὅλου τοῦ γένους· οἱ γὰρ ἐντυχόντες ὁποτερωσοῦν ἔχουσι περὶ πάντων ὑπολαμβάνουσιν ὡς ἁπάντων ἐχόντων.......*

(D) Ib. § 11. *Διὸ καὶ χαριέστατα λέγουσι τῶν ἀρχαίων οἱ φάσκοντες αἴτιον εἶναι τοῦ πλείω ζῆν χρόνον τὸ μὴ ἔχειν χολήν, βλέψαντες ἐπὶ τὰ μώνυχα καὶ τὰς ἐλάφους· ταῦτα γὰρ ἄχολά τε καὶ ζῇ πολὺν χρόνον. Ἔτι δὲ καὶ τὰ μὴ ἑωραμένα ὑπ᾽ ἐκείνων ὅτι οὐκ ἔχει χολήν, οἷον δελφὶς καὶ κάμηλος, καὶ ταῦτα τυγχάνει μακρόβια ὄντα. Εὔλογον γάρ, κ.τ.λ.*

(E) The elephant and man are mentioned together as long-lived animals (*De Long. et Brev. Vitæ*, IV. 2, and *De Generat. Animal.* IV. 10. 2.)

The following is the import of these passages:

(*A*) "Of viviparous quadrupeds, the deer, roe, horse, mule, ass, seal, and some of the swine have not the gall-bladder. ...

The elephant also has the liver without gall-bladder, &c."

(*B*) "The external parts of man are well known: the internal parts are far from being so. The parts of man are in a great measure unknown; so that we must judge concerning them by reference to the analogy of other animals. ..."

(*C*) "Some animals are altogether destitute of gall-bladder, as the horse, the mule, the ass, the deer, the roe...But in some kinds it appears that some have it, and some have it not, as the mice kind. And among these is man; for some men appear to have a gall-bladder on the liver, and some not to have one. And thus there is a doubt as to the species in general; for those who have happened to examine examples of either kind, hold that all the cases are of that kind."

(*D*) "Those of the ancients speak most plausibly, who say that the absence of the gall-bladder is the cause of long life; looking at animals with uncloven hoof, and deer: for these are destitute of gall-bladder, and live a long time. And further, those animals in which the ancients had not the opportunity of ascertaining that they have not the gall-bladder, as the dolphin, and the camel, are also long-lived animals."

It appears, from these passages, that Aristotle was aware that some persons had asserted man to have a gall-bladder, but that he also conceived this not to be universally true. He may have inclined to the opinion, that the opposite case was the more usual, and may have written ἄνθρωπος, in the passage which I have been discussing. Another mistake of his is the reckoning deer among long-lived animals.

It appears probable, from the context of the passages (*C*) and (*D*), that the conjecture of a connection between absence of the gall-bladder and length of life was suggested by some such notion as this :—that the gall, from its bitterness, is the cause of irritation, mental and bodily, and that irritation is adverse to longevity. The opinion is ascribed to "the ancients," not claimed by Aristotle as his own.

W. W.

TRINITY LODGE,
April 13, 1850.

III. *On Impact on Elastic Beams. By* HOMERSHAM COX, ESQ., B.A. *of Jesus College.*

[Read *Dec.* 10, 1849.]

THE modern use of iron girders for Railway Bridges has given great interest and importance to the investigation of the strength of Elastic Beams when subjected to various statical and dynamical tests. Unfortunately the theoretical examination of the dynamical question is very difficult, and to supply additional knowledge respecting them an extensive series of experiments has been instituted under the authority of the Royal Commission appointed to inquire into the Application of Iron to Railway Structures.

Part of these experiments, related to the effects of impact in producing the deflection and fracture of beams. In most cases the impact was given in a horizontal direction by means of heavy balls moving in circular arcs as pendulums. The ball was suspended by a fine inextensible cord from a point of support vertically over the centre of the beam which was fixed in a horizontal position by bearings near its two ends. The ball descended by its own weight, and when it reached the lowest point of its arc, struck the centre of the beam horizontally in a direction perpendicular to its length. Care was taken to prevent the ends of the beam from yielding to the blow, and to carefully mark the degree of deflection produced at the centre. The velocity acquired by the ball before impact, was computed by a well known formula from the chord of the arc of descent. The course of experiments comprised great variations of the strength and size of the beam, the velocities of impact, and the relative masses of the beam and ball.

There were two ways in which these experiments might be made practically useful: either (firstly) as confirmations of an independent theoretical investigation of the laws of impact on beams—or (secondly) supposing such an independent investigation impossible, as empirical data susceptible of theoretical generalisation.

It is to the first of these purposes that the experiments will here be applied, and it will be attempted to be shewn that the observed results might have been predicted with a very great degree of confidence and accuracy from known dynamical principles.

A most elaborate paper by Poisson, entitled *Sur l'Equilibre et le Mouvement des Corps Elastiques*, is given in the Memoirs of the French Academy of Sciences, Tome VIII., and the eighth chapter of the *Traité de Mécanique* of the same author, relates to vibrations of elastic rods. It is assumed however in these investigations, that the initial velocity and position of every particle are known. But the essential difficulty of the problem here to be discussed, is to determine that which, for the purposes of the illustrious writer referred to, might be considered as known—namely, the initial effects upon the several points of the system produced by assigned external causes of motion.

The subject of the present investigation may be stated as follows:

An elastic beam of uniform density and section throughout its length, abuts at each extremity against a fixed vertical prop, and is impinged upon at its centre by a ball moving horizontally, with an assigned velocity in a direction perpendicular to the length of the beam before collision, and subsequently moving in contact with the beam throughout its deflection. It is required to determine the deflection of the beam produced by the impact.

The dynamical circumstances of the problem may be divided into two stages. The first consists in the sudden alteration of the velocity of the ball, at the instant of collision; the second, the effect of the elastic forces developed in the beam by deflection, in destroying the *vis viva* which the system has immediately after collision.

(1) In order to determine the first part of the problem, it will be assumed for the present that the ends of the beam remain in contact with the fixed abutments, and that the side of the beam which is struck, begins to take the form of a curve concave in every part. The case in which the ends of the beam recoil from their bearings after impact, requires a different method of investigation. Now in the first case referred to, the curve will not, while the deflection is small, differ considerably from the elastic curve of a beam deflected by statical pressure at its centre.

As the beam begins to take a curvilinear form, it begins to move in different parts with different velocities. But while the deflection is indefinitely small, the velocities are parallel to the direction of impact and proportional to the spaces described.

D'Alembert's Principle holds for simultaneous percussions as well as for finite forces and, as it reduces every dynamical problem to a statical form, may here be combined with the principle of virtual velocities. The legitimacy of combining the two principles is specially shewn by Poisson in the ninth chapter of his *Traité de Mécanique*, numero 535.

Let it be supposed that the arbitrary displacement is that which actually occurs during motion. To construct the equation of virtual velocities, this displacement of each particle must be multiplied by its quantity of motion; *i. e.* by its initial velocity multiplied by its mass. The sum of the products so formed must be put equal to the external impulsive force of the ball multiplied by its virtual velocity.

If dx be an element of the length of the beam and μdx its mass, initially at rest, and y the small distance through which it moves in the indefinitely small time t, parallel to the direction of impact, $\frac{y}{t}$ is the velocity, $\mu dx.\frac{y}{t}$ is the "quantity of motion," and $\frac{\mu dx y^2}{t}$ is the product of the quantity of motion by the virtual velocity. Also let P be the force of impact, and f the value of y at the centre of the beam. Pf is the product of the blow and its virtual velocity. Hence combining the principles above referred to, we have the equation

$$\frac{\int \mu dx y^2}{t} = Pf \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where the integral includes the product of the quantity of motion of every particle of the beam by its virtual velocity.

In forming the equation of the equilibrium of the quantities of motion and the force of impact, the quantities of motion produced by the finite elastic forces do not appear as, that period being indefinitely small, those quantities are indefinitely small also and disappear in the limit.

In order to effect the summation of the products of the quantities of motion of the several particles by their respective virtual velocities, it is necessary to ascertain the form which the beam assumes at the instant after impact. We set out with the hypothesis, that the side of the beam, which is struck, begins to take the form of a curve concave in every part. The ordinates of this curve are the distances which the several parts of the beam described in an indefinitely short time. Consequently the ordinates are proportional to the initial velocities of the several particles, and by substituting their values from the equation to the curve, the summation required is reduced to the summation of a function of a single variable. The curve in question is here assumed to be the elastic curve, as determined by Poisson and Prof. Moseley to be the form assumed by a uniform beam deflected by a pressure applied perpendicularly at its centre. The accuracy of the computation does not, however, depend essentially on the selection of this particular curve, for the quantity to be computed from it is involved in such a manner, that if the curve had been assumed to be a portion of a circle or parabola, the final results would not be very widely different.

Poisson has shewn (*Traité de Mécanique*, Tom. I. p. 641) the equation to the elastic curve to be

$$y = \frac{f}{a^3}(3a^2x - 4x^3),$$

where a is the whole length of the beam, and x the ordinate measured from one end along the beam when undeflected. This equation is the same, *mutatis mutandis*, with that arrived at by different methods by Professor Moseley, and given in his *Principles of Engineering*.

Squaring both sides of the equation, and integrating between limits $x = \frac{1}{2}a$, and $x = 0$,

$$\int_0^{\frac{a}{2}} \mu y^2 dx = \frac{17}{70} f^2 \mu a.$$

For the whole length of the beam the integral will have double the above value,

$$2\int_0^{\frac{a}{2}} \mu y^2 dx = \frac{17}{35} f^2 \mu a \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2),$$

and therefore equation (1) becomes

$$\frac{\frac{17}{35} f^2 \mu a}{t} = Pf.$$

Or, dividing by f, putting v for the velocity at the centre of the beam, and μa the mass of the beam $= M$,

$$P = \frac{17}{35} Mv \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

Now by the ordinary principles of impact, if u be the velocity of the ball before impact, and m its mass, the blow is equal to the quantity of motion lost by the ball, or since v is the same for both beam and ball after impact,

$$P = m(u - v).$$

Substituting this value of P in equation (3),

$$m(u-v)=\frac{17}{35}Mv.$$

And consequently

$$v=\frac{m}{m+\frac{17}{35}M}\cdot u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

From (2) it appears that the total *vis viva* after impact or

$$2\int_0^{\frac{a}{2}}\frac{\mu y^2 dx}{t^2}=\frac{17}{35}\frac{f^2\mu a}{t^2}=\frac{17}{35}Mv^2,$$

and the total *vis viva* of the beam and ball together is

$$mv^2+\frac{17}{35}Mv^2=(\text{from } 4)\,\frac{m}{m+\frac{17}{35}M}mu^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Adopting, then, the elastic curve to represent the initial velocities of the several parts of the system, effecting the integration and supplying the numerical calculations, we find ultimately that rather less than one half the inertia of the beam may be supposed to act initially to resist the ball; or, to speak more precisely, that at the instant after impact the impinging ball loses as much of its motion as it would have done if it had impinged on another free ball having 17-35ths of the mass of the beam. From this conclusion it is easy to infer, as in the accompanying formula, the total *vis viva* of the system after impact.

(2) The second part of the problem consists in determining the effect of the elastic forces developed in the beam by deflection. At the end of the deflection the whole system is supposed to be brought to rest simultaneously, and by the principle of the conservation of *vis viva*, the whole work done by the elastic forces is equal to half the *vis viva* destroyed.

It seems safe to assume that the elastic forces are functions of the distances between the particles of the beam and not of their velocities. This assumption is made in investigations of vibrating cords and rods, of which the results are confirmed by experiment.

If, then, the elastic forces of the beam vary as the extension and compression directly, the work done in bending the beam into a particular form will be the same whether the particles move intermediately with a greater or less velocity. Now when the beam is deflected statically through a certain distance at its centre, the deflecting pressure is to the distance of deflection in a nearly constant ratio which is usually determined by ascertaining experimentally the number of pounds weight which will statically maintain a deflection of one inch.

Let a be that weight, f the deflection in inches, af is the pressure necessary to maintain, and $\frac{1}{2}af^2$ the work necessary to produce the deflection.

Consequently $\frac{1}{2}af^2$ will be the work done in deflecting the beam after impulse, if f be the central deflection, and the final form of the beam be that which it would statically assume. Therefore from (5) by the principle of *vis viva* as explained

$$\frac{m}{\frac{17}{35}M+m}\cdot mn^2=af^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Now u is the velocity of the ball before impact: if this velocity be produced by a vertical descent h, $u^2 = 2gh$, where g is the force of gravity. Whence (6) becomes

$$2\,mgh \, . \frac{m}{\frac{17}{35}M + m} = af^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7).$$

Also by the geometrical properties of the circle, if r be the radius and c the chord, $h = \dfrac{c^2}{2r}$. So that

$$mg\frac{c^2}{r} \, . \frac{m}{\frac{17}{35}M + m} = af^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8).$$

The principal mathematical formula arrived at by the above methods may be enunciated as follows. Divide the weight of the ball by itself + $\frac{17}{35}$ths of the weight of the beam. Multiply the resulting fraction by twice the product of the weight of the ball in pounds by the vertical distance of descent; the result is equal to the square of the deflection multiplied by the number of pounds which statically maintain one inch deflection.

Hence we conclude that for a beam of assigned mass and elasticity struck by a ball of given weight the deflection varies as

1st. The velocity of impact directly; from (6),

2nd. The square root of the vertical distance of the ball's descent; from (7),

3rd. The chord of impact directly; from (8).

All these results are confirmed by the experiments above referred to. The comparison of theory and observation in the accompanying table is extremely satisfactory, and has been made for beams and balls of very different dimensions, and the agreement of the results under widely-varying circumstances is so close as to leave nothing to be desired.

When the beam is very flexible and subjected to great velocity of impact, parts of it will recede with the blow and parts move in contrary directions. In this case the above investigations do not apply, and the problem becomes excessively difficult: but the difficulty is the less to be regretted because in practice beams of great rigidity are always employed.

COMPARISON OF THEORY AND EXPERIMENT.

THE length of the wire suspending the ball in all cases = 210 inches.

(TABLE I. *Report on Railway Structures*, p. 39.) Statical pressure to maintain 1 inch deflection = 500 lb. Weight of beam 403 lb.

Weight of Ball 151 lb.

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
$\frac{1}{2}$	20.207	20.812	$-\ \frac{1}{33}$
1	40.41	40.625	$-\ \frac{1}{200}$
$1\frac{1}{2}$	60.62	58.875	$+\ \frac{1}{33}$
2	80.83	79.000	$+\ \frac{1}{44}$
$2\frac{1}{2}$	101.04	99.125	$+\ \frac{1}{50}$
3	121.24	119.750	$+\ \frac{1}{48}$

Weight of Ball 75½ lb.

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
$\frac{1}{2}$	35.82	35.437	$+\ \frac{1}{250}$
1	71.65	70.375	$+\ \frac{1}{59}$
$\frac{1}{2}$	107.47	105.625	$+\ \frac{1}{59}$

Weight of Ball 600 lb.

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
$\frac{1}{2}$	7.64	7.875	$-\ \frac{1}{33}$
1	15.28	15.750	$-\ \frac{1}{31}$
$1\frac{1}{2}$	22.91	23.125	$-\ \frac{1}{115}$
2	30.55	30.750	$-\ \frac{1}{153}$
$2\frac{1}{2}$	38.19	38.500	$-\ \frac{1}{96}$
3	45.83	46.125	$-\ \frac{1}{153}$
$3\frac{1}{2}$	53.46	53.812	$-\ \frac{1}{134}$
4	61.10	61.187	$-\ \frac{1}{763}$

(TABLE II. *Report*, p. 42.) Weight of beam 262 lb. Statical pressure to maintain 1 inch deflection 157 lb.

Weight of Ball 75½ lb.*

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
$\frac{1}{2}$	17.03	15.5	$+\ \frac{1}{11}$
1	34.06	30.3125	$+\ \frac{1}{9}$
$1\frac{1}{2}$	51.09	44.125	$+\ \frac{1}{8}$
2	68.12	59.75	$+\ \frac{1}{8}$
$2\frac{1}{2}$	85.15	73.875	$+\ \frac{1}{7}$
3	102.18	87.625	$+\ \frac{1}{7}$
$3\frac{1}{2}$	119.21	101.75	$+\ \frac{1}{7}$
4	136.24	116.75	$+\ \frac{1}{7}$

Weight of Ball 151 lb.

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
1	20.12	20.1875	$-\ \frac{1}{287}$
2	40.23	39.875	$+\ \frac{1}{100}$
3	60.35	58.875	$+\ \frac{1}{40}$
4	80.46	79.625	$+\ \frac{1}{100}$
5	100.58	99.5	$+\ \frac{1}{100}$
6	120.70	118.5	$+\ \frac{1}{57}$

(Ibid.) Bars half the length of the preceding. Weight of bar 130 lb. Statical pressure to maintain 1 inch deflection 1275 lb.

Weight of Ball 603 lb.

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
$\frac{1}{2}$	11.23	11.3437	$-\ \frac{1}{112}$
$\frac{3}{4}$	16.84	17.125	$-\ \frac{1}{57}$
1	22.46	22.719	$-\ \frac{1}{76}$
$1\frac{1}{4}$	28.07	28.156	$-\ \frac{1}{280}$
$1\frac{1}{2}$	33.61	33.625	$-\ \frac{1}{168}$

Weight of Ball 75½ lb.

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
$\frac{1}{2}$	41.22	41.875	$-\ \frac{1}{68}$
$\frac{3}{4}$	61.83	63.125	$-\ \frac{1}{52}$
1	82.45	84.3125	$-\ \frac{1}{40}$
$1\frac{1}{4}$	103.06	105.531	$-\ \frac{1}{42}$
$1\frac{1}{2}$	123.67	126.0625	$-\ \frac{1}{52}$
$1\frac{3}{4}$	144.28	147.875	$-\ \frac{1}{41}$

Weight of Ball 151½ lb.

Defn. inches.	Chord of Impact.		Difference nearly.
	Formula.	Experiment.	
$\frac{1}{2}$	25.54	26	$-\ \frac{1}{52}$
$\frac{3}{4}$	38.31	39	$-\ \frac{1}{55}$
1	51.08	51.75	$-\ \frac{1}{74}$
$1\frac{1}{4}$	63.86	65.125	$-\ \frac{1}{54}$
$1\frac{1}{2}$	76.63	77.625	$-\ \frac{1}{77}$
$1\frac{3}{4}$	89.40	89.875	$-\ \frac{1}{179}$
2	102.17	100.75	$+\ \frac{1}{71}$

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* In this case the velocities were so great, and the bar was so flexible that the experimental results do not agree accurately with those of the formula, which is applicable only when the beam is very rigid.

IV. *On the Symbols of Logic, the Theory of the Syllogism, and in particular of the Copula, and the application of the Theory of Probabilities to some questions of evidence.* *By* AUGUSTUS DE MORGAN, *Sec.* R.A.S., *of Trinity College, Cambridge, Professor of Mathematics in University College, London.*

[Read *February* 25, 1850.]

THREE years ago I communicated to the Society some developements of the theory of the Syllogism, which I have since embodied, with additions, in a work* on *Formal Logic*. I now proceed to consider the subject still further, with particular reference to the application of symbols, and the tendency which such application has to develope what I must call the *algebra* of the laws of thought.

It will be necessary for me to refer, in various ways, to the literary topics of a controversy in which the paper above-mentioned involved me: and this I can do without dwelling on the part of it which is personal to myself or to my opponent. That controversy turned upon the connexion between two systems of syllogism. The first, Sir William Hamilton's alteration of the Aristotelian system by the *invention* of forms of predication, so as to assign either of the two modes of quantity, universal or particular, to either of the terms of a proposition, subject or predicate, in either of the different species of propositions, affirmative or negative. The second, my own *numerically definite* system, in which the number of objects of thought that are spoken of, whether under subject or predicate, and also all that exist, are numerically signified, either by the specific or general symbols of arithmetic. But in the present paper I have nothing to do with the numerically definite system, except as it may be alluded to in illustration of the others. Still, I shall have to compare two systems. The first, that of Sir William Hamilton above alluded to. The second, the other and prior of my own two systems, in which the extension of the Aristotelian system is made by the application of *contrary terms* to all the usual forms of predication, without any *direct* invention of modes of applying quantity.

And I may further state, that the methods of this paper have nothing in common with that of Professor Boole, whose mode of treating the forms of logic is most worthy the attention of all who can study that science mathematically, and is sure to occupy a prominent place in its ultimate system.

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* Referred to, throughout this paper, by the initials *F. L.*

SECTION I.

ON THE APPROXIMATION OF LOGICAL AND ALGEBRAICAL MODES OF THOUGHT.

Throughout this paper I use the word logic in the purely technical sense. The progress of algebra as distinguished from arithmetic, is marked by the gradual approach to the following theorem, that *every pair of opposite relations is undistinguishable from every other pair, in the instruments of operation which are required.* So early did this principle gain some practical acceptance, that no attempt was ever persisted in (even if made, which is more than I know) to signify different oppositions by different pairs of symbols: + and − were found instrumentally adequate to all the wants of the mechanism of the science. I do not say that this was a benefit: I only state it as a fact. An algebraist, who may be required, should the proper problem occur, to interpret $-(-(+(-a)))$ as the *removal* from an expression of all traces of a *loss* incurred in an *ascent* made at a time *prior* to a certain epoch—may have gained the power of making such interpretation very slowly, in consequence of never having sufficiently distinguished differences, as a preliminary to, or a concurrent with, abstraction by observation of resemblances. There may be many for whom it would have been better that the above symbol had been $-(^{\bullet}(\dagger(\ddagger a)))$, or the like, until identity of rules had suggested identity of symbols: and I am sure that I was of the number.

The forms of thought which have not immediate relation to magnitude have been otherwise treated: consideration of differences has predominated, that of resemblances has been almost ignored. In the single case in which algebra was forestalled, the old maxim that *two negatives make an affirmative*, so loose was the treatment that the penalty of algebra was incurred. The two negatives, which are only instrumentally the equivalent of an affirmative, are two signs as different in origin and character as the two negative signs in $-(-a)$. In "man is not (not-animal)," the first negative disconnects, the second describes the predicate disconnected.

In many cases, the difference of symbols, so much wanted by the beginner in algebra, far from encouraging abstraction of resemblances, stimulated differences of interpretation, as in 'not unwilling,' which means less than willing: the double negative has, in common language, deteriorated into an affirmative of a lower degree.

The suggestions of symbolic notation have led me to more recognition than is usually made of harmonies which exist among various pairs of opponent notions common in logical thought. I select the following;—affirmative and negative—universal and particular—the subjective distinction of possible and impossible—the objective distinction of existent and non-existent—necessary and not necessary—sufficient and insufficient—conjunctive and disjunctive—convertible and inconvertible—conclusive and inconclusive—singular and plural—definite and indefinite: omitting *true and false*, the most general of all, as most obviously capable of forming one element of the distinctive definition of any pair. I believe that any of these oppositions might be interchanged and used for each other: but not always without what would be called *forcing*. This, however, is not a conclusive objection: a forced analogy may only deserve that name because we have not been accustomed to the comparisons which it suggests, or to the language, or to the order of ideas, &c. The following phrase of Sir William Hamilton's system, 'All A is not some B' is very forced, both in order and phraseology; one who sees it for the first time finds it hard to make either English or sense of it. The meaning is, 'Each A is not any one among certain of the Bs:' and in its place in the system alluded to, the uncouth expression helps to produce system, and the perception of uniform law of inference.

I now take an instance from the preceding list, in which it will appear that an obscurity of expression, if not absolute error, which has often occured, would have been avoided if the mind had been *forced* to an analogy or an analogy had been *forced* until the mind readily saw it. I say that the distinction of *universal* and *particular* may be replaced by that of *conclusive* and *inconclusive*. In comparing 'every X is Y' and 'some Xs are Ys,' the first is conclusive with respect to every case which can be brought to the proposition for settlement by it. Certain Xs, determinable or indeterminable, are proposed; are they Ys, or not? The universal is conclusive on this point, the particular inconclusive. Had this comparison been always made, we should not have had so many[*] writers who have expressed themselves as if the 'some' of a particular proposition excluded every other some.

I will now go through all the cases I have named, making *universal and particular* the standard case which is to be compared with all the rest.

1. The distinction of universal and particular may be replaced by that of affirmative and negative. The universal *affirms* the right to assign one or other or both of two names to every object of thought in the universe of the proposition: the particular *denies* it. Thus, x being the contrary name of X, the proposition 'Every X is Y' affirms that Y and x fill the universe; and 'some Xs are not Ys' denies it.

2. The distinction of universal and particular is that of possible and impossible, or of existent or non-existent, according as the proposition speaks subjectively or objectively. Let PQ signify the name compounded of P and Q, and belonging to every object which has a right to both names. Then 'every X is Y' sets down Xy as impossible or non-existent, and 'some Xs are not Ys' sets down Xy as possible or existent.

3. On the connexion of universal and particular with necessary and not-necessary (contingent) see the doctrine of modals, *passim*: on the correlative connexion with sufficient and not-sufficient, see *F. L.* p. 73.

4. The distinction of universal and particular is that of conjunctive and disjunctive; the universal speaks conjunctively, the particular disjunctively, of the same set. The Xs being

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[*] No small number of the elementary writers do this. But that their leaders had no such error in their heads, is clear enough. Ludovicus Vives, speaking of the maxim that *Differentia et proprium non accipiunt magis et minus*, remarks that to be fit for a physician or a sailor are *propria* of man, and yet one man may have *more fitness* for either than another. But this is wrong, and the error consists in introducing into the *proprium* that which is not part of it, as *proprium*: for one beast is *more fit* for a sailor than another; and the *proprium* of man is to have as much as is wanted. The same maxim, *non accipit magis et minus*, was practically and consistently applied to particular quantity, though not stated of it, that I am aware of: but the writers I speak of use language which, but for their subsequent proceedings, would lead any one to suppose that they accepted *magis et minus* as indefinitely applicable to particulars, and definitely to the relation of universal and particular.

distinguished as X_1 X_2 X_3 &c., the universal 'Every X is Y' affirms that X_1 is Y, *and* that X_2 is Y, *and* that X_3 is Y, *et cætera*. But the particular 'some Xs are not Ys' only declares that *either* X_1 is not Y, *or* that X_2 is not Y, *or* that X_3 is not Y, *aut cætera*. Nor do I here narrow the meaning of the particular: as used in logic, this species of proposition does not necessarily affirm nor deny of more than one.

5. The distinction of universal and particular may be made that of convertible and inconvertible. This is the only case in which I have had to search for a meaning to make the system good: in all the other cases, perception of the instance preceded that of the general analogy; I believe my work on logic will shew this of nearly all. Now convertibility and inconvertibility are only expressions of identity and non-identity: and it may be easily shewn that the universal is the identity of (followed by the right to convert) two names; and the particular the non-identity. Let U be the name of everything in the universe of the proposition: and let X,Y be a name which includes everything that is either X, or Y, or both. Then the universal 'Every X is Y' affirms the convertibility of x,Y and U; while 'some Xs are not Ys' denies it.

6. On the connexion of universal and particular with conclusive and inconclusive, I have already spoken.

7. The distinction of universal and particular is that of singular and plural. All books of logic affirm that the singular proposition is universal. But the manner in which logicians have treated the *universal* proposition as *singular*, in effect, if not in name, will be the material of a curious chapter in the history of logic, when written. Some of them have seen their own tendency, and have made 'Man is animal' to be a proposition of a distinct species from '*Every* man is *an* animal.' The universal proposition treats the subject collectively, and makes one singular notion of the whole: the particular makes, or may make, two groups, of indefinite proportions to the whole, and affirms or denies of one, neither affirming nor denying of the other.

8. Sir William Hamilton has very effectively forced the attention of logicians to the manner in which their universal and particular are *definite* and *indefinite*. I shall presently insist on this same distinction as that of *indefinite* and *definite*, and that with particular relation to Sir William Hamilton's system.

If, instead of taking universal and particular as the standard relation, I had chosen affirmative and negative, the principle for which I contend would have appeared more clearly, perhaps: but at the same time it would have appeared to state nothing but what everybody knows. Surely, one would remark, all oppositions stand to one another in an affirmatory and negatory relation, so that affirmation and negation are the root of them all, and as things which are connected with the same are connected with one another, it follows that all opposite relations are connected with one another. This is perfectly true, and fully admitted: nevertheless, true and easy as it is, writers on logic are not yet masters of it, nor were writers on algebra till recently. In this last science, all oppositions are instrumentally reducible to addition and subtraction: let gain, ascent, prior time,.... give or require addition, then loss, descent, posterior time,.... give or require subtraction. This easy key to the generalization of the meanings of + and −, is modern as to clear perception and full acceptance: D'Alembert denied its universality.

I think it reasonably probable that the advance of symbolic logic will lead to a calculus of opposite relations, for mere inference, as general as that of + and − in algebra. On the advantages or disadvantages of its introduction it would be vain to speculate beforehand. I now proceed to another point of the approximation of logic and algebra.

When the dry and lifeless instrumental forms of syllogism are placed before a student who has already familiarized himself with their use without thinking about them, it may easily happen that they are received with disgust, and it often *has* happened. That the noble act of the mind called by us *inference*, should be defined as consisting in mere transformation and substitution, appears* ridiculous. And the definition is truly so, unless it be confined to the *instrumental* part of inference, the part of the process which might be done by a machine. Algebra might be just as unworthily treated, by confining it to those few general rules in which its operative part really consists, and elevating this part to the dignity of a whole. At the highest, we can but compare the forms of logic in reasoning with the laws of linear perspective in painting: and the presence of these forms with the incidental lines which perspective requires, and which are rubbed out, not merely before the design is finished, but before the higher art of the process begins. And the parallel holds still further. Many great painters have disfigured their work by too much neglect of the instrumental laws of perspective; many have wilfully and skilfully violated them to produce the effect they wanted;—and so has it been with reasoners.

Speaking instrumentally, what is called *elimination* in algebra is what is called *inference* in logic. If there be four $\genfrac{}{}{0pt}{}{\text{equations}}{\text{assertions}}$ in $\genfrac{}{}{0pt}{}{\text{algebra}}{\text{logic}}$ involving any number of objects of $\genfrac{}{}{0pt}{}{\text{equation}}{\text{assertion}}$, it is possible from the four to produce one $\genfrac{}{}{0pt}{}{\text{equation}}{\text{assertion}}$, excluding three objects of $\genfrac{}{}{0pt}{}{\text{equation}}{\text{assertion}}$ from among those in the originals. The $\genfrac{}{}{0pt}{}{\text{resulting equation}}{\text{inference}}$ is free from the three $\genfrac{}{}{0pt}{}{\text{eliminated quantities}}{\text{middle terms}}$.

The logicians confine themselves in the first instance to the simple syllogism, which is the elimination of one middle term between two assertions. In like manner the algebraist asserts that all elimination may be reduced to successions of eliminations of one quantity between two equations. And just as all direct power of elimination, exclusive of what are called *artifices*, depends upon our being able to find one quantity in terms of others with which it is involved in an equation—so all our power of expressing inference depends upon our being able to describe one object of thought in terms of others, by means of an assertion in which they are all involved. When I say "'John met Thomas in the street'—'John shook hands with Thomas'—that is, 'John shook hands with what John met in the street'"—there is an elimination of 'Thomas' perfectly answering in process to '$a + x = b$, $y = c + x$, therefore $y = c + b - a$'.

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* The two propositions *Omnis dives est sapiens*, and *Solus sapiens est dives* are *logical equivalents!*

Several remarkable matters connected with inference may be made to arise out of this view, of which some will be noticed. But I now proceed to observe that this perfect sameness of logical and algebraical process does not continue. Whenever close resemblances exist which are rarely or never noted, we may be pretty sure that the attention has been diverted by differences as remarkable or more so. In algebra, in which the result is quantity, equations are perfect identities. If x be 10, every x that can be produced under precisely the same circumstances, is entitled to the copular sign (=) in connexion with any abstract 10 that we can imagine. The consequence is, that a complete conversion of all the processes of elimination can always be made. If $y = \phi x$, $z = \psi x$ give $z = \chi y$, then $z = \psi x$ and $z = \chi y$ always give $y = \phi x$ as one at least of certain alternatives. From $y = c + x$, $y = c + b - a$, we can recover $a + x = b$, with which we began: but from 'John shook hands with Thomas' and 'John shook hands with what he met in the street', we cannot of right recover 'John met Thomas in the street.'

When an assertion becomes complicated, we may be, as in algebra, in the position of finding the reduction impossible: or else we may make the beginner's mistake of disentangling the subject (solving the equation) by the process of *ignotum per ignotum*. For instance 'John met Thomas near his own house:' if we describe John as '*a man* who met Thomas near *his own* house,' we may not have a solution, for the *pronoun* is logically the *noun*. But we have one if we refer '*his own*' to '*a man*.'

The following is an imitation of an elimination of two quantities between three equations:

John met Thomas; John and Thomas live in the same street;

John is richer than Thomas. The elimination gives either of three forms, of which this is one; the algebraic process is here fully imitated.

'A person who met one not so rich as him whom he met' and 'that same person not so rich as him whom he met' live in the same street.

These assimilations may appear ludicrous, but it will be presently seen that the ideas which they suggest may help to free logic from being to the higher processes of thought what algebra would have been to its present state if it had discussed no forms more complicated than $x = y$, $y = z$, &c. The world at large is more advanced in constructive process than the books on logic; in which arbitrary separations are declared by old authority to contain all the forms of thought.

There are many processes of inference which are not syllogisms nor reducible to them, and they all belong to a form of elimination which is not precisely that of *common* algebra. In this last science we use identities because we can command identities. We have not much occasion to employ $x > y$ or $x < y$, because, from our experience of identities, we find it more convenient to suppose $x = y - b$, or $x = y + b$; making the indeterminate character of b supply the want of precision which appears in $x > y$, as compared with $x = y$. Nevertheless, elimination between *inequalities* is sometimes required: and then we know that in $x > y$ our right of substitution is, that we may for x write an equal or a greater, for y an equal or a less. In $x > y$,

y is used after the manner of a universal term in logic, x after the manner of a particular. We must know the whole of y; but we may be enabled to make the assertion by knowing a part of x. A little consideration suggests as a necessary rule of inference, the right to substitute a larger term used particularly for a smaller one, however used, and a smaller, used in either way, for a larger used universally. What we may affirm or deny of *some* or *all* men, we may affirm or deny of some *animals:* what we may affirm or deny of *all* animals, we may affirm or deny of *all* or *some* men. The second part of the rule is the *dictum de omni et nullo;* the first part has not, within my reading, been added to it: both might well be incorporated in one under the name of the *dictum*[*] *de majore et minore.* Observing that every inference was frequently declared to be reducible to syllogism, with no exception unless in the case of mere transformation, as in the deduction of 'No X is Y' from 'No Y is X,' I gave a challenge in my work on formal logic to deduce syllogistically from 'Every man is an animal' that 'every head of a man is the head of an animal.' From the total absence of attempt to answer this challenge, I conclude[†] that no one has succeeded in whose way it has fallen.

I shall presently have occasion to pursue this subject a little further: I now proceed to a new section.

SECTION II.

ON THE FORMATION OF SYMBOLIC NOTATION FOR PROPOSITIONS AND SYLLOGISMS.

The commonly received method of denoting the affirmative and negative universals by A, E, and the particulars by I, O, is rather mnemonical than instrumental: it suggests to nothing but the memory. Of all the systematic deductions of the valid forms of syllogism, not one came into general use: inductive selection and exclusion were employed, when anything more than a mere declaration of results was given. The figure being given, the above symbols are sufficient: thus AEI in the first figure can be nothing but the invalid mode 'Every Y is Z, no X is Y, therefore some Xs are Zs.'

In my former paper, and in my work since published, I borrowed from the above notation the use and meaning of A, E, I, O, and added symbolic distinctions. So far I have nothing to change: any system must use some mode of expressing its relation to the language of so many centuries. I also adopted a more detailed mode of expressing propositions: and here it would have been better if the detail had been greater at first, that it might have been ultimately made still less.

A *fundamental* symbol should not be of compound meaning: that is, should not expressly signify more than one thing. Composite expressions should be reserved for symbols which are avowedly in abbreviation of combinations of fundamental symbols. Again, the complex symbol should not be prior in invention to the simple ones; nor should it be invented until the simple ones have had their chance of good suggestion.

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[*] Every syllogism in the Aristotelian system is a direct use of this dictum.

[†] This would be a very unsafe conclusion from the absence of printed answer. But any one who writes on a controverted subject gains a number of private correspondents, with and without names. When I advanced, in my discussion with Sir William Hamilton, that a person kept close to Aristotle's forms could not prove that some must have both coats *and* waistcoats, if a majority have coats, and a majority waistcoats, I had various ingenious attempts to disprove my assertion.

For instance, in the old notation, the letters A, &c. are of compound meaning; A is universal and affirmative. If A and E had stood simply for affirmative and negative, and two consonants, as B and N, for universal and particular, the distinction of figure might have been symbolized. *Barbara* does not suggest the first figure, nor *Camestres* the second, except by memory. But if the premise-consonants had been made to imitate the middle term in location, the figure would have been seen in the word which would have resulted, whether unmeaning letters had been added for euphony or not. Thus AE occurring as premises in the second figure, in which the middle term is predicate of both premises, would have given $ABEBEB$; and so on. If some choice of liquids had been allowed for designation of particulars, and of other consonants for designation of universals, euphonic words might have been invented, which would have been, by this time, as venerable as *Bokardo* or *Felapton*. Nor would it have been difficult to have fitted on letters symbolic of the method of reduction into the first figure.

This suggestion, however, comes a few centuries too late: the following one is more to our purpose.

Symbols in which *relative position* is the whole or part of the symbol, whatever their advantages may be in other respects, lie under one great disadvantage: abstraction is not suggested, and can only be done awkwardly. The exponential symbol in algebra has this defect: we cannot describe it independently of others, except by $(\ )^{(\ )}$, or some such contrivance. In the more detailed notation of my former paper this fault was committed: thus XY, by mere position of the letters, was made to indicate 'Some Xs are Ys.'

The distinctive characters of the proposition are made to be, usually, the terms, the copula (affirmative or negative), and the quantity of the subject. But if the quantity of the *predicate* be also symbolized, notice of the terms is not *distinctively* necessary: for every proposition used has neither more nor less than two terms, and a term need not enter except to have its quantity noted. The symbols of the terms, in fact, are only pegs on which to hang distinctions: so that it is desirable that they should not be essential, though capable of introduction.

The quantities of the terms give name to the proposition: which is usually called universal or particular after its subject. This is arbitrary; and it is open to us, by the same license, to make the proposition take the name of its predicate. The enlargement of the proposition, whether Sir William Hamilton's or my own, will probably require a new word to express the distinction of propositions: for both have their double universals, and their double particulars, which are not in the Aristotelian set. At present, however, I am not prepared to suggest a term which would apply to both systems.

Let the subject and predicate, when specified, be written before and after the symbols of quantity. Let the inclosing parenthesis, as in $X)$ or $(X$, denote that the name-symbol X, which would be inclosed if the oval were completed, enters universally. Let an excluding parenthesis, as in $)X$ or $X($, signify that the name-symbol enters particularly. Let an even number of dots, or none at all, inserted between the parentheses, denote affirmation or agreement; let an odd number, usually one, denote negation or non-agreement. Thus $X))Y$ means that all Xs are Ys; $X(.(Y$ means that some Xs are not Ys: but)) and (.(specify the characters of the propositions; as do also ((and).). We must conceive ourselves at liberty to read either way: thus $X))Y$ and $Y((X$ both denote that every X is Y.

A syllogism may be denoted by juxtaposition of the symbols of the premises, taking the order XY, YZ, XZ. Thus 'Every X is Y, some Zs are not Ys, therefore some Zs are not Xs' may be stated thus,

$$X))Y).)Z \text{ gives } X).)Z \text{ or }))).) \text{ gives }).).$$

In the Aristotelian system, and in my extension, the canon of formation of the inference, when there is one, is;—*Erase the symbols of the middle term, the remaining symbols shew the inference.* Thus, if there be a valid inference from).((.) it is).·) or)). Thus also $X()Y))Z$ expresses the premises 'Some Xs are Ys, and every Y is Z:' erase Y and its accompaniments, and we have $X()Z$ for the conclusion, or 'Some Xs are Zs.' In Sir William Hamilton's system, this law is not quite universal in the symbolic deduction of the inference; but a certain variation, which I shall presently suggest, will make it so.

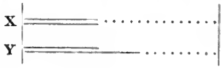
If we wish to read by *distinction* of figure, that is, by Aristotelian figure, we may contrive it thus. Let the subject of each proposition have its quantity denoted by a thicker or larger parenthesis. Then the first figure, in which we read *through* the concluding terms, would present the appearance ‖‖; remembering that when the distinction of major and minor term of conclusion is preserved, we read the second premise first. Thus

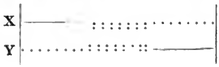
$$X))Y))Z = X))Z, \text{ which is now symbolized in the first figure, is } Y))Z + X))Y = X))Z.$$

The second figure, in which we read *to* the middle term, is as seen in ‖‖; the third, in which we read *from* the middle term, is as seen in ‖‖; and the fourth, in which we read *through* the middle term, is as seen in ‖‖.

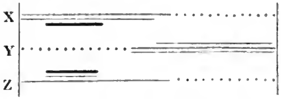
All notation, no doubt, is both pictorial and arbitrary: nevertheless there are cases in which one or the other character decidedly predominates. The arbitrary character decidedly predominates in the preceding notation: but the syllogism admits of a graphical representation which is as suggestive as a diagram of geometry. This was partially adopted by Lambert and Euler (*F. L.* p. 323), and may be more completely shewn, and without new types or woodcuts, in the following way.

Let all the instances in the *universe* of the syllogism be represented by the points of a definite straight line: but to avoid confusion, let this straight line be repeated as often as it is necessary to introduce a name. Let the division of this straight line into a continuous and a dotted portion signify the distribution of the universe into a name and its contrary. When a proposition is asserted, let a second line run over so much of the extent of each name as is declared by the proposition to be in agreement or disagreement with the whole or part of the other: extents which fall *under* one another being taken as in agreement. Thus in the following diagrams we see the propositions 'Every X is Y,' and 'Some things are neither Xs nor Ys,'





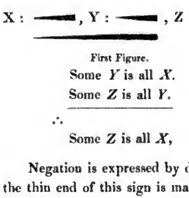
Let the middle name of a syllogism be placed in the middle, and when the two premises are formed, let the extents which enter into the conclusion be signified by thicker lines, or thicker dots. Thus it will appear what the conclusion is, and also whether the quantities in the conclusion be those which entered into the premises, or whether, by the character of the inference, either be curtailed. Thus in the diagram before us

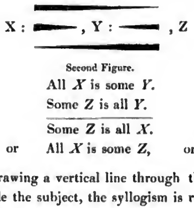


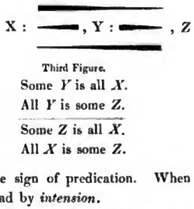
(·))·) = ()

we see the pictorial and arbitrary notation for the following syllogism :—Everything is either X or Y; some Zs are not Ys; therefore some Zs (as many as entered the premise) are Xs (not necessarily as many as in the premise). We also see that the real middle term of agreement is a portion (or what may be only a portion) of the extent of y: and that the affirmative form of the syllogism is $X((y()Z = X()Z$. References to figure might easily be added.

I should here close this section, if it had not been that Sir William Hamilton's scheme of notation has been published by an acute writer, with such commendations*, that I must not appear to shun the comparison. This system is certainly so simple, that a person who knows the premises *and inference* well, would write down any case of it immediately. I exhibit one case of it in the three figures: premising that Sir William Hamilton rejects the distinction of major and minor, and draws two conclusions in the second figure and in the third: but does not permit the fourth figure to append itself to the first, nor to appear in any way.







First Figure.	Second Figure.	Third Figure.
Some Y is all X.	All X is some Y.	Some Y is all X.
Some Z is all Y.	Some Z is all Y.	All Y is some Z.
∴	Some Z is all X.	Some Z is all X.
Some Z is all X, or	All X is some Z, or	All X is some Z.

Negation is expressed by drawing a vertical line through the sign of predication. When the thin end of this sign is made the subject, the syllogism is read by *intension*.

It would appear at first that this notation is almost identical with what I have proposed above, as to principle. Leave out the lines of predication, and the above syllogism would be

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* "A mode of notation proposed by Sir William Hamilton, is, beyond doubt, one of the most important contributions to pure Logic which has ever been made since the science was put forth; and I am fortunate in being permitted to annex it. Its excellences are—that it is very simple, that it shews the equivalent syllogisms in the different figures at a glance, that it shews as readily the convertible syllogisms in the same figure, that it enables us to read each syllogism with equal facility according to extension and intension,......" *Outline of the Necessary Laws of Thought.* By William Thomson, M.A. (2nd. Edition, 1849, p. 265.) A work to be strongly recommended.

$X:,\ Y:,\ Z = X:,\ Z$, instead of $X))Y))Z = X))Z$. And I am bound to attribute[*] to Sir William Hamilton the expression of the two distinct quantities, universal and particular, as two distinct matters of phraseology and thence of notation, ***when the quantities are not numerically conceived and expressed.*** (*F. L.* p. 301.)

If then the preceding notations were as much alike as they appear to be, I should call mine (which was first used in trying Sir W. Hamilton's system, *F. L.* p. 302), so far as affirmative syllogisms are concerned, Sir William Hamilton's with what I judge to be a more convenient mode of expression. But there is, in fact (independently of the omission of term-symbols), a very material difference, and one which makes the notation I have given more suggestive[†], and its symbolic rules more easy to some extent in my own system, to a greater in Sir William Hamilton's. In his mode of notation, the symbols of universal and particular are absolute (: and ,): in mine, the ***universality of the subject*** has the same symbol as the ***particularity of the predicate***, and *vice versâ:* thus in $X))Y$, the same symbol [ )] is applied to the universal subject, and the particular predicate.

The notion on which this mode of symbolizing quantity was tried and found to succeed, was as follows: The most natural mode of predication, because the easiest premise for inference, is the affirmative, and the syllogism of affirmative premises is the one to which all other cases are naturally reduced. And here the predicates are always particular, while the subjects are either universal, or, which is the same reality in inference, take the whole extent named in the premise into the conclusion. Whenever this is the case, the invention of a name will shew that the inference is of the same kind as one in which the term of the premise and of the conclusion are both universal. In representing "All X is in Y, all Y is in Z, therefore all X is in Z by $X))Y))Z$ there is something pictorial, and the particular character of the predicate has its symbol of particularity invented out of the relation to the subject from which we deduce that character.

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[*] And I may also note an inaccuracy of expression used by Mr Thomson (pp. 265, 266): "Many of the different elements of the notation are not new, but the novelty lies in the completeness and simplicity of the whole scheme." Not so; for though the notation had failed entirely both in completeness and simplicity, there would have remained the most remarkable novelty which there now is, namely, notation of quantity in both subject and predicate.

[†] It would seem that the forms given by Sir William Hamilton have not suggested any rules. "In the negative modes the distribution of terms will remain exactly the same as it was in the affirmatives from whence they were respectively formed, with some few exceptions in which the conclusion has a term distributed which was not when it was affirmative." Thomson, *Op. Cit.* p. 267. Under a suggestive notation, the rule which regulates the exceptions will be visible; and it will be seen that these so called exceptions are in the rule, and that cases of the *affirmative* syllogism are the exceptions.

Section III.

ON THE SYMBOLIC FORMS OF THE EXTENSION OF THE ARISTOTELIAN SYSTEM IN WHICH CONTRARIES ARE ADMITTED.

This system is, *by consequence, not by assumption*, one in which any term, be it subject or predicate, may have either kind of quantity, universal or particular, in any proposition, affirmative or negative. Sir William Hamilton's system has the same peculiarity, as the *basis of invention for the forms of predication:* that is, the accidental form of my system is the substantial form of his, so far as these terms are applicable.

I did not make this point of agreement between the two systems prominent in my work on formal logic, for the following reasons. In the memoir printed by this Society, in which all my interest in *novelty* of quantification was directed to the algebraical form of numerically definite propositions, this complete distribution of all the quantifications, existing in the system of contraries, was overlooked. So much so, that no one could conclude from my words * more than that, with *eight* forms of predication, and knowledge of the doctrine of combinations, I must have seen the necessity of this alternative—either two forms of predication with the same terms and the same quantities, or a distribution of all possible pairs of quantifications. But I have no remembrance of even this alternative suggesting itself.

When the discussion with Sir William Hamilton turned all my attention to the question whether he had or had not the numerical system, and, subsequently, to the comparison of his system (*F. L.* pp. 300—302) with the numerical one, it became evident of course, that the complete distribution of quantifications is incidental to the system of contraries. But I did not mention this explicitly in my work (pp. 63, 293): because, as the controversy was then unfinished, I neither wished to dwell upon an irrelevant quantification (that which is assumed and constructed, not that which can be derived, being the subject-matter of the dispute), such as might mislead the reader of the controversy, nor to appear as insinuating that I had published to the Society, before I had had any correspondence with my opponent, a system containing by derivation the whole extent of quantification, the invention of which was in the subject-matter of the discussion. Such insinuation would have been untrue: for though the system I now write upon *does* contain that extent of quantification, and though it *was* published (to the Society) before I had any knowledge even of the fact of Sir William Hamilton having a system of his own, yet I can most distinctly affirm that all my perception of complete quantification of both terms was derived from the algebraical form of numerical quantification.

The universal and particular affirmative may be made the bases of all the modes of predication: the others arising out of the various substitutions of contraries in them. The following are then the eight forms, with reference to the order *XY*.

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* As follows, in the fourth page of my paper "...every proposition speaks in different ways of each term and its contrary; making one particular or universal, according as the other is universal or particular....And of the two terms and their contraries, each proposition speaks universally of two, and particularly of two."

Universals.

Notation of my			Notation now proposed.	Proposition expressed in common language.
Former Memoir.	Work on Logic.	Both.		
A	A_1	$X)Y$	$X))Y$	Every X is Y
a	A^1	$x)y$ or $Y)X$	$x))y$ or $X((Y$	Every Y is X
E	E_1	$X)y$ or $X.Y$	$X))y$ or $X).(Y$	No X is Y
e	E^1	$x)Y$ or $x.y$	$x))Y$ or $X(.)Y$	Everything is X or Y or both.

Particulars.

I	I_1	XY	$X()Y$	Some Xs are Ys
i	I^1	xy	$x()y$ or $X)(Y$	Some things are neither Xs nor Ys
O	O_1	Xy or $X:Y$	$X()y$ or $X(.(Y$	Some Xs are not Ys
o	O^1	xY or $Y:X$	$x()Y$ or $X).)Y$	Some Ys are not Xs.

I think that the words universal and particular can be better described for this system than in the usual way, both as to terms and propositions. A *term* enters *universally*, when, in order to verify the proposition by induction, every instance of the name *must* be examined: *particularly*, when the verification *may* be attained without it. A *proposition* is *universal*, when every instance in the *universe* must be examined before it can be inductively verified: *particular*, when the verification *may* be attained without. So that the phrases 'universal proposition,' and 'particular proposition,' refer to all things in the universe of the proposition; while 'universal term' and 'particular term' refer to all things contained in a term, or portion of that universe. The introduction of contraries does, in fact, introduce a third term into the proposition; the *universe*, or *summum genus*, be it the whole universe of thought, or a conceivably separable portion of it. And it is to be particularly remembered, that every term is supposed to be *part* only of the universe: that is, to have an existing contrary in that universe.

In the inductive examination above alluded to, we suppose that we do not know beforehand which instances of the universe are Xs and which are Ys; nor, in selecting* an instance from the universe, can we say that that instance is not an X, till we have examined it with all the Xs. In verifying $X))Y$, we have then to examine every instance of the universe to see whether it be X, but only when it is X need we examine further to see whether it be Y: Ys may never be ascertained to be such, since, by reason of their not being Xs, the examination may never take in those Ys. Hence $X))Y$ is a universal proposition in which X is universal and Y particular. Again, in $X)(Y$ or 'some things are neither Xs nor Ys,' the ***first*** instance of the universe which is carried through all the Xs and all the Ys, and found to agree with no one instance of either, verifies all that the proposition asserts: hence the proposition is particular, and its terms both universal.

I can now correct a symbolic want of my former writings, though the thing to be signified was often matter of explanation. Expression for the *contrary* of a *contrary* was wanting in the *subject* of the proposition: whence it arose that only three equivalents appeared in the *detailed symbols*, though four existed in thought and in the more compressed symbols. Thus I had A_1 of XY, E_1 of Xy, A^1 of xy, E^1 of xY, as equivalents: while only $X)Y = X.y = y)x$ appeared in the detailed symbols. I now have

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* That is, we do not suppose, as it were, that every instance is ticketed in its place in the universe as X or x, Y or y, but that each instance taken from the universe has its own universe-mark, and that we must then, in order to find if that instance be an X, examine a separate index of Xs, to see if that mark occur. The ticketing of the instances, first mentioned, is the representative of the supposition on which some have discovered that the syllogism concludes in what was already known before the premises could be asserted.

$$X))Y = X).(y = x((y = x(.)Y$$
$$X(.(Y = X()y = x).)y = x)(Y.$$

The rule of transformation is;—To use the contrary of a term, without altering the import of the proposition, alter the curvature of its parenthesis, and annex or withdraw a negative point.

We may now say that the *quantitative contrary* of 'every X' is 'some xs,' and of 'some Xs,' 'every x.' Thus, when I say, 'some Xs are Ys' I deny something of every x: namely, that any one of them is one of those Ys. Again, 'Every X is Y' denies of some xs that they are Ys: for Ys must not fill the universe. And so on.

The distinction of affirmative and negative, in the usual sense, is abandoned: for any affirmative proposition, as $X()Y$, is also negative, as seen in its equivalent $X(.(y$. My two new forms of predication were properly called, (.) negative, and)(affirmative, and were derived in the forms $x).(y$ and $x()y$ as 'no not-X is not-Y' and 'some not-Xs are not-Ys.' Nevertheless, stated in reference to X and Y, the first appears affirmative 'Everything is either X or Y', and the second negative, 'Some things are neither Xs nor Ys.' But the first obeys the rules for the indisputable negatives, and the second those for the affirmatives. There enters an extension of an old maxim;—it is that *three negatives make a negative.* There are three positive ideas, X, Y, affirmation; opposed to the three correlative negatives, x, y, negation. Negative propositions present an *odd* number of the negatives; positive ones an even number (or none). Thus in all the equivalent forms of $X(.)Y$, nothing but an odd number of the negatives will occur; as seen in

$$X(.)Y = x))Y = X((y = x).(y.$$

There is yet one more opposition; the quantitative parentheses may turn the same or different ways: which are to be considered as positive and negative cases. By this, and the opposition of affirmation or negation, the extent of the proposition is determined. When these oppositions are none or two, the proposition is universal: when one, particular. Thus $X))Y$ having none, and $X(.)Y$ having two, are universals: but $X()Y$ having one, is particular.

In a universal proposition, any *one* quantity may be altered, either from universal to particular, or from particular to universal; and the result is always a true deduction, though not an equivalent. Thus $X))Y$ gives both $X()Y$ and $X)(Y$.

Contrary propositions (usually called *contradictory*) of which one must be true and one false, differ both in quantities and copula. Thus $X))Y$ and $X(.(Y$ are contraries. The *concomitants* of a universal, to which it is perfectly indifferent, differ from it in quantities, or in copula, not in both. Thus $X))Y$ coexists either with $X((Y$ or $X).)Y$. The superior universals of a particular, or the inferior particulars of a universal, are made by altering one quantity only: thus $X)(Y$ has $X((Y$ and $X))Y$ for its superior universals; and $X(.)Y$ has $X).)Y$ and $X(.(Y$ for its inferior particulars. The alteration of one quantity, and the copula, turns a universal into another and inconsistent universal, and a particular into another and a consistent particular: as $X))Y$ into $X(.)Y$, or $X(.)Y$ into $X).)Y$. An alteration of either of the terms into its contrary, must count as alteration of the quantity of that term, and the copula.

To carry this a little further, observe that though all the particulars consist with one another, or may exist together, yet they take distinctions, when looked at as to probability, which have some resemblance to those existing among the universals either as probabilities or as certainties. Take the universal $X))Y$, and change both quantities: it becomes $X((Y$. If the latter be true, it is in favour of $X))Y$ rather than otherwise. Do the same with $X()Y$, it becomes $X)(Y$: if the second be true, it is also rather a presumption for $X()Y$ than otherwise. The more things there are which are neither X nor Y, the smaller the number of instances of the universe within which all the Xs and all the Ys are contained; and the greater the probability of $X()Y$. Now take the universal $X))Y$ and alter one quantity and the copula: we have $X(.)Y$ and $X).(Y$, both absolutely inconsistent with $X))Y$. Do the same with $X()Y$: we have $X).)Y$ and $X(.(Y$, neither inconsistent with $X()Y$, but both diminishing its probability. This observation is of some worth in classification: it may justify us in extending the general name of concomitants to particulars in which both quantities differ, and opponents to those in which one quantity and the copula differ; and will help us to system.

Now let a proposition be considered as having two sides, on each of which a change of quantity counts as *one*, and a change of copula as *one* for each side. Changes of quantity then may be represented by $1|0$ and $0|1$, change of copula is $1|1$, and changes of term are $2|1$ and $1|2$. The utmost amount of change is $4|4$, which restores the original: thus $x(..(y$ is $X))Y$ affected by $4|4$, and it is $X))Y$ itself. And it will be found that $1|0$ and $0|1$ always produce a superior universal or an inferior particular, that $1|1$ produces a concomitant, $2|1$ or $1|2$ an opponent, $2|2$ a contrary; and so on.

When contraries are allowed, all inference may be made to consist in declaration of agreement. The disagreement of two things, because one agrees and the other disagrees with a third, may be made to fall under the other and more simple case; not indeed, as the easiest rule of thought, but as the best basis of a classification. This X agrees with this Y; this Z does not agree with this Y; therefore this X does not agree with this Z—may be stated as, This X agrees with this Y, something not this Z agrees with this Y, therefore this X agrees with something not this Z. And the universe, explicitly divided into this Z and other things, is a provision for the definite understanding of the manner in which the second proposition is transformed.

Accordingly, agreement between Xs and Ys, Ys and Zs leads to agreement between Xs and Zs, if the number of agreements be altogether more than there are Ys. Restricted as we are now to the case of the universal and the indefinite particular, one of the propositions must mention all the Ys. So that the fundamental syllogisms, from which all the rest are derived by every introduction of contraries, seem to be

$$\begin{array}{ll|ll} A_1A_1A_1 & X))Y))Z = X))Z \text{ or })))) =)) & X(\,)Y))Z = X(\,)Z \text{ or } (\,))) = (\,) & I_1A_1I_1 \\ A^1A^1A^1 & X((Y((Z = X((Z \text{ or } ((((= ((& X((Y(\,)Z = X(\,)Z \text{ or } (((\,) = (\,) & A^1I_1I_1 \end{array}$$

$$A^1A_1I_1 \quad X((Y))Z = X(\,)Z \text{ or } ((\,)) = (\,).$$

But in truth this basis is excessive, in three distinct ways.

1. ((((and)))) really belong to one of the sets which is to be formed; and the proceeds of the two will be identical. One only of them should be taken.

2. The sets from ( ))) and ((( ) only differ in respect to the species of figure[*] which I employ. I keep to XY, YZ, XZ; and one of these sets is the other referred to ZY, YX, ZX: the refusal of all distinction of figure obliges me to retain both sets.

3. In (( )) = ( ) we see only *a strengthened* syllogism: that is, stronger in the premises than is required to produce the conclusion. It is either ( ))) or ((( ) with the particular middle term universalized.

The number of variations produced by combining X or x, Y or y, Z or z, is eight, taking one out of each pair. And as these variations do not affect the character of the proposition, we must have as follows. *Eight universal* syllogisms, with two universal premises and a universal conclusion, derived from)))) =)). *Sixteen particular* syllogisms, each with a particular conclusion, derived from one universal and one particular premise; *eight* commencing with a particular, from ( ))) = ( ); *eight* commencing with a universal, from ((( ). *Eight strengthened* syllogisms, in which two universals conclude with a particular, derived from (( )) = ( ).

The canon of *validity* is as follows. The change of Y into y will alter the middle parentheses by changing them both, and therefore will not affect their relative character. Consequently, if the middle parentheses turn the same way, ((or)), any two propositions which give this arrangement allow of an inference, if one at least be universal; that is, if the number of oppositions in the parentheses, and the number of negative dots, put together, be an odd number in all, or an even number (0 included) in each proposition. But when the middle parentheses turn contrary ways, there need but be two universals, or an even number (0 included) on each side.

The canon of *inference* is merely this;—Strike out the middle parentheses, and *two* negative dots, if there be two; the remaining symbol shews the inference.

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[*] I thought I had sufficiently expressed this in my work by stating that I abandoned the *distinction* of figure. But I suppose it was hardly clear: for a learned reviewer observes, "That is not new—but is Sir William Hamilton's avowed rule." This is not correct: Sir William Hamilton's method is equally indifferent to all figures; mine holds by one, and recognizes no other, in its classifications. I have one establishment, and tolerate no distinction of sect. Sir William Hamilton has none, but tolerates three sects: Aristotle had three state-churches, Galen (they say) founded a sect of dissenters, which, after some centuries of toleration, was made a fourth establishment.

In the following table, the syllogisms are arranged in the manner described in the headings.

Terms employed.	Particular syllogism with *first* premise particular.	Universal syllogism strengthened by the *first* quantity of the particular.	Particular syllogism strengthened by the second quantity of the particular.
X Y Z	$I_{,}A_{,}I_{,}$ ()))=()	$A_{,}A_{,}A_{,}$))))=))	$A'A_{,}I_{,}$. (())=()
x y z	$I'A'I'$)(((=)(	$A'A'A'$ ((((=((	$A_{,}A'I'$))((=)(
x Y Z	$O'A_{,}O'$).)))=).)	$E'A_{,}E'$ (.)))=(.)	$E_{,}A_{,}O'$).())=).)
X y z	$O_{,}A'O_{,}$ (.(((=(.(	$E_{,}A'E_{,}$).(((=).(	$E'A'O_{,}$ (.)((=(.(
X y Z	$O_{,}E'I_{,}$ (.((.)=()	$E_{,}E'A_{,}$).((.)=))	$E'E'I_{,}$ (.)(.)=()
x Y z	$O'E_{,}I'$).)).(=)(	$E'E_{,}A'$ (.)).(=((	$E_{,}E_{,}I'$).().(=)(
X Y z	$I_{,}E_{,}O_{,}$ ()).(=(.(	$A_{,}E_{,}E_{,}$))).(=).(	$A'E_{,}O_{,}$ (().(=(.(
x y Z	$I'E'O'$)((.)=).)	$A'E'E'$ (((.)=(.)	$A_{,}E'O'$))(.)=).)

Terms employed.	Particular syllogism with *second* premise particular.	Universal syllogism strengthened by the *second* quantity of the particular.	Particular syllogism strengthened by the first quantity of the particular.
X Y Z	$A'I_{,}I_{,}$ ((()=()	$A'A'A'$ ((((=((	$A'A_{,}I_{,}$ (())=()
x y z	$A_{,}I'I'$)))(=)(	$A_{,}A_{,}A_{,}$))))=))	$A_{,}A'I'$))((=)(
x Y Z	$E_{,}I_{,}O'$).(()=).)	$E_{,}A'E_{,}$).(((=).(	$E_{,}A_{,}O'$).())=).)
X y z	$E'I'O_{,}$ (.))(=(.(	$E'A_{,}E'$ (.)))=(.)	$E'A'O_{,}$ (.)((=(.(
X y Z	$E'O'I_{,}$ (.)).)=()	$E'E_{,}A'$ (.)).(=((	$E'E'I_{,}$ (.)(.)=()
x Y z	$E_{,}O_{,}I'$).((.(=)(	$E_{,}E'A_{,}$).((.)=))	$E_{,}E_{,}I'$).().(=)(
X Y z	$A'O_{,}O_{,}$ (((.(=(.(	$A'E'E'$ (((.)=(.)	$A'E_{,}O_{,}$ (().(=(.(
x y Z	$A_{,}O'O'$))).)=).)	$A_{,}E_{,}E_{,}$))).(=).(	$A_{,}E'O'$))(.)=(.(

The universal and strengthened syllogisms appear twice, once from each set of particular syllogisms. The *contranominals*, or those in which all the terms used are contraries, are written together: they are formed from each other by alteration of all the parentheses.

Altering one quantity in a particular proposition, even though a universal term be changed into a particular one, ***strengthens*** the proposition, or converts it into a universal. In fact, in this system strength and weakness are terms which, like affirmation and negation, have nothing but opposition of character left: the less the strength of a proposition with respect to a term, the greater with respect to its contrary: so that what I have called strengthening a proposition, in only altering the distribution of strength.

The ***opponents*** of a syllogism, meaning those in which one premise and the contrary of the conclusion produce the contrary of the other premise, are thus formed. Substitute the contrary of the conclusion for one premise and change the order of reading in the other, or let the symbol of the other revolve about an axis. Thus the two opponents of).)).(are (.((.) and (.)).(; and those of ( ))) are ( )).(and).(((

The opponents of the strengthened syllogisms are universals weakened in their conclusions.

The change of the figure from $XY\ YZ\ XZ$ to $ZY\ YX\ ZX$ is merely the writing of the symbol from the other end: it turns)))) into ((((and ( )).(into).(( ). Those syllogisms which agree in their parentheses and differ only in their affirmations and negations are the *concomitants* of my work (pp. 88, 89, 94), or the *coexistents* in a complex syllogism. Thus $A_1A_1A_1$, $O'A_1O^1$ and $A_1O^1J^1$ are))))).))) and))).). These syllogisms coexist in my complex syllogism $D_1D_1D_1$; and *perhaps*, then, the best notation for this last might be derived from).))).), indicative of the only one of the four combinations which is *not* valid as a simple syllogism.

I need not enter further into this subject, as what is here given on notation may be easily applied throughout my work.

SECTION IV.

ON THE SYMBOLIC FORMS OF THE SYSTEM IN WHICH ALL THE COMBINATIONS OF QUANTITY ARE INTRODUCED BY ARBITRARY INVENTION OF FORMS OF PREDICATION.

This system, which belongs to Sir William Hamilton, has not yet been published in detail by its learned author, except in lectures; in which, I believe, it was first published in 1840 or 1841. There is some account of its forms, communicated by him, and printed with his sanction, in Mr. Thomson's *Outlines*, already cited. See also *F. L.* pp. 300—302.

The modes of predication in this system, are, by hypothesis, as follows, at least when the language of *extent*, preferred by Sir William Hamilton, is changed into that of numeration of instances. The symbols attached are dictated by the quantities of the terms, with reference to the order XY.

All Xs are all Ys	) (	$A_1 + A^1$.	No Xs are Ys	).(	E_1.
Some Xs are some Ys	( )	I_1.	Some Xs are not some Ys	(.)	—.
All Xs are some Ys	))	A_1.	No Xs are some Ys	).)	O^1.
Some Xs are all Ys	((	A^1.	Some Xs are no Ys	(.(	O_1.

Some authors had gone so far (*F. L.* Appendix II.) as to adapt expressed quantity to the predicate, for the purpose of procuring convertible forms: and Mr. Thomas Solly (*Syllabus of Logic*, 1839, p. 47) gave the above eight forms, with his reasons for reducing them to four. But Sir William Hamilton is the first who published the idea of taking all phases of usual quantification, and making them the basis of a system of syllogism.

It will be observed that I have not called this system an *extension* of that of Aristotle. That it is more extensive, in one sense, I admit; namely, in so far as it includes all which Aristotle included, and more. But a mathematician cannot therefore call it an extension, accustomed as he is to a very precise use of that term. With him enlargement is not extension, unless the wider extent be governed by the laws of the narrower one. The name *multiplication*, conceived by aid of integer numbers, is properly allowed to be *extended* to fractional ones, because, among other reasons, in every problem in which integers demand integer multiplication, fractional data demand what is *thence* called fractional multiplication. Botany was once agriculture, but in its present state it cannot properly be called an extension of agriculture: the union of England and Scotland was not an *extension* of England. This refusal to use the word extension, in the present case, is not the assertion of any defect in the system, but rather the contrary: it is quite open to inquiry whether the best form of syllogism be what I call an extension of Aristotle, or contain the incorporation of new fundamental principles.

Perhaps some may ask why I have called my own system an *extension* of Aristotle. I answer that no new laws are propounded, though the application of the old ones to an enlarged subject-matter of predication introduces some new forms of expression, and some striking points of view from which to look at the old ones. Every one of my syllogisms can be reduced to an Aristotelian form, without any addition except that of contraries to the matters of predication. For example, one of my new syllogisms,)) ((=)(, or 'All Xs and all Zs are Ys, therefore some things (namely, all that are not Ys) are neither Xs nor Zs—is reducible to ordinary form. With X Y Z it is new: but with X y z it is *Fesapo*, being

$$X).(y))z = X).)z.$$

The syllogism)) ((=)(can thus be made Aristotelian: but in my system, a plain man who sees clearly that some things are proved to be neither men nor mice, were it only because they do not eat cheese, may rest content that his knowledge, even in the form of the light of nature, can be made science, without the necessity of having recourse to the following very venerable, but very unsatisfactory, form:

No man is a non-eater of cheese
All non-eaters of cheese are other things than mice
Therefore some other things than mice are also not men.

Now take one of Sir William Hamilton's peculiar syllogisms;—

Some men are soldiers, Some animals are not men,
Therefore some soldiers are not some animals.

This syllogism cannot in any way be made Aristotelian, either with the terms as they stand, or with any others derived from them by a method independent of the syllogism itself; for instance, the derivation of the contrary from the direct term. Sir William Hamilton's system is therefore an independent addition to that of Aristotle; and the addition must be discussed on its own merits.

The forms)) (( ).) (.(()).(exist in the old system, in that of contraries, and in that of invention of predicates. The peculiar propositions of the second and third may therefore be compared as follows, and under the same[*] symbols:

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[*] Looking at the fact that the system of contraries admits premises both negative, and that of invention of predicates admits premises both particular, with other analogies which I do not describe here,—I strongly suspect that the two systems have some correlative formation in which the distinction of affirmative and negative appears in the first with the same laws of form under which that of universal and particular appears in the second; and *vice versa*. If this be the case, then (·) though it have a particular form in common language, will be a universal, and)(a particular. This is a hint for the consideration of the reader: I have not been able to make anything out of it, as yet.

Introduction of Contraries.	*Invention of predicates.*
(.) Universal negative, with particular terms, and affirmative form in common language.	(.) Particular negative with particular terms, not used in common language.
All things are either Xs *or* Ys.	*Some* Xs *are not some* Ys.
)(Particular affirmative, with universal terms, and negative form in common language.	)(Universal affirmative with universal terms, being declaration of identity in common language.
Some things are neither Xs *nor* Ys.	*All* Xs *are all* Ys.

My objections to this system as promulgated by Sir William Hamilton (*F. L.* p. 302) may be developed as follows:

First, the fundamental propositions of a logical system should be independent of each other, so that no one of them should be a compound of two others. Now $X)(Y$, or 'X and Y are identical names,' is really compounded of $X))Y$ and $X((Y$. If we once grant a *complex* proposition, why this one only, when there are others, out of which, as I have shewn, a separate system of complex syllogism may be constructed?

To say that the mode of inventing propositions yields no other, is not an answer: for it is the mode itself which is attacked in its results. Every syllogism in which)(occurs, is either a strengthened form, or the resultant of *two* other syllogisms.

Secondly, one object of *formal* logic being to provide form of enunciation for all truth, and form of denial for all falsehood, it is clear that every falsehood which can be enunciated as a truth should be deniable within the forms of the science. Now the simple denial of)(is the disjunctive assertion ').) or (.('. Though it happen that I can prove one of these to be true, without knowing which, yet the power of denying in an elementary form the elementary proposition)(is refused me. A philologist asserts the Greek words A and B to be identical in meaning: he says "All A is all B." One passage of Homer, and one of Hesiod, both contain the doubtful word C, having two possible explanations, the first of which makes Homer assert that some As are not Bs, while the second makes Hesiod assert that some Bs are not As. The premises being admitted, the resulting denial of the *simple proposition* of Sir William Hamilton's system is only obtainable by a *dilemma*, or, as it were, *metasyllogism*.

Thirdly, the proposition (.), or 'Some Xs are not some Ys,' has no fundamental proposition which denies it, and not even a compound of other propositions. It is then open to the above objection: and to others peculiar to itself. It is what I have called (*F. L.* p. 153) a *spurious* proposition, as long as *either* of its names applies to more than one instance. And the denial is as follows: 'There is but one X, and but one Y, and X is Y.' Unless we know beforehand that there is but one soldier, and one animal, and that soldier the animal, we cannot deny that 'some soldiers are not some animals.' Whenever we know enough of X and Y to bring forward 'Some Xs are not some Ys' as what *could be conceived* to have been false, we know *more*, namely, 'No X is Y,' which, when X and Y are singular, is true or false with 'Some Xs are not some Ys.'

These difficulties[*] lie on the surface; and the first objector is sure to seize them. I expect a powerful consideration of them in Sir William Hamilton's forthcoming work, from the known learning and acuteness of the author, with some weight given to his assertion that his system has been "adequately tested and matured." And I should not be surprised at a *successful* explanation: for, though I cannot give one myself, as long as the system stands on its inventor's ground, yet I can prevent the appearance of the objections by shifting that ground.

The occurrence of eight forms, corresponding in their modes of quantification with those which I had obtained, and by coincidence which did not arise from any sameness in the path of investigation, struck me as exceedingly remarkable. I could not entirely declare against the possibility of sufficient reason for a system which, independently of the habitual acquaintance of its promulgator with the logician's mode of thinking in every age, had, as we shall see, strong symbolic claims to being *something*. Symbolic language gives the expression of the laws of thought in their purest forms: and it has never deceived those who were willing to be its servants before they claimed to be its masters. In the present case, there seemed something resembling a system of algebra with a singular form in it. Formal Logic must teach how to enunciate all definitely conceivable truth and falsehood, just as symbolic algebra must teach how to enunciate all definitely expressible quantity: and 'some *X*s are not some *Y*s' appeared to partake very much of the indeterminateness of $\frac{0}{0}$. An algebraist has not profited by the history of his science, if he dogmatically reject what appears incapable of interpretation in connexion with the rest of its system. Thinking on this, I tried whether there might not be some view of predication which would make Sir William Hamilton's eight forms self-consistent: that is, make them contradict each other four and four. The thing required is that)), ((,).), (.(,).(, and ( ) should remain related to each other as at present: and that)(and (.) should be a double universal and a double particular, destructive each of the other.

We might put this question to any person, When you say 'every man is an animal,' do you speak of *all* men, or of *one* man, of as many animals as there are men, or of *one* animal? Is your proposition *cumular*, or what I will call *exemplar?* I apprehend the general first answer would be in favour of the cumular view, but not the universal one. Some would say, I speak of *one* man, being *any one I can select*, and of one animal, but not *any one I please*, for upon *what man* I select, depends *what animal* I select. Some would say that the article *an*, which denotes one animal, confines the subject to one man: how else can every man be *an* animal? And in truth they are etymologically right, for *every* is *each*, not *all*, in meaning.

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[*] Sir William Hamilton is, I have no doubt, the first *advocate* of the form (·): but it, and the peculiar syllogism derived from it (with two particular premises) have been seen and *rejected*, by myself at least, probably by others. I mention the following as a curious coincidence. Sir William Hamilton states that he had in his own mind arrived at the form, 'most *Y*s are *Z*s, most *Y*s are *X*s, therefore some *X*s are *Z*s' before me, and thrown it away, unpromulgated, as a cumbrous and useless subtlety. He had thus made the approach of a single instance towards the numerically definite syllogism. Now I, on my part, had made and published, as a true inference, but not within the forms of predication, one syllogism in the new part of Sir William Hamilton's system; in the following words: "The weakest syllogism from which such an inference [particular negative] can be drawn would then seem to be as follows. Some *X*s are *Y*s, some *Z*s are not *Y*s, therefore some *Z*s are not *X*s.... But here it will appear on a little consideration, that the conclusion is only thus far true, that those *X*s which are *Y*s cannot be *those Z*s which are not *Y*s.... (*First Notions of Logic*, 1839, reprinted (with slight alteration) as the introductory chapter of *F. L.*)

Some would be inclined to say they were sometimes cumular, and sometimes exemplar: and a great many would hesitate to affirm perfect identity of mode of thought as existing between 'Every man is an animal' and '*all* men are *animals;*' and would admit, when it was put to them, that there is more or less of this difference—the former *tells off*, the latter *sums up:* the former is a completed *induction*, the latter a transformation preparatory to *deduction*.

The next question would be, Are the general propositions of any science, *as actually proved*, cumular or exemplar: and here the answer must be, that what is proved is the exemplar proposition. Euclid, meaning to prove that *all* isosceles triangles have equal angles at the bases, fulfils his meaning by proving that *any one* isosceles triangle is so, and the perception of the goodness of his proof is entirely dependent on the perception of the force of the word *any* before *one*. A student who looks at the definition of proportion sees nothing but a hopeless heap of conditions between him and the knowledge of any possible proportion, until he sees a case established in which *any one* multiple of the first is shewn to lie among the multiples of the second as *that same* multiple of the third lies among those of the fourth.

If the cumular proposition can, generally speaking, only be proved by the help of the exemplar, it follows that the exemplar proposition must precede in order of thought: and it is justifiable to propose it as the basis of a logical system. The distinction of the two modes exists in every language in which I can form the sentence: if there be one in which both forms do not exist, the study of the minds of those who speak that language would be curious.

Of our language and many, I suppose most, others, it must be obvious that the exemplar and cumular forms of expression are much more apt to be confounded as to subject than as to predicate, as to affirmation than as to negation. The logician, who must have *forms*, has to make a choice; and he has invented cumular expressions which do not suit the genius of common thought or common language. "All man is not fish" is the form in which a logician denies that any man is a fish: Sir William Hamilton says, "All man is not all fish." Common language would deny the first by saying, "No, nor any part of him." Even "All men are not fishes" only means, in common language, "some men are not fishes," with emphasis upon the great number that are implied to be so; and would therefore be held false. The predicate of a negative must be exemplar: it is, "Every man is not any one fish." The examination of the following table will shew that there is much less forcing of common expression in a list of nothing but exemplars, than in a list of nothing but cumulars.

Quantity now becomes *mode of selection of the example: universal* is replaced by *wholly indefinite*, particular by *not wholly indefinite*, having some, no matter how much nor how little, limitation on the right of selection, under conditions known or unknown. The oppositions of logic must always be mere contradictions, of which one must exist (*F. L.* p. 59). Accordingly, *definite* is to mean *not wholly indefinite*. The terms of selection may be *any one* and *some one*.

Now apply the two selective forms in every way, and the following exemplar forms of predication will occur, arranged in contrary pairs and presenting a system of predication free from the objections which I have urged against the cumular forms, so far as contradiction is concerned.

$A_{\prime}^{\prime}$	)(	Any one X is any one Y	*giving*	There is but one X and one Y, and X is Y.
$O_{\prime}^{\prime}$	(.)	Some one X is not some one Y	*giving*	There can be found some one X and some one Y which are not the same.
$A_{\prime}$	))	Any one X is some one Y	*giving*	All Xs are Ys.
$O_{\prime}$	(.(	Some one X is not any one Y	*giving*	Some Xs are not Ys.
$A^{\prime}$	((	Some one X is any one Y	*giving*	Every Y is X.
$O^{\prime}$	).)	Any one X is not some one Y	*giving*	Some Ys are not Xs.
$E_{\prime}^{\prime}$	).(	Any one X is not any one Y	*giving*	No X is Y.
$I_{\prime}^{\prime}$	()	Some one X is some one Y	*giving*	Some Xs are Ys.

The detailed notation needs no explanation. The form given to the old notation may be explained thus. In the system of contraries, the accents refer to the relation of standard terms and contraries. Thus $O_{\prime}$ being X (.(Y, $O^{\prime}$ is x (.(y or Y (.(X. In the exemplar system, the accent refers to the example, *sub*accent to *sub*ject, superaccent* to predicate. Every accent which occurs shews that its term is named after the proposition. Thus in $I_{\prime}^{\prime}$ both subject and predicate are like the proposition, particular or not wholly indefinite: but in $A_{\prime}$ only the subject is, like the proposition, universal or wholly indefinite. The $E^{\prime}$ and $I^{\prime}$ of the system of contraries cannot exist here, but $E_{\prime}^{\prime}$ and $I_{\prime}^{\prime}$ have double accents, and we have the new propositions $A_{\prime}^{\prime}$ and $O_{\prime}^{\prime}$, which have similar selective forms in both terms.

The non-entrance of contraries keeps the following rules of syllogism intact. The middle term must be indefinite in one of the premises—negative premises give no conclusion. But on the other hand, the system of contraries acknowledges the rule that particular premises give no conclusion—which is not a rule of the exemplar system—and does not acknowledge the two first named.

Taking the 64 cases of combination of premises, in any one figure, we must then reject 16 cases of negative premises, and 12 others in which, the premises not being both negative, both the middle terms are not indefinite. There remain 36 cases for examination, *and all admit of inference:* so that the canon of validity is;—one affirmative premise and one indefinite middle term. This is also the case when exemplar forms are abandoned in favour of cumular ones: but the conclusions in the second (or Sir William Hamilton's) form are not always the same as those in the first.

The validity of every case in which there is one indefinite middle and one premise is affirmative, immediately appears: the *any one* Y of one premise may be taken to be the *some one* Y of the other: and the same is then compared with two others, both giving agreement, or one agreement and one disagreement. Further it will appear that in every case, the selective forms of the terms of the conclusion are the same as they had in the premises. Thus 'any one X is some one Y, any one Y is some one Z' gives 'any one X is some one Z,' for whatever some one Y the first premise can allow, the second premise makes it some one Z: and similarly of others.

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* Subject, as used in reference to a proposition, is liable to some objections. He would be a bold man who would dare to substitute *subdict* and *superdict* for subject and predicate: or, retaining *subject*, to call the predicate the *superject*: but a word on the rashness of this imaginary individual may help the reader in remembering the use of the accents.

Hence the symbolic canon of inference is the same as in the system of contraries, namely, the erasure of the middle parentheses gives the symbolic form of the conclusion.

But in the cumular system, this canon of inference is modified. In the exemplar system, the proposition $X)(Y$ or 'Any one X is any one Y,' though it appear to be of the widest character, is restricted by its *double* indefiniteness: doubly indefinite produces doubly singular. This will explain the mode in which conclusions are admissible in the exemplar form which are not admissible in the cumular. For example, in the former,)))(gives)(or 'Any one X is some one Y, and any one Y is any one Z,' gives 'Any one X is any one Z.' Here, if all we knew were that Y and Z are identical, all we could infer would be, 'Any one X is *some* one Z;" but because we know from $Y)(Z$ that there is but one Z, we may say that any one X is *any* one Z. In the cumular system, then,)))(gives)): and the modification of the canon of inference which the cumular system requires is;—Erase the middle parentheses, but when they both *turn one way*, any parenthesis of indefiniteness which turns the *other* way must itself be turned, unless it be *protected* by a negative point. Thus ( ))(gives ( ), but))).(gives).(.

To collect the syllogisms, we may observe that there are two doubly indefinite forms $A_{,}'$ and $E_{,}'$, each of which has three less indefinite varieties; the first $A_{,}$, A', and $I_{,}'$; the second, $O_{,}$, O', and $O_{,}'$. A syllogism then in which both premises are double, admits, in original and weakened forms, of 16 varieties; and there are three such double syllogisms which are valid. Classifying the syllogisms by the degrees of relaxation in their forms, we have the following list:

$\mathrm{A_{,}'A_{,}'A_{,}'}$	)()(=)(	$A_{,}A_{,}'A_{,}'$	)))(=)(	$I_{,}'A_{,}'A'$	())(=((	$\mathrm{I_{,}'A_{,}I_{,}'}$	()))=()
		$\mathrm{A'A_{,}'A'}$	(()(=((	$\mathrm{A_{,}A_{,}A_{,}}$	))))=))	$\mathrm{A'I_{,}'I_{,}'}$	((()=()
		$\mathrm{A_{,}'A_{,}A_{,}}$	)())=))	$\mathrm{A'A_{,}I_{,}'}$	(())=()		
		$A_{,}'A'A_{,}'$	)(((=)(	$\mathrm{A'A'A'}$	((((=((		
				$A_{,}'I_{,}'A_{,}$	)(()=))		
$\mathrm{A_{,}'E_{,}'E_{,}'}$	)().(=).(	$\mathrm{A_{,}E_{,}'E_{,}'}$	))).(=).(	$\mathrm{I_{,}'E_{,}'O_{,}}$	()).(=(.(	$\mathrm{I_{,}'O'O_{,}'}$	()).)=(.)
		$\mathrm{A'E_{,}'O_{,}}$	(().(=(.(	$\mathrm{A_{,}O'O'}$	))).)=).)	$\mathrm{A'O_{,}'O_{,}'}$	(((.)=(.)
		$\mathrm{A_{,}'O'O'}$	)().)=).)	$\mathrm{A'O'O_{,}'}$	(().)=(.)		
		$A_{,}'O_{,}E_{,}'$	)((.(=).(	$\mathrm{A'O_{,}O_{,}}$	(((.(=(.(		
				$A_{,}'O_{,}'O'$	)((.)=).)		
$\mathrm{E_{,}'A_{,}'E_{,}'}$	).()(=).(	$O\,A_{,}'E_{,}'$	).))(=).(	$O_{,}'A_{,}'O_{,}$	(.))(=(.(	$\mathrm{O_{,}'A_{,}O_{,}'}$	(.)))=(.)
		$\mathrm{O_{,}A_{,}'O_{,}}$	(.()(=(.(	$\mathrm{O'A_{,}O'}$	).)))=).)	$\mathrm{O_{,}I_{,}'O_{,}'}$	(.(()=(.)
		$\mathrm{E_{,}'A_{,}O'}$	).())=).)	$\mathrm{O_{,}A_{,}O_{,}'}$	(.())=(.)		
		$\mathrm{E_{,}'A'E_{,}'}$	).(((=).(	$\mathrm{O_{,}A'O_{,}}$	(.(((=(.(		
				$\mathrm{E_{,}'I_{,}'O'}$	).(()=).)		

The cases in which the cumular system requires the above-mentioned modification of the law of inference have their capital letters in Italics. The three fundamental syllogisms give, each of them, four cases with one relaxation, five with two, and two with three. Many different rules of formation might be produced, and many analogies, but it is not necessary for my present purpose to examine them.

One however may be mentioned, as giving the mode of arrangement preferred by Sir William Hamilton. The preceding is a table of double entry, and the passage from compartment to compartment in the same column shews that all the negative syllogisms are formed from the affirmative ones by simple insertion of the negative distinction into one or the other premise and the conclusion. Thus, in the second column, at the head of the three compartments, stand))) (=) (and))).(=).(and).)) (=).(. In the cumular system, the modified modes of inference occur with certain *affirmative* propositions, which are therefore exceptional.

Looking at the preceding as one way of meeting the formal difficulties of Sir William Hamilton's system, it will easily be conceived possible that there may[*] be others, for one or more of which we are to look to its inventor, and one more of which I shall presently give myself. But, in the manner in which it has been given out, up to the present time, the defects which I have pointed out exist unanswered.

The exemplar system and that of contraries have 21 syllogisms in common. Of the remaining 15 of the first, 7 correspond to 7 of the syllogisms with both premises negative in the second. The remaining 8 of the first belong, a pair to each of the four syllogisms of the second which have both middle terms particular, one of them being the remaining syllogism with premises both negative. When a symbol taken from the exemplar system is invalid if read in the other, first try it by annexing the negative sign to each premise which is affirmative: if this do not make it valid, alter the curvature of both the middle parentheses, and then either the addition or removal of a negative symbol will make it valid.

Section V.

ON THE THEORY OF THE COPULA AND ITS CONNEXION WITH THE DOCTRINE OF FIGURE.

The analysis of the so called *simple* proposition, or judgment, shews that it is, in every one of its particulars, of a complex character. This is readily admitted in regard to the terms, and to the quantity in its connexion with them. But the copula has always been considered as the extreme both of simplicity and generality; and any attempt at the resolution of the copular relation into its elements, is likely to be misunderstood.

In algebra, as it now stands, the *forms* born and educated in arithmetic have left their parent and set up for themselves. Any meanings which obey certain specified laws may be adopted as the means of giving expression to the forms: and the results must be accepted as true in every instance in which the combinations used are consistent and intelligible under the meanings given. Thus we may have, and have, a calculus in which the symbols are in frequent cases not magnitudes at all, but directions how to operate: and though this calculus, when pushed beyond a certain point, has, for the present, its unintelligible symbols, which play the part of the impossible quantities of the older algebra, we have good reason to think that some day the victory over these will be cited as encouragement under the difficulties of a yet more advanced stage of progress.

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[*] Of course I might instance my own numerically definite system, which, as Sir William Hamilton acknowledges, contains his system, and which is not open to any charge of inconsistency. The acknowledgment was made in the assertion that my numerical system, so far as it gives expressed quantity of all kinds to the predicate, was *derived from information furnished by him*. I am still waiting for two *citations*: one from Sir William Hamilton's communication, containing information; one from my own subsequent writings, containing use made of it. I want, in fact, an exemplar proposition of the form)(: having hitherto obtained nothing but)), or ( ): I have had enough of single indefiniteness, and want an instance of the *double* character which gives singularity to the things to be compared.

In my work on *Formal Logic* (pp. 46—54) I followed the hint given by algebra, and separated the essential from the accidental characteristics of the copula, thereby shewing the conditions of invention for a copula different from the ordinary one, or for a copula which, being substituted for the ordinary one, shall leave all the forms and conditions of inference unaltered. On this a learned critic remarks that I claim the abstract copula as an improvement: adding, that some of my modes of making the copula are less abstract, none more so, than *is* and *is not*. As this remark either embodies or suggests the whole, or nearly the whole, of what would be said by a logician, technically so called, it will be worth while to dwell upon it.

By an "abstract copula" of course is meant a formal mode of joining two terms which carries no meaning, and obeys no law except such as is barely necessary to make the forms of inference follow. Any concrete or actual copula, fulfilling other conditions, is, to the extent of those conditions, less abstract. The best proof of the perception of an abstraction, is the invention of an abstract designation, which must be a technical symbol, if convenience be to dictate. The mode of denoting terms by *letters* fully shews us that the abstract *term* is arrived at: but as there is no symbol for the copula except the verb *is*, we are left to ascertain from the use of that word how the matter stands with respect to the copula.

If an abstract copular symbol had been used, the copular conditions would have been expressly laid down. They are two in number; together sufficient for all forms of inference, but not both necessary in all. The first is what I shall call *transitiveness*, symbolized in

$$X\text{——}Y\text{——}Z = X\text{—}Z\,;$$

meaning that if X stand in the relation denoted by — to Y, and Y to Z, X therefore stands in that relation to Z. Very many copulæ exist in which this transitive relation is seen; as is,—rules,—lifts,—draws,—leads to,—is superior to,—is ancestor of,—is brother of,—joins,—depends upon,—is greater than,—is equal to,—is less than,—agrees with (in a given particular), &c.

The second condition is *convertibility*, symbolized in

$$X\text{——}Y = Y\text{——}X,$$

in which the relation is its own correlation. Of those mentioned above, convertible relation is seen in—is,—is brother of,—joins (if a middle verb),—is equal to,—agrees with. As instances of the convertible without the transitive character we may take—converses with,—is in the habit of meeting,—is cousin of,—is in controversy with, &c. &c.

The connexion between the affirmative and negative copula is merely that of contrariety: in $X\text{—}Y$ and $X - - Y$ it is supposed that one or the other must be. It is not necessary that one should be the denial of the other, as I may say, over the whole universe of thought: it is enough if within the universe of the syllogism, either X—Y or X – – Y must be true. Thus 'agrees in colour,' 'agrees in size,' may be alternative copulæ in a universe the alternative of which is that any X is either of the colour or of the size of some Y.

When contraries are introduced, the copular condition further required is that either X—Y or X——y should hold for any X.

The least abstract of all copulæ is the *is* and *is not* of the logicians, when employed, whether subjectively or objectively, in the sense of identity. But logicians rarely confine it to that sense. It takes many meanings in their modes[*] of speaking and in their examples: and perhaps may be stated as generally signifying agreement in some understood and, *pro vice*, unvarying particular. I have never seen it carried further, formally at least: and I feel safe in affirming that *is*,—used as merely satisfying the transitive and convertible conditions, with a consequent right, for example, to treat it as standing for 'is brother to,'—is a more abstract copula than the *is* of the logicians.

I should almost presume this from the silence of the writers who distinguish between reading the *extension* and reading the *intension* of a proposition, as to any other change of the copula being possible. 'Every man is an animal' is read either in extension, 'All man is animal,' or in intension, as by saying that the notion of animal forms part of the notion of man; every attribute of an animal is an attribute of man. Without at all denying the existence of the distinction, or its *metaphysical* value, I feel convinced that it is not another reading of the proposition, but another proposition, derived *inferentially*, though not *syllogistically*, by aid of the *dictum de majore et minore*. For a term used universally we may substitute a lesser term; and the attribute of *all* animals is the attribute of all *men*. The proposition only says that man is the lesser term; and intensive reading is thus derived from the *dictum*. Invent a substantive, *A*, such that *ex vi termini*, the '*A* of an animal,' is one in which the word *animal* must be construed universally; and then we may affirm that '*A* of an animal' is '*A* of a man.' And though, undoubtedly, the word *attribute*, used metaphysically, will be the one most often employed in place of *A*, yet the formal process of reading by intension will be only a case of that principle on which we say that 'Englishman who does not use tobacco' is included in 'European who does not take snuff,' obtained by enlargement of the particular and diminution of the universal.

Formally speaking, the change from extension to intension is as follows; and it will be seen that a new distinction is introduced, and further, that the two modes of reading are not convertible; the extensive mode gives the intensive, but not *vice versa* in all cases.

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[*] But not in their formal definitions and enunciations. The *substantive verb* is the most common definition. The Port Royal Logic, which I choose from those lying before me, as a work which dares to differ, or to revive, at pleasure, makes its definition very distinct. "Non sufficit hos duos terminos percepisse; sed debent mente vel conjungi vel separari. Et hæc mentis Operatio in Propositione notatur verbo, *Est*...." To take two extremes of the part of time in which scholastic logic and printing flourished together, we have *Paulus Venetus* (1474), "Copula semper est verbum substantivum scilicet sum, es, est,"—and *Crackanthorp* (4th ed. 1677), "Et hæc [*sc.* copula] est in omni Propositione Verbum hoc Substantivum, *Sum*, *es*, *fui*, aut aliqua persona illius."

Extension given.	*Intension deduced.*
Every X is Y.	Every *existing* part of *all* Ys is an *existing* part of *all* Xs.
No X is Y.	No *sufficient*[*] part of any Y is an *existing* part of any X.
Some Xs are Ys.	Every *existing* part of all Ys is an *existing* part of some Xs.
Some Xs are not Ys.	No *sufficient* part of any Y is an *existing* part of some Xs.

Intension given.	*Extension deduced.*
A *sufficient* part of some one Y is an existing part of every X.	Every X is Y.
An existing part of any Y is not an existing part of any X.	No X is Y.
A sufficient part of some Ys is an existing part of some Xs.	Some Xs are Ys.
An existing part of any Y is not an existing part of some Xs.	Some Xs are not Ys.

By treating the predicate itself as one attributive notion, the modes of reading both kinds of propositions intensively may be assimilated: but not without losing sight of an important distinction. In the affirmative, any portion of the intension of the predicate may be affirmed of the subject; in the negative, it is not true that *any* portion of the intension of the predicate may be denied of the subject. Thus 'no planet moves in a circle' gives us a right to deny any *constitutive* attribute of circular motion to that of a planet, but not *any* attribute; not, for instance, the progression through every longitude.

Leaving the distinction of extensive and intensive reading entirely out of view, as not connected with the theory of the copula, I proceed to inquire into the connexion of the doctrine of figure with that of the copula; a connexion which I take to be the most important part of the former doctrine. And first, I take the system from which contraries are excluded. Looking at the two copular conditions, the transitive and the convertible, I ask what inferences hold good when *only* the transitive condition holds: for this one must hold, as long as there is but one species of copula; we shall presently see that it is the *condition of permanence* of the copula. As a representative of a transitive but inconvertible copula, let us take the verb *give*, assuming that he gives who gives that which gives. Then 'X gives Y, Y gives Z, therefore X gives Z' is legitimate, being the mere expression of the transitive hypothesis. And the opponent forms of this must be legitimate. Thus 'X gives Y, X does not give Z' leads to Y does not give Z: for if Y gave Z, X giving Y, X would give Z: this is in the third figure. Again, 'X does not give Z, Y gives Z' leads to 'X does not give Y:' for if X gave Y, Y giving Z, X would give Z: this is in the second figure. When these propositions are made cumulative, with quantities proper for inference, the usual inferences are obtained. Accordingly, + meaning affirmative, and – negative, all + + syllogisms in the first figure, + – syllogisms in the second figure, and – + syllogisms in the third figure (reading premises in the Aristotelian order) do not need a convertible copula. They are

I.	II.	III.
Barbara))))	*Camestres*).(((	*Felapton* (().(
Darii ()))	*Baroko* (.(((	*Ferison* ()).(
		Bokardo (((.(

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[*] Meaning part sufficient to determine it to be Y. I do not say *attribute*, because the inference does not depend upon the metaphysical meaning of that word.

Turning to Sir William Hamilton's system, I find that his peculiar syllogisms will not follow this rule: but in the *exemplar* system it is preserved throughout. The fourth figure is incapable of inference, as long as one *inconvertible* copula, and that only, is used.

The copula being inconvertible, we may complete the modes of inference by allowing the correlative copula; as ‘X gives Y’ ‘Y is given by X’. That is, whenever conversion is necessary to turn the syllogism into + + of the first figure, + − of the second, or − + of the third, the correlative copula must somewhere be introduced.

The following addition to the poetry of logic may be euphonized by any one to his own taste: the letter *g*, following a vowel, denotes that the premise (or conclusion) denoted by that vowel takes the correlative copula.

Barbara, Celagrent, Darii, Ferigoque prioris
Cesareg, Camestres, Festinog, Baroko secundæ
Tertia Darapgi, Disagmis, Datigsi, Felapton
Bokardo Ferison habet. Quarta insuper addit
Bramantigp, Camegnes, Dimarigs, Fegsapo, Fregsison.

This means, for example, that *Fesapo* may be read in the *fourth* figure under the additional condition seen in *Fegsapo*, namely, that the copula of the first premise is to be correlative of that used in the second premise and conclusion: and Darapgi in the third, if the second copula used be correlative of the other two. Thus ‘all the piles supported arches; all the piles were supported on gravel, therefore gravel did then support arches’ is a good syllogism, and not capable of reduction to an Aristotelian syllogism. The proposition ‘support of support is support’ is necessary to the inference, which inference can only be obtained from a sorites, and not then except by help of the *dictum* already quoted. It may be said that this is *more* than an Aristotelian syllogism; I maintain it to be *less*, *if either*. The outstanding copular relation (always implied) of an Aristotelian syllogism is ‘*is that which is*, gives *is*:’ of the preceding case, ‘support of support is support.’ The former demands transitive and convertible meaning for *is*, or that *is* shall be its own correlative: the latter demands transitive meaning for *support*, and the allowance of its correlative.

The following table shews the way in which any combination of premises is to be read in any figure, it being presumed, as in Sir William Hamilton's system, that every syllogism may be read in any figure.

I.	+ + +	(+) − −	− (+) −
II.	(+)+ +	+ − −	− + (−)
III.	+(+)+	+ − (−)	− + −
IV.	++(+)	+ (−) −	(−) + −

These cases may be remembered as follows. Let + +, + −, − +, and − −, be called the *primitive forms* of the four figures: the fourth figure taking its root in an inconclusive form. In every primitive form the correlative copula need not appear. When *one* premise of a primitive form is altered, the necessity of a correlative copula is thrown upon *the other*: when *both*, upon the *conclusion*.

The principles laid down in Section I. enable us to make a still further enlargement. The logician, followed by the algebraist, has restricted himself to one copula: the former uses *is*, the latter $=$; and both are used in some variety of sense. The algebraist, indeed, sometimes goes a little further, and introduces the correlatives $>$ and $<$, which might be generalized for the purposes of logic into symbols of correlative copulæ in general. In every kind of logic, formal and applied, the transitive copula is insisted on. This however is not necessary: inference may be seen without it. The perception of relations by means of relations does not require us to use only one relation. If I can see that

Every X has a relation to some Y
and Every Y has a relation to some Z,

it follows that every X has a compound relation to some Z. Be the premised relations what they may, there is a concluding relation, which may or may not be expressible by one word. Thus if John can persuade Thomas, and Thomas can command William, we cannot infer that John can either persuade or command William: but if we express by one word the process of gaining an end by persuading one who can command—say we choose to use the word *control** for this purpose—then John can control William. We have then a bicopular syllogism, in which the intransitiveness of the individual copulæ is supplied by the invention of a compound copula for the conclusion. This is the step by which we ascend to the general theory of the copula: but those who proceed from a general copula to a particular one, will merely see how to read the general conditions in the particular case.

Can a bicopular syllogism be reduced to a compound process of unicopular syllogism? In the case before us the conditions are

Postulate. *Control* includes the influence exerted over the governed by one who can persuade the governor.

Premises. John can persuade Thomas.
Thomas can command William.

Conclusion. John can control William.

The only way in which this can be reduced to unicopular syllogism is by the following sorites.

Given premise. John is {one who can persuade Thomas}.

Postulate. {One who can persuade Thomas} is {one who can control all whom Thomas commands.}

Second given premise, and dict. de maj. et min. $\left\{\begin{array}{c}\text{One who can control all whom}\\ \text{Thomas commands.}\end{array}\right\} \text{ is } \left\{\begin{array}{c}\text{One who can}\\ \text{control William}\end{array}\right\}.$

Conclusion. John is {one who can control William}.

The algebraical equation $y = \phi x$ has the copula $=$, relatively to y and ϕx: but relatively to y and x the copula is $= \phi$. This is precisely the distinction of 'John can persuade

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* This is not an improper word; for the compound relation, as well as the original ones, enters in a particular, not a universal, manner. It is enough that persuading one who can command is *one* of the ways of controlling. And let there be any number of forms of persuasion, and of command, it is enough that one of the forms of persuasion procuring one of the forms of command, should be one form of control. And thus it may happen that, legitimately, the copular word of the conclusion may be that of the premise in a different sense; a very common thing in the logic of common life. But in the negative copula, the copular word must be universal.

Thomas' and 'John is {one who can persuade Thomas}.' The deduction of $y = \phi\psi z$ from $y = \phi x$, $x = \psi z$ is the formation of the composite copula $= \phi\psi$. And thus may be seen the analogy by which the instrumental part of inference may be described as *the elimination of a term by composition*[*] *of relations.* For though in ordinary inference the concluding copula is usually identical with those premised, yet it is no less true that the composition must have taken place: *X* is *Y*, *Y* is *Z*, therefore *X* is that which is (= is) *Z*.

Upon two given relations, there is but one species of affirmative syllogism, and two species of opposing negative syllogisms; that is, if the alternative relation be merely negative, and no correlative be permitted. If —— and $\ldots\ldots$ be the copular symbols, and $\overline{\ldots\ldots}$ that of the compound relation; and if — — and $\ldots\;\ldots$ and $\overline{\ldots}\;\overline{\ldots}$ represent the negatives of the first relations, the three syllogisms are

First figure.	Second figure.	Third figure.
$\begin{cases} Y \text{ —— } Z \\ X \ldots\ldots Y \end{cases}$	$\begin{cases} Z \text{ —— } Y \\ X \,\overline{\ldots}\;\overline{\ldots}\, Y \end{cases}$	$\begin{cases} Y \,\overline{\ldots}\;\overline{\ldots}\, Z \\ Y \ldots\ldots X \end{cases}$
$X \,\overline{\ldots\ldots}\, Z$	$X \ldots\;\ldots Z$	$X \text{ — — } Z$

The terms are here singular: if we make them cumulative, we introduce the usual laws of quantity. And the three figures shew their primitive forms + +, + −, − +. If the correlative copulæ be introduced, the preceding theory of figure may be extended to the bicopular syllogism. And we now see as follows;—

First, the fourth figure is dependent for its existence upon the entrance of the *copular correlative.* I have always looked with surprise upon the arguments for and against the fourth figure. If the inferences therein made be good inferences—if the man who can establish the premises, *does* establish the conclusion, of *Dimaris* or *Camenes*, how can it be said that the fourth figure is not to be used? *there it is*, and it cannot be reasoned out of existence. But in the above system, the fourth figure yields no *true inference* until the copular correlative is allowed: this may be excluded from *data* by hypothesis, but on what principle of reason can true consequences of data be excluded by hypothesis, and what is the correct English of the word hypothesis, in such case? There ought to be no bounds to any science, except such as are determined by its data.

Secondly, it appears that the affirmative syllogism gains its conclusion by composition, the negative by resolution: *the negative premise has a compound copular relation, which is to be resolved.* We see this plainly when *different* relations are compounded: just as a beginner in algebra sees properties of $a \times b$ which would be much clouded by taking his initial examples from $a \times a$. Indeed, the forms of logic, stinted down to the Aristotelian syllogism, much resemble those of an algebra in which all the letters are equal, and all equal to unity. For example, taking our previous instance of composition, from '*Z* can command *Y*, *X* cannot control *Y*' we deduce *X* cannot persuade *Z*. And we must deduce it by resolving control into persuasion to command; and saying 'X cannot persuade any one who can command Y' or X cannot persuade Z.

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[*] Or *composition and resolution*, but these two words never finally mean more than one. Composition of an opponent is resolution. Whether the words shall both occur or not depends upon whether we begin on a narrow or a wide basis. Arithmeticians begin with *multiplication* and *division*, and end by declaring them identical. The old geometers began with *composition* of ratios, and never arrived at *resolution*, for they had all resolutions in their compositions.

But, it may be said, this cannot be shewn on the ordinary syllogism 'Z is Y, X is not Y therefore X is not Z;' nor on any syllogism in which the negative premise is not expressly composite. This objection I proceed to examine.

When the copula *is* means pure identity, as in 'that horse *is* that horse' (in which case I shall write it in capitals) I own that the resolution is so purely formal, and the reality represented under the compound form so elusive of our attempts to mean something more than was meant by the simple form, that I should compare the resolution to that of 1 into 1 × 1 in arithmetic. And though this arithmetical resolution be often useful, and even requisite, as a preliminary to clear notion of some higher development of thought, yet the logical type of it might well wait until the time comes when the same may be said of *it*. "That horse 'IS that which IS' that horse" may therefore be abandoned to the objector. But when, as more frequently happens, the copula *is* only denotes some agreement, or something[*] transitive and convertible which is not pure identity, I maintain the actual resolution to be part of the process of inference. Let us take a material instance, which may represent anything objective, as will be admitted: and I think that the process may find its analogue in purely subjective matters. Let *is* represent agreement in *colour*. Then it will be instantly admitted that 'Z is Y' and 'Z is that which is Y' are two different propositions. The question is, how do we infer from 'Z is Y, X is not Y.' Our second premise is X has not 'the colour which Y has:' and for the last words we substitute, as an *identity*, 'the colour which Z has,' the identity being purely subjective. Here the resolution was made: our second premise became 'X is not that which Y is'.

Logical writers do not collect their copulæ: and their processes seem reducible to common form in no way but this. They postulate that the copulæ they use (which they take good care shall be transitive and convertible) have the inferential force of the identity IS: they examine and dwell upon syllogisms of identity, thereby giving the opponents of logic some reason for their scoffs at the syllogism, and return with their results to other copulæ, by aid of the postulate. To this there can be no objection as to the truth of its results, but much in every other way. This subject however is too wide a one; and requires the preliminary consideration of the manner in which the *term* has been used, and a disentanglement of the confusion of ideas under which logicians speak, in one sense, of man as a part[†] of the notion animal, and in another of animal as part of the notion man. The "integrate or mathematical whole" which, according to Sir William Hamilton, "the philosophers contemned," and which is seen in 'All men are animals', must take its proper place: and "philosophy", which "tends always to the universal and the necessary" must be taught that the *universal* belongs to the mathematical whole, as surely as the *necessary* to the metaphysical whole.

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[*] In the *IS* of identity, the transitive and convertible relations are a kind of zero forms. There is not much transition in travelling from York to York and from thence to York: nor much to see, in spite of the Italics, in the proposition that from York to *York* is the same distance as from *York* to York.

[†] In all that relates to quantity, there is no occasion for the mathematical logician to pay the least deference to the Christian followers of Aristotle; the master himself was a mathematician, as were some of his Greek successors. Those who could contentedly put *number* and *speech* together, as discrete *quantity*, in opposition to *length*, as continuous, do not deserve to be listened to, as quantitarians. But this it will be said is in the writings of the master. It is so; but the question is, whether it be not an obvious interpolation. The case stands thus: In the sixth chapter of the Categories, quantity is made discrete or continuous; the discrete is *ἀριθμὸς καὶ λόγος*, the continuous is length, surface, &c. Farther on, *λόγος* is changed into *φωνή—λέγω δὲ αὐτὸν τὸν μετὰ φωνῆς λόγον γινόμενον*—and the reason given is that it is *measured* by long and short syllables. On this I say first, that Aristotle would probably have seen what his determined disciple Crackanthorp afterwards saw, that the *time of speaking* is not *speech*; secondly, that it is incredible that he should have used a wrong word and have afterwards corrected himself by a synonyme instead of an erasure; thirdly, that *ἀριθμὸς καὶ λόγος*, in the language of the time, are integer number, and fractional ratio or ratio of

The admission of relation in general, and of the composition of relation, tends to throw light upon the difference between the invented syllogism of the logicians and the natural syllogism of the external world. The logician, tied to a verb of identity, from which if he wander it is never quite out of sight, is bound to subject and predicate of the same class; objective both, or subjective both. He cannot say the rose *is* red, for his *is* would require the inference that some red is the rose. He has nothing but a method of reducing his predicate to an object: the rose is a red thing; some red thing is a rose. The common man uses a copula which ties the object up in relation to a more subjective predicate; not reading inversely by intension, not dwelling on redness as an attribute of the rose, but directly by extension, thinking of the family *rose* as his external object, and the sensation red as one condition under which it appears to his senses. Again, an ordinary person says that the rose is red, and red is pretty, so that the rose is pretty. The logician's pupil, when not far advanced, will interpret him as saying that 'The rose is a red thing; a red thing is a pretty thing, &c.' thus committing him to an opinion upon red cabbage which perhaps was not within his meaning. The logician himself will substitute another process: 'The colour of the rose is red colour; red colour is pretty colour; therefore the colour of the rose is pretty colour:' he thus reduces it to the equation of two terms by comparison with a third; when most probably the real form of thought (and logic professes to be the study of existing forms) is nothing but the composition of relations. The rose *is* (in its relation to one sense) red; red *is* (in the relation of the perceptions to the judgment) pretty; therefore the rose is (to one sense that which is to the judgment) pretty. This distinction of copulæ is more nearly present to the mind than any one of the transformations by which the technical form is arrived at: it is visible, more or less distinctly, that the first premise is matter of *fact*, the second matter of *taste*; here is a recognition of difference between *is* of the first premise and *is* of the second; which leads, with proportionate strength, to the recognition of the composite character of *is* in the conclusion. Nor does the logician's form avoid it, except by a pure assumption that the copulæ are the same; an assumption not true in fact. Nor can an answer be given by saying that logic takes no cognizance of the matter, but only of the form; for it is bound, by the laws of form, not to represent differences by agreements, except when it is formally shown that the differences cannot affect the object to be gained.

In one very material consideration, the logicians have evidenced the manner in which their

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commensurables. An editor who was not a mathematician, seeing λόγοι, which to him was only communication in its most general sense, may have excogitated the absurdity above alluded to, by help of φωνῇ. Proclus has been used in a similar way. Barocius makes him inform us that the predecessors of Euclid had written on inexplicables (*quæ non explicari possunt*): Proclus says ἄλογα, and means incommensurables.

But how can λόγοι be called discrete? Before Euclid it must have been so. Euclid's definition, which brings incommensurables under the same law with commensurables, gives the term ratio a title to continuity, which it had not before: just as in modern times, the interminable decimal, when admitted as a distinct conception of arithmetic, gives continuity to *number*.

copula '*is*' partakes of the verb of identity on which, as above remarked, they test all their inferences. When the copula of identity is employed, it is of necessity that one subject example can only agree with one predicate example: any one man IS, merely because he IS *one* man, only *one* animal. Extend the copula to a meaning short of perfect identity, and the copular relation may belong, in an affirmative, to more than one example of the predicate. For instance, when the copula is agreement in colour, 'Every X is Y' may mean that each X is of the colour of one or more of the Ys. The particular character of the predicate, that is, the description of its extent as being 'one or more, possibly all' may belong to it, independently of the number of instances of the subject, in right of, and by derivation from, *one* instance of the subject. And the proposition 'one X——Every Y' may be true without our being obliged to say that there is therefore only one Y in existence, by reason of the relation indicated by——existing between that one X and every Y in existence.

Hence follows an extended mode of interpreting Sir William Hamilton's system into consistency. Let the quantity of the predicate be determined by the least number of instances of the predicate which are declared to stand in the copular relation to each instance of the subject: it being understood that the declaration of the copular relation existing between a certain X and a certain Y does not deny that the relation may exist between that X and other Ys, or between other Xs and that Y. Thus, "All Xs——all Ys," or $X)(Y$, now means that every X stands in the copular relation to each and every Y. If there were ten Xs and ten Ys, Sir William Hamilton's proposition would assert ten agreements: the form here proposed declares a hundred. And it is now contradicted by $X(.)Y$ or "some Xs – – some Ys:" for either each X and each Y agree, or some Xs do not agree with some of the Ys.

This view of the subject meets another objection of mine to the system as proposed. It can no longer be asserted that $X)(Y$ is a proposition compounded of two others of the system, $X))Y$ and $Y))X$: for each X may agree with some of the Ys, and each Y with some of the Xs, without every X agreeing with every Y.

There is however a distinction to be drawn. When the quantity of the affirmative predicate is determined solely by the number of instances which *each* instance of the subject is related to, let us say that the predicate has *exemplar quantity.* But in the common system, in which the quantity of the predicate is cumulated from all the instances of the subject, let us say that it has *cumular quantity:* this second system permitting, like the first, each instance of the subject to be related to more than one instance of the predicate. Thus, quantity being exemplar, we must say of 'Every man is an animal' that the predicate is singular: quantity being cumular, that the predicate is arithmetically coextensive with the subject. But when it is "Every man—agrees with in being carnivorous—a brute," the predicate is not singular, even when its quantity is exemplar.

[An extension of the numerically definite system here arises, which I shall not stop to investigate the consequences of. If ξ be the whole number of Xs in the universe, η the number of Ys, and n the number of Ys (at least) to which each of m Xs is related; them mn is the least number of relations asserted. If mn be greater than $(\xi - 1)\eta$, then $mn - (\xi - 1)\eta$ at least of the η Ys must be related to ξ Xs, that is, to all the Xs. Now as n cannot exceed η, m must be ξ: and a necessarily convertible proposition may then exist, if n exceed $\eta - \eta \div \xi$.]

By merely allowing a complete exemplar form of quantification we procure the extension of Sir W. Hamilton's system which I have described in the last section: we now annex to this admission, a predicate quantity of more than one example.

And the forms with their meanings are as follows, each universal with its particular contradiction.

$X)(Y$	Each X is related to all the Ys.
$X(.)Y$	Some Xs are not related to some of the Ys.
$X))Y$	Each X is related to one or more Ys.
$X(.(Y$	Some Xs are not related to any Ys.
$X((Y$	Some Xs are (among them) related to all the Ys.
$X).)Y$	No X is related to some one or more Ys.
$X).(Y$	No X is related to any one Y.
$X()Y$	Some Xs are related to some one or more Ys.

This system as it stands, is free from all objections which I can raise. Every proposition has its contradictory, the form (.) is not generally spurious, no numerical dictum is established[*] by the mere assertion of a proposition. The symbolic law of inference does not present any modified cases; and, transitive copular relation existing, though inconvertibly, the use of the correlative relation, in the places previously marked out, will allow of inference. The manner in which the introduction of more general copular relation allows of more general inference, and the alterations in the import of the forms)(and (.), with the value of those alterations in the expression of the forms of thought, would admit of considerable discussion, if the length to which this paper has attained did not forbid.

Taking the predicate as both exemplar and cumulative in quantity, the copular relation as both transitive and convertible, we see a system which differs from that of Sir William Hamilton only in the admission of a wider alternative of copular agreement: instead of 'one Y or no Y at all' it is 'one or more Ys or no Y at all.' Circumstances require me, I think, to point out my reasons for concluding that this extension of alternatives has *not* been made by Sir William Hamilton. They are first, his description of his own use of quantity, made in terms[†] so technical, that it is impossible to suppose he varies from the ancient *modes* of quantity, in extending the *application* of them: secondly, that the author of the "Outlines," an accredited expounder of the details of this system, not only does not give the smallest hint to the contrary, but adopts the conclusions[‡] suggested by the old modes, as "of course."

I shall now proceed to the application of correlative copulæ to the system of predication by contraries. I insert a slight and preliminary notice of this extension more as an example of the study of extension than because of its representation of any very common modes of thought. At the same time it should be remembered that we cannot appeal to existing modes, and to their agreement with the Aristotelian forms, as conclusive evidence of the natural character of those forms, or of any restriction which results from their use. For many centuries, and in particular during those centuries which saw the relations of the modern and ancient languages finally settled, every man who learnt to read was alphabetically drilled in the Aristotelian syllogism. If technical forms could produce such an effect as either to supplant more natural ones, or to prevent the natural growth of the most natural ones, the Aristotelian forms have had every advantage given them, and may have done it. Consequently, no modes of thinking which are not easily reconcileable with the Aristotelian forms ought to be suppressed because they do not seem to have any usage in their favour.

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[*] If indeed we were to declare that the number of correlated instances should never exceed 10, then)(would imply that there are not more than 10 Xs nor more than 10 Ys. It is the refusal to admit of more than one instance of each in relation which reduces)(to double singularity in the exemplar form first discussed.

[†] "The *first* scheme is that which logically—confines all expressed quantity to the *Subject*, presuming the *Predicate* to be taken—in *negative* propositions, always determinately in its *greatest* and *least* extension (universally and singularly), in *affirmative* propositions, always indeterminately in *some part* of its extension (particularly). The *second* scheme is that which logically—extends the expression of quantity to *both* the propositional terms, and allows the *Predicate* to be of *any quantity*, in propositions of *either quality*....The first doctrine is the common or Aristotelic; the second is mine;..." "Letter, &c." subsequent to my "Statement," pp. 31, 32.

[‡] Mr Thomson agrees with me as to the *spuriousness* of "Some Xs are not some Ys" in Sir Wm. Hamilton's system, except "of course" as a denial of singularity and identity: "Except of course they represent individuals." "Outlines, &c." pp. 188, 189.

The copula to be extended is the word *is*, in its widest sense: the extended copula may be said to be *has relation to*, the relation being expressed or understood, but distinctly conceived as *transitive*. The correlative copula may be expressed by *has correlation to:* and it is demonstrable, or is rather seen to accompany than to follow, that all transitive relations have transitive correlatives. As a short mode of speaking, we may say that our relations are *gives to* and *receives from*. The terms *active* and *passive* might be generalized into words of description for the correlatives; and *middle* into a word of description for relations which are their own correlatives.

In the common system, if a certain X be a certain Y, that Y is that same X: but the correlative (then called converted) relation need not be much dwelt upon. Of the two 'X is Y' and 'Y is X,' either may be called direct, and the other inverted. In the extension, each instance may stand both in direct and inverted relation to something: so that every X gives something, every Y gives something; every X receives from something, every Y receives from something. And it must be that no X gives to nor receives from, any contrary, x, either directly or by transition. If, for instance, a certain X give to one or more Ys, no one of those Ys can give to any x: but as each of those Ys must *give*, and either to Xs or xs, it must be that they give to Xs, but not necessarily to those from which they received. Nor must the same X give to both Y and y, &c.

We have then for any one X either $X_{,}Y'$ and $Y_{,}X'$ or $X_{,}y'$ and $y_{,}X'$: meaning that either an X selected gives to Ys, and each of those Ys to Xs; or to ys, and each of those ys to Xs. And the subscript and superscript accents may be applied to the parentheses: thus $X_{,}))'Y$ may stand for 'every X gives to one or more Ys.'

[Accordingly[*], if we adopt this hypothesis, namely, that a direct and inverted relation always exists, it follows that all syllogisms are valid, put what marks of relation we may. For $X_{,}))'Y$ never exists without $X'))_{,}Y$: and the same of all the other forms. And $X_{,}))'Y$ has $X_{,}).('y$ and $y_{,}))'x$ and $x_{,}(.)'Y$ among its equivalents. From this most complete of all possible assignments of relation, we may descend by two steps.

First, we may dismiss the condition that every existing relation has its correlative: and then we may ask under what relations each syllogism is valid, there being one existing between each of the pairs XY, YZ, and one (inferred) between XZ. Here each relation is supposed to extend over the terms and their contraries: thus from $X_{,}))'Y$ we are to infer $X_{,}).('y$, and may use it. Secondly, we may dismiss the condition that the relation extends over the contrary, and then we arrive at the modification of the common or Aristotelian syllogism, already discussed: in this case, the inference can be obtained without any use of the contrary, or extension of the relation.

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[*] The paragraphs between [] are substituted for those originally given: their date is July 1, 1850.

I now give the heads of the results which arise from the hypothesis that one of the correlations is assigned or inferred, and the other not admitted. The equivalent propositions are those of the ordinary system, with the relations unaltered: thus

$$X_{,}))'Y = X_{,}).('y = x_{,}(('y = x_{,}(.)'Y$$
$$X_{,}()'Y = X_{,}(.('y = x_{,})('y = x_{,}).)'Y,$$

Here $x_{,}(.)'Y$ must be read as follows; in every instance, either x gives, or Y receives, or both: and $x_{,})('y$ means that there are some instances in which neither x gives, nor Y receives.

These equivalences may be easily proved by those who remember the meaning of the relative symbol. The assertion $X_{,}))'Y$ is that of a relation in which every X and every x is a giver, and every Y and every y is a receiver; but so that no Y nor y can receive both from a X and from a x, no X nor x can give both to a Y and to a y. Accordingly, $x_{,}(('y$ immediately follows: for every y receives either from a x or from a X, but not from the latter, because then a Y and a y would both receive from that X.

The law of entrance of the relations is as follows. When the syllogism is read in an Aristotelian figure (the first or the third) which begins with the middle term, all that is requisite is that the major and concluding relation should be the same: when in a figure (the second or the fourth) which does not begin with the middle term, all that is requisite is that the major and concluding relation should be correlatives. Each syllogism therefore can be read in two ways, according as the major and minor relations are the same or correlative.

For example, the syllogism $A_{,}A'I'$, or $))((=)($, read in the first figure, is

$$Y((Z+X))Y = X)(Z,$$

which may be either

$$Y_{,}(('Z+X_{,}))'Y = X_{,})('Z, \quad \text{or} \quad Y_{,}(('Z+X'))_{,}Y = X_{,})('Z.$$

To prove the first, remark that since *every* X supplies a Y, there are but the xs to supply the remaining Ys and all the ys. But since the Zs are all supplied by Ys, all the ys supply zs, being themselves supplied by xs. Hence, by the transitive character of the relation, we have $x_{,}()'z$, or $X_{,})('Z$. To prove the second, observe that every y supplies a x, and if any such x supply a Z, the y which supplies that x also supplies that Z; but every Z is supplied by a Y. Consequently $x_{,}()'z$, or $X_{,})('Z$, as before.

In each syllogism, one or other of its two cases must involve the direct entrance of contraries: so that the Aristotelian form never allows two cases. For instance,

$$Y_{,}))'Z + X_{,}))'Y = X_{,}))'Z$$

is obviously true without consideration of contraries; every X supplies a Y, and every Y a Z. But $Y_{,}))'Z + X'))_{,}Y = X_{,}))'Z$ must be proved thus. If any X supplied a z (here the contrary enters), a Y would supply that z, and would therefore supply both a Z and a z.

The remarkably simple modification which occurs in the Aristotelian forms depends upon those forms being confined to particular predicates in affirmatives, and universal ones in negatives. When all relations have their correlatives, the reading by figure is a matter of indifference; when the two correlatives are not allowed *in the same proposition*, the figures are grouped in pairs, the first and third, the second and fourth; when no term is allowed its alternative, or contrary, the four figures become distinct in their properties.

The primitive forms of the four figures being + +, + –, – +, – –, the mode of reading by thickened parentheses may be connected with them as follows:

$$\text{I. }\ \overset{+}{\pmb{|}}\ \overset{-}{|}\ \overset{+}{\pmb{|}}\ \overset{-}{|} \qquad \text{II. }\ \overset{+}{\pmb{|}}\ \overset{-}{|}\ \overset{+}{|}\ \overset{-}{\pmb{|}} \qquad \text{III. }\ \overset{+}{|}\ \overset{-}{\pmb{|}}\ \overset{+}{\pmb{|}}\ \overset{-}{|} \qquad \text{IV. }\ \overset{+}{|}\ \overset{-}{\pmb{|}}\ \overset{+}{|}\ \overset{-}{\pmb{|}}$$

in which, over the thick lines in I, occur + +; in II, + –; in III, – +; in IV, – –.

All these relations may, no doubt, be connected; but a general demonstration of the law of relations, when two correlatives are not admitted in one proposition, would be requisite: this demonstration is very easy.

In my written communication to the Society, having seen that the + +, + –, and – + syllogisms of the first three figures required no correlative, I concluded that, on investigation, the – – syllogisms of the fourth figure would require none. And as this turned out to be true, I looked no further, at the time. But I afterwards found that, though this be all correct, the entrance of the contraries under which alone a – – syllogism gains existence, gives rise to the validity of *every* syllogism without any correlative.]

When an intermediate relation exists, which is equally related to both the correlatives, as in the case of *greater*, *equal*, and *less*, that intermediate relation may be employed for either. The effect, if any, upon the conclusion, is easily connected with the law by which the strengthening or weakening one of the premises produces its effect on the conclusion. But the length of this paper compels me to omit the detail of this and other points. The combination of the system of invention of predicates with that of contraries remains for consideration.

SECTION VI.*

APPLICATION OF THE THEORY OF PROBABILITIES TO SOME POINTS CONNECTED WITH TESTIMONY.

EVERY application of the numerical theory of probabilities requires and presupposes an hypothesis on the cases enumerated in the problem: they are, or they are not, equally probable. If not, the assignment of their several probabilities is, as it were, equivalent to a further subdivision of the cases, each of which is made to consist of several: all the individuals of the ultimate resolution being supposed equally probable. This last arrangement I call the *primary distribution.*

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* This section was forwarded as a separate communication, and was dated Nov. 19, 1849.

Not only do a great many acknowledged errors arise from mistaken modes of making this primary distribution: but it is a fair matter of inquiry whether diversity of method, without error, may not be forced upon the mind by its own legitimate act.

A little consideration will shew, with regard to the mathematical part of the theory of probabilities, that the primary distribution is *out of the subject*, as much as the matter of a premise is out of the subject in logic, or the material substance of a solid out of the subject in geometry. Numerical application may be made to a false distribution, as well as to a true one. The well-known mistake once[*] made by D'Alembert, was one of primary distribution: it could not be in the power of a mathematician, as such, to convince him of his error. The replies of Lacroix and Laplace both amount to nothing more than a perfectly correct denial of D'Alembert's primary distribution, and the proposal of another.

The primary distribution is a mental act. It matters nothing that the circumstances of the problem appear to dictate it. When it is stated that an urn contains 100 white and 100 black balls, and that therefore there is an even chance of drawing a white ball, it is the want of sufficient reason for any other allotment which produces a provisional assent. Experience may shew sufficient reason, and may dictate a different distribution. Thus, should it turn out that 2000 drawings produce 1800 black balls, that circumstance alone would demand the change. Both distributions may be true: that is, true exponents of the rational result of the existing knowledge of the party whose mind is addressed, at two different times.

I was led to consider the following question;—What is the primary distribution of the mind in regard to a proposition and its contradiction, antecedently to the production of any evidence in favour of either. In the writings of logicians, although no formal exposition of their ideas upon probability is made, I thought I had detected a leaning to the notion that 'Every X is Y,' and 'some Xs are not Ys,' are *à priori* of equal probability. And by *à priori*, I mean antecedently to the production of the specific subject and predicate. Say that opposite to X and Y are to be written at hazard, by two persons selected at hazard, and not in communication, the verbal descriptions of two objects of thought. Which is most likely to turn out true, that every X is Y, or that some Xs are not Ys? We should pronounce without hesitation in favour of the latter, and should even say perhaps, that its extreme case, no X is Y, far exceeds in probability all the others put together.

Nevertheless, writers on logic, in their tacit references to authority and its effects, seem to admit that a given authority, producing a universal proposition, has his weight in no material degree lowered by the very great antecedent improbability of his statement. Their admission is quite correct, but, so far as I know, not yet explained by the theory.

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[*] I say *once made*, because, though never mentioned, it is pretty certain he saw his error before he died. The second edition of his *Opuscules* (and the first also, I believe) contains the reflexions on the Theory of Probabilities prefixed to his dissertation on the effects of inoculation for the small-pox. Herein is contained, as cited by Lacroix and others, the celebrated argument that the probabilities for *head* at or before the second toss are two to one, the three possible, (and according to D'Alembert, equally probable) cases being H, TH, and TT. But in the collection of D'Alembert's works, published by Bastien (1805, Paris, 18 vols. 8vo.) in the fourth volume of which the memoir on inoculation and its preliminaries are somewhat recast and augmented, though the paradox of the Petersburg problem and some other objections are reproduced, all allusion to the problem above described, and all objection to its ordinary solution, are struck out.

On looking at this theory, we find it appear that almost every sound process of mathematical application has results counterpart to its own among the results of the operations of unassisted thought. I should wonder at no one who suspected that a manifestation of the secrets of the brain would exhibit something more like a calculating machine for the appreciation of probabilities than any one, as matters stand, could venture to mention without ridicule, or to maintain without deserving it. Kant has gone so far as to call our mental organization, in this respect, *a weighing machine with unstamped weights.*

The verifications of theory above mentioned will lead all who feel their force to have confidence in the converse, namely, in the theory ultimately confirming every widely observed result.

Now we have before us the following phænomena, gathered from observation:—

1. That an authority, or a witness, of whose value we have a previously acquired notion, produces an effect upon us by his testimony which partly depends upon the preconceived probability of his statement: so that, whatever his general credit may be (short of imputed infallibility), his particular credit with regard to any one statement is a function of the probability of that statement, as well as of his previous character.

2. That the universal proposition is, *à priori*, very much less probable than the contrary particular: that, X and Y being terms on the connexion of which evidence has never been offered, it is *very much* (not to say *infinitely*) more probable that some Xs are not Ys than that every X is Y.

3. That an authority, stating a proposition on the connexion of the terms of which we have no previous opinion, does not produce any very marked difference in our disposition to trust him, by stating the particular rather than the universal. We receive his statement, one of us with another, with much the same reliance, whether it be 'every X is Y,' or 'some Xs are not Ys.'

The difficulty derived from the apparent incompatibility of the third phænomenon with the first two, and the necessity of admitting that the universal and its contrary particular are each made to be of an even chance *à priori*, was with me long antecedent to the explanation. This admission I imagined to have been made by the logicians: nor was I singular here. I have heard more than one person versed both in mathematics and in technical logic, express himself for the superior antecedent probability of the particular proposition, in terms which implied that he thought himself in opposition to general opinion.

An observation which contains the spirit of the true answer would be triumphantly met by many from the common maxims of the theory. Suppose a person in whose accuracy (accuracy of statement, a compound function of veracity and judgment) we have ordinary confidence, to draw a card from the pack and to announce that it is the seven of spades. We cannot conceal from ourselves that we believe he *has* made a correct statement as much as we believed that he *was* *going* to make one: and yet he has ventured an assertion against the coming truth of which it *was* 51 to 1. But a common person says at once, Why not the seven of spades as well as any other? against which the student of the theory is tempted perhaps to retort, Yes, but why the seven of spades rather than some one or another out of the 51 others?

The observation, Why not the seven of spades as well as another, is a sound one: it reminds us, that in our absence of all knowledge of motive or bias, it is as hard to believe in error having fallen exactly on the seven of spades, as it is to believe in the seven of spades having been actually drawn: if I may speak so chemically, these difficulties combine and neutralize each other, and disengage our original belief in the witness. The reply is fallacious: it rubs out the distinctive marks from the other 51 cards, and writes on each of them 'not the seven of spades' as its only exponent.

The difference of these two cases is admirably elucidated by Laplace, in two successive problems (*Th. des Prob.* 3rd edit. pp. 446—451,) but the effect of the contrast is destroyed by a strange remark. First, there are n counters, each of which is marked with a number; and a witness of veracity p and judgment r announces that n° i was drawn. The probability that it was so drawn is made to be

$$pr + \frac{(1-p)(1-r)}{n-1},$$

in which it will be observed that how great soever the number of counters, that is, how improbable soever the event announced, *à priori*, the probability which the testimony gives cannot be less than pr. In the second problem, $n-1$ balls are black and one white, and the same witness announces that the white ball has been drawn. The result is, q representing $pr+(1-p)(1-r)$, that the probability of the event is

$$\frac{q}{q+(1-q)(n-1)},$$

so that the best witness living might be incredible, if n were great enough.

On this last problem Laplace remarks as follows;—'Ainsi l'on voit comment les faits extraordinaires affaiblissent la croyance due aux témoins; le mensonge ou l'erreur devenant d'autant plus vraisemblable, que le fait attesté est plus extraordinaire en lui-même.'

Without denying all the conclusion, we may see that a comparison of these two problems shows it by no means sufficiently arrived at. If the counters were 10^{10} in number, all differently marked, the production of, say n° 5000, is just as improbable beforehand, as the production of the white ball when one only is white, and $10^{10}-1$ black. And the first case more nearly represents our mode of primary distribution than the second. If an astronomer were to tell us that he had seen in the telescope a dragon fly off the moon, we should certainly never think, *pro hac vice*, of dividing all possible events into that of a dragon flying from the moon—*and others*. From among other events, we should select and give prominence to the possibilities of a dream, a defect in the object glass, an atmospheric phænomenon, a fly in the telescope, &c. &c. &c.

The case in which error of judgment is distinguished from wilful falsehood, need hardly be entered upon, except in a manner subsequently mentioned. Let the *general credibility* of a witness, with reference to a particular event, be the measure of our previous belief that, should that event happen, he will state it to have happened, if he make any statement at all. Should his general credibility be the same with reference to each event which can happen, we have the case to which calculation is usually applied. But it is necessary to recognize the distinctions which exist in fact, between our opinion of a witness as to one event, and as to another. We may have reason to think we know beforehand that, according as A_1 or A_2 shall happen, the narrator will be a willing or unwilling witness, a sagacious or a foolish expositor.

Let the *particular credibility* of the witness be the measure of our belief in his statement, after it has been made, and we know what it is. Let A_1, A_2, $\ldots A_n$ be the names of the events any one of which may have happened, and one of which, A_k, the witness asserts to have happened. Let ν_1, ν_2, $\ldots\nu_n$ be the probabilities of these several events, before the assertion, in the mind of the receiver of the testimony. Let p_q signify, in the same mind, the previous probability that if A_q should happen, the witness will state A_p. Hence the general credibility of the witness, before any statement, is $\mu = \nu_1 1_1 + \nu_2 2_2 + \ldots + \nu_n n_n$; after statement of A_k, it is k_k. But the previous probability of his asserting A_k is $\nu_1 k_1 + \nu_2 k_2 + \ldots + \nu_n k_n$.

The particular credibility, after assertion of A_k, is thus found. Either A_k has happened, and he has announced it, of which the previous probability was $\nu_k k_k$; or A_1 (if k be not $= 1$) has happened and he announces A_k, of which the previous probability was $\nu_1 k_1$; or A_2 (k not being $= 2$) has happened, &c. &c. Hence the particular credibility of the witness is

$$P_k = \frac{\nu_k k_k}{\nu_1 k_1 + \nu_2 k_2 + \ldots\ldots + \nu_n k_n}.$$

If there be only two cases, then, if $1_1 = 2_2$, we have $\mu = 1_1$; and $1_1 + 2_1 = 1$, $1_2 + 2_2 = 1$, give $1_2 = 2_1 = 1 - \mu$. Hence the particular credibility is, for the statement A_1, $\nu_1\mu$ divided by $\nu_1\mu + (1 - \nu_1)(1 - \mu)$, which agrees with the result of Laplace's second problem, in which $\nu_1 = 1 \div n$, $\mu = q$.

The general credibility as to A_k remains unaltered by the statement, if $\nu_k = \Sigma\nu_s k_s$. That is to say, the witness is unaltered by the statement, with reference to the event stated, if the asserted event, and the assertion of that event, be *à priori* equally probable; if what he says be just as likely as that he should have said it. And according as the first is more or less likely than the second, the witness is raised or lowered by his assertion.

Again, let $\Sigma'\nu_s k_s$ stand for $\Sigma\nu_s k_s$ with $\nu_k k_k$ omitted. Then $\nu_k = \Sigma'\nu_s k_s \div (1 - k_k)$. When $k_k = \mu$, particularly when the immediate source of this equation is $1_1 = 2_2 = \ldots = n_n$, this last result is of a well known and admitted character. For $\Sigma'\nu_s k_s \div (1 - \mu)$ then represents the probability that of all *erroneous* announcements any given one shall be A_k: and we thus see that the credit of the witness is unaltered by his statement if the previous probability of the event stated be equal to the previous probability of its erroneous statement; that is, its previous probability in the minds of those who know that some error is coming.

When a witness asserts an extraordinary thing, his credit may be maintained, in spite of his assertion, by our previous feeling of the great unlikelihood of *his* making such an assertion. The strangeness of the assertion is, as it were, balanced by the strangeness of *his* making it. In the formula $\nu_k = \Sigma \nu_s k_s$, if ν_k be very small, and $k_1, k_2 \ldots k_n$ also very small, these circumstances may produce the balance mentioned, so that $P_k = k_k$.

If $1_1 = 2_2$ &c. $= \mu$, the particular credibility is

$$P_k = \frac{\nu_k \mu}{\nu_k \mu + \Sigma' \nu_s k_s}.$$

If there be no *particular bias* towards inaccuracy, then, $\Sigma' s_m$ being $1 - m_m$, s_m is

$$(1 - m_m) \div (n - 1), \text{ or } (1 - \mu) \div (n - 1),$$

except only when $s = m$. Hence, when the chance of accuracy is the same for every event, and no bias whatever towards one inaccuracy rather than another, the particular credibility, after the assertion of A_k, is ($\Sigma' \nu_s$ being $1 - \nu_k$)

$$P_k = \frac{\nu_k \mu}{\nu_k \mu + (n - 1)^{-1} (1 - \nu_k)(1 - \mu)}.$$

Here $P_k = \mu$ when $\nu_k = 1 \div n$, the *mean probability*. So that, when we know nothing of any particular bias, the particular credibility exceeds, equals, or falls short of the general credibility, according as the previous probability of the assertion exceeds, equals, or falls short of, the mean probability.

When n is very great, the preceding is very near to unity, unless $\nu_k \mu$ be of the corresponding order of smallness. For a given value of μ, the supposition of absence of particular bias will make a very bad witness almost an infallible authority. Practically, we allot more than his general credibility to the particular statements of any witness, when we see no reason to suppose a particular bias. But experiments upon extreme cases cannot be made: for in fact, the want of particular bias is almost the sufficient reason for a growth of general habit of accuracy: so that the preceding case is almost always one in which μ is not small.

In human affairs it generally happens that a great majority of the cases have probabilities of the same order of magnitude as $1 \div n$; and these are the *ordinary events*. Cases of a probability much differing from $1 \div n$ are comparatively few. Hence, when we disbelieve the dragon flying from the moon, above supposed as an assertion, it is not *because* the probability is small, for so, generally speaking, is that of an ordinary event. But the probability of the asserted event is small compared with $1 \div n$; or $n \nu_k$ is small.

Any one event which we were not expecting, and for reasons, will be such as, *à priori*, we should call improbable. And in the common run of occurrences, things improbable (but still ordinary) are happening one after another. D'Alembert pronounced the occurrence of 100 successive tosses of head to be *metaphysically* possible, and *physically* impossible. In our day, we should translate his phrases into *subjectively* possible, and *objectively* impossible; conceivable, but unattainable.

In the remarks made upon this assertion, whether with or without reference to D'Alembert, there are several different points to notice; and some matters irrelevant to my main subject must be touched upon, to clear the way for the consideration of the manner in which the question of *primary distribution* enters.

The *objective* impossibility has disappeared before thought and experience. All who either think or try are now satisfied that repetition of trials will produce any event, however rare. Buffon, in 2048 trials of the Petersburg problem, found only one case in which the first head was deferred as late as the 9th throw. In the other instance of as many trials, mentioned in my *Formal Logic*, p. 185, head was once deferred till the 16th throw. A third trial of 2048 sets was made by a gentleman in the country, who communicated the results to me: he also had one case in which head was deferred to the 16th throw. Both these cases are extraordinary: for, *à priori*, more than 22,000 trials must be undertaken, to have an even chance of seeing head deferred beyound the fifteenth throw.

It is sometimes replied that any given order of head and tail in 100 throws is just as unlikely as a hundred consecutive heads: and is therefore objectively as impossible. But this reply does not attend to the circumstance that the fulfilment of precedent conditions is the extraordinary event counted upon as impossible.

And it is to be noted that, in looking out for the way of judging the probability of what may take place in 100 throws, the primary distribution of our minds does not take in 2^{100} cases, on the one hand; neither do we divide into the case of 100 heads—*and others*. We group the cases in thought into something between these two extremes: in fact, we imagine collections of the same degree of remarkability. Different minds will do this in different ways: if we could stamp the weights in some one mind, we might find perhaps only 2^5 groups, many of them involving each more than 2^{95} cases. We should thus have ordinary events, each of a probability of 2^{-5} nearly; and cases more or less extraordinary, varying in probability down to 2^{-100}.

Those who have not attended to arithmetic would make the primary distribution in a manner essentially wrong. Nothing would be called ordinary which very much differs from alternate head and tail: so that they would be surprised at almost any thing that might happen.

This method of distribution into groups, the members of each group being of equal notability, is that which prevails in our methods of judging: and it serves to explain the conclusions which we actually, though unconsciously, draw on the subject of evidence.

The preceding remarks may now be applied to the case of an authority asserting a universal proposition.

If we suppose $n-1$ to be the number of Xs in existence, and if we admit into our thoughts all the varieties of the numerically definite proposition, from 'Every X is Y' through 'Every X but one is Y' &c. down to 'no X is Y:' then, if all these n cases be of equal probability *à priori*, that is, before any consideration whatever of the connexion of the terms, it follows that the credit of the witness is neither raised nor lowered by the assertion of any one of them. And the same, if we make groups of equal notability, m in number, provided that the probability of the universal *à priori*, is looked upon as being exactly or nearly $1 \div m$. The two universals would each form a group, the idea of necessity, as distinguished from that of contingency, not only securing them this character, but perhaps giving them rather a higher share of probability than deliberate consideration would approve of. I do not profess here to *stamp the weights*: but only to show that the application of the theory explains the apparent inconsistency of the three phænomena above noted.

Let us now suppose that the statement of the witness is a denial that A_k happened, and nothing more. Then either A_k happened and he denies it, of which the previous probability is $\nu_k(1-k_k)$: or A_1 (k not being $=1$) happened, and he denies A_k, of which the previous probability is $\nu_1(1-k_1)$, &c. So that, from his assertion,

$$\frac{\nu_k(1-k_k)}{1-\Sigma\nu_s k_s} \text{ and } \frac{1-\nu_k-\Sigma'\nu_s k_s}{1-\Sigma\nu_s k_s}$$

are the particular probabilities that A_k *did* and *did not* happen. The first is the probability of his inaccuracy after the statement; and the previous probability that he shall inaccurately deny A_k is $1-k_k$. So that, looking at the previous force of $\Sigma\nu_s k_s$ when equal to ν_k, we see that his previous reputation for *inaccuracy* as to A_k, is increased, unaltered, or diminished, according as the probability of the event *denied* is greater than, equal to, or less than, the probability against the event, the *affirmation* of which would have left his reputation for *accuracy* unaltered.

The particular proposition being considered as more probable than the universal, the denial of the universal A_k, which is the affirmation of the contrary particular, generally makes the universal still more improbable. But we can hardly suppose a formal affirmation of the particular, except in opposition to some amount of belief gained for the universal, of a larger amount than its *natural* probability. About such an hypothesis there is nothing particular to examine.

The following is the most plausible supposition as to the bias of inaccuracy. The receiver of the testimony supposes that the probabilities of the n events are $\lambda_1, \lambda_2, \ldots \lambda_n$ in the mind of the witness; and, k_k being still the previous probability of accuracy when k happens, he divides $1-k_k$, the probability of inaccuracy, among the events which may be inaccurately stated, in proportion to the presumed probabilities of these events in the mind of the witness. That is, since $1_m+2_m+\ldots+n_m=1$, he makes $p_m=\lambda_p(1-m_m)\div(1-\lambda_m)$. On these suppositions, the particular credibility of the assertion A_k is

$$P_k=\frac{\nu_k k_k}{\nu_k k_k+\lambda_k\Sigma'\left(\frac{1-s_s}{1-\lambda_s}\nu_s\right)}.$$

Here, for a given probability in the receiver's mind, both as to the event and the witness stating it if it happen, the more incredible he is supposed to think it, the more must the receiver be inclined to believe it.

If $\lambda_s=s_s$, for all values of s, that is, if the previous probability of the witness asserting an event when it happens, be exactly the previous probability in his own mind that it shall happen, then $P_k=\nu_k$, and his testimony is nothing either way. According to the point of view from which this result is looked at, it appears exceedingly rational or exceedingly absurd. That the more extraordinary he thinks an event, the less likely is he to state it, even if it happen, would by itself make us trust him: but then, on the other hand, the more likely he thinks an event, the more likely is he to state it when it does not happen; which by itself lowers his credit. The balance of the two is perfect, on the definite supposition above: and thus an indisposition to state what he thinks extraordinary, is not, by itself, always a *prima facie* indication of a good witness to an extraordinary event.

If instead of $\lambda_s = s_s$ we take $s_s = \theta\lambda_s$, we find that $P_k = \nu_k$ when

$$\theta = \theta\nu_k + \Sigma'\left(\frac{1-\theta\lambda_s}{1-\lambda_s}\nu_s\right).$$

This is the case in which the disposition to accurate statement varies as the witness's previous expectation of the event. But if $1 - s_s = \theta(1-\lambda_s)$, that is, if the bias against inaccuracy vary as the previous expectation that the event shall not happen, we find

$$P_k = \frac{\nu_k(\theta\lambda_k + 1 - \theta)}{(1-\theta)\nu_k + \theta\lambda_k}.$$

This is ν_k when $\theta = 1$, as before.

Let there be two distinct sources of mistatement, by the double action of which an *incorrect inaccuracy* may be a correct statement: as in intentional and unintentional inaccuracy. Let all the preceding symbols apply to error of judgment, that is, let p_q signify the chance of his believing A_p when A_q happens; and let p_q' represent the previous probability in the mind of the receiver, that the witness will wilfully state A_p when *he believes* A_q happens. The statement A_k having been made, the possible originators of this statement are the compound events of which the probabilities are thus stated.

Let A_m happen, let the witness judge that A_v happened, and state that A_k happened. The previous probability of this combination is $\nu_m v_m k_v'$: and the denominator of the particular probability P_k is $\Sigma\Sigma(\nu_m v_m k_v')$ for all combined values of m and v. Select only those cases in which $m = k$, and we have $\Sigma(\nu_k v_k k_v')$ for the numerator. Accordingly

$$P_k = \frac{\Sigma(\nu_k v_k k_v')}{\Sigma\Sigma(\nu_m v_m k_v')}.$$

If s_s and s_s' be always p and r, as in Laplace's case, and $\nu_s = 1 \div n$, and if the biases both to falsehood and error of judgment be all equal, we have

$$m_s = (1-p) \div (n-1), \text{ and } m_s' = (1-r) \div (n-1);$$

$$\Sigma\nu_k v_k k_v' = n^{-1}pr + (n-1)\left\{n^{-1}(1-p)(1-r)(n-1)^{-2}\right\}$$

$$\text{the rest of } \Sigma\Sigma(\nu_m v_m k_v') = (n-1)\left\{\frac{1}{n}\frac{1-p}{n-1}\left(r + \frac{n-2}{n-1}(1-r)\right)\right\} + p\frac{1}{n}\frac{n-1}{n-1}(1-r)$$

$$= \left\{(1-p)r + (1-r)p + \frac{n-2}{n-1}(1-p)(1-r)\right\}\frac{1}{n},$$

whence the whole denominator is $1 \div n$, and

$$P_k = pr + \frac{(1-p)(1-r)}{n-1}, \text{ as given by Laplace.}$$

There is, however, but one case of the above which need be especially considered. When we suspect intentional falsehood in an assertion, it is usually because there is some particular reason why want of veracity should lead to that one particular assertion. Taking the extreme case, let us suppose A_k to be the only assertion which has any probability of being falsely made, so that $m_s' = 0$, except only when $m = k$. So far the formula is unaffected: but as it generally happens that the previous bias towards falsehood is towards the assertion itself, and nearly or altogether independent of what may really have happened; let $k_k' = 1$, and $k_1' = k'_2 = \ldots = \kappa$. Then we have

$$P_k = \frac{\nu_k \{k_k + \kappa(1 - k_k)\}}{\Sigma\{\nu_s(k_s + \kappa(1 - k_s))\}},$$

$$= \frac{(1 - \kappa)\nu_k k_k + \kappa\nu_k}{(1 - \kappa)\Sigma.\, \nu_s k_s + \kappa},$$

lying, as of course might have been predicted, between $\nu_k k_k \div \Sigma\nu_s k_s$ and ν_k.

If there be no probability of error of judgment, $k_k = 1$, and $k_s = 0$ when s is not $= k$. We have then

$$P_k = \frac{\nu_k}{(1 - \kappa)\nu_k + \kappa}.$$

The formula for any number of witnesses may be constructed as follows: Let p_q^m represent the previous probability that the m^{th} witness, A_q having happened, will judge that A_p has happened; and $p_q'^m$, the previous probability that the m^{th} witness, judging A_q to have happened will state A_p. If the statements made by the several witnesses be A_k, A_l, A_m, &c., the denominator of the probability thence arising in favour of A_t, is the sum of all terms of the form

$$\nu_u w_u^1 k_w'^1 \times x_u^2 l_x'^2 \times y_u^3 m_y'^3 \times \ldots\ldots$$

for all combinations of values of u, w, x, y, &c.: the numerator is the collection of all the terms of the denominator in which $u = t$.

If all the witnesses be of the same character in every respect, and all agree in asserting A_k, the coefficient of ν_u is

$$\Sigma(w_u x_u y_u \ldots \times k_w' k_x' k_y' \ldots)$$

for all combinations of values of w, x, y, &c.

In all that has preceded, it is to be remembered that application can immediately be made to the case in which the n events are not the only possible ones: that is, in which $\Sigma\nu_s$ is not equal to unity. For this is but the supposition of one case more, namely, no event at all, with the probability $1 - \Sigma\nu_r$.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON,
January 7, 1850.

ADDITION.

It may be useful to advert to the manner in which logicians, who all contend more or less explicitly for the sufficiency of ordinary syllogism, meet the cases in which premises give inference which cannot be reduced to one ordinary syllogism *with those premises.* I pass over the following, as extra-logical. First, giving the inference a bad name, as calling it a subtlety, or the like; logic is the science of the *necessary* laws of thought, subtle and not subtle. Secondly, affirming that the inference is very easy: which is just as much the case with the favoured modes of syllogism. I proceed to one which is both logical and sufficient, but which again applies just as much to the ordinary syllogism as to the cases which will not fall under it.

In every inference, there is an act of the mind, which we may perform with or without consciousness of reference to the general, self-evident, and indemonstrable postulate under which the validity of that act might be maintained. Useful as reference to the postulate may be, it is not, or need not be, formally necessary, since the act of the mind by which we refer the instance of inference to the postulate is, logically considered, of the same kind as that by which we draw the inference at once from the premises. Nevertheless, the formal syllogism, in the first case, is never anything but that easiest of all cases, the syllogism of *principle and example*, (*F. L.* p. 257). And thus we have for the ordinary syllogism, two forms of argument: the common one, and the reference to the postulate. Now this postulate always has had, perhaps always must have, a *composition of relations*, such as I have generalized in Section V. Consequently, I assert that the logicians have, when compelled to declare the ordinary syllogism incapable, had recourse to instances of the composition of relations out of which I have constructed the bicopular syllogism; though in truth they have not seen the extension of the theory, merely because their reference of the example to the principle may be made under the form *Barbara*.

First, I take a common syllogism, say $X))\,Y).(Z = X).(Z$. Using the terms of my work (*F. L.* ch. xiv., in which every syllogism is thus reduced, though composition of relations is not, as it ought to have been, the leading idea,) we see that X is a species of Y which is an external of Z, so that X is a species of an external of Z. Now if we merely compound the relations, we see that species of external is external; whence X is an external of Z. If we choose to make our last step in the greatest form, we have

Every species of an external (of any notion) is an external (of that notion),
X is a species of an external of Z;
Therefore X is an external of Z.

If I were to attack the syllogistic theory in a manner analogous to that in which logicians have, in certain isolated cases, supplied its deficiencies, the attack would involve the assertion that the preceding transformation is the legitimate form.

I now give instances of the supplementary use of this method. Reid justly remarks that "*A is equal to B, and B to C*, therefore *A is equal to C*, cannot be brought into any syllogism in figure and mode." On which Sir W. Hamilton's note (p. 702) is as follows;—"Not as it stands; for, as expressed, this reasoning is elliptical. Explicitly stated, it is as follows:—

What are equal to the same, are equal to each other;
A and C are equal to the same (B);
Therefore, A and C are equal to each other."

I am quite at a loss to see how the second is an *expanded* form of the first. But I see distinctly the composition of relation, 'equal of equal is equal,' expressing the transitiveness of the copula *equals*, which, with its convertibility, renders it of equal validity with *is* as the copula to be employed in a syllogism.

Again, a recent and learned editor of Aldrich, the Rev. H. L. Mansel, observes of the preceding, that "the reasoning is elliptical, and therefore, *as it stands*, material; though owing to the suppressed premise being self-evident, its deficiency is apt to be overlooked." This suppressed premise is the major given above. But does it not then follow that the ordinary syllogism is elliptical? Is there not always the suppressed premise (as I feel bound, on these authorities, to call it), "What *are* the same *are* each other?"

There is some want of distinctness in the use of the word *material*, as distinguished from *formal*. When the last named writer makes it "material and therefore extra-logical" that Alexander was the son of Philip because Philip was his father, he uses the word historically. The formal connexion of relation and correlation exists, though Aristotle did not recognize it as copular except when the relation is its own correlation, and then only in a limited case. Certainly the matter, in the case of *father* and *son*, supplies the knowledge of the correlative relation existing, but not the mode of using it in inference, when known to exist. In this and a great many other instances, matter is opposed by writers, not to form, but to what is recognized as form in the school of Aristotle: the assumption of course being that that school exhausts the forms of thought. Historically speaking, the copula has been material to this day: this I must continue to believe until it be pointed out where the formal conditions have been separated from the matter, and made the instruments of inference, independently of the separated matter.

A certain indifference to close description of the copular relation is manifest in many logical works, accompanied by negligence in its application; and this may produce strange consequences in literal translations. The proposition '*X* is *Y*,' may bear expression in French as '*X* doit être *Y*,' and in Latin as '*X* debet esse *Y*,' though both are rather negligent forms for a strict work: but the English '*X* ought to be *Y*' is of a different signification. Both* the English translations of the *Port Royal Logic* give a paralogism for the instance cited in illustration of the definition of the word *reasoning*: "having judged that true virtue ought to be referred to God, and that the virtue of the heathens was not referred to him, we thence conclude that the virtue of the heathens was not true virtue." Had the copula been as fully treated of as the terms, such inaccuracies could hardly have occurred.

July 3, 1850.

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* A new one has recently been published by T. S. Baynes, Edinburgh, 1850.

V. *Mathematical Exposition of some Doctrines of Political Economy. Second Memoir.* *By* W. WHEWELL, D.D., *Master of Trinity College.*

[Read *April* 15, 1850.]

1. THERE have appeared of late years many works upon the subject of Political Economy in which the reasoning, illustrated by numerical examples or in other ways, is of such a kind that a person of mathematical habits of mind, in reading the works, is naturally led to reflect upon the possibility of putting the reasoning into a general algebraical form, and upon the consequences which would result from such a mode of treating the subject. It is evident that such a mathematical mode of investigation would, when the fundamental principles of the subject were once distinctly stated and expressed in algebraical symbols, give the results with a certainty and simplicity of method which would be more satisfactory than special numerical examples,—at least to a mathematician. And among many other advantages of such a process, we may expect that we should obtain these:—that we should see both how the doctrines may be generalized, and how they must be limited, much more easily and clearly than we could do by the light of special numerical examples only. Accordingly, attempts have not been wanting to make such an application of mathematical methods to Political Economy: among which I may refer to a paper of my own, published in the *Transactions* of this Society[*].

2. It would, however, be to take a very erroneous view of the consequences of this application of mathematics to Political Economy, to suppose that it can add anything to the certainty of the fundamental principles. There is perhaps in some persons a propensity to believe that any subject, when clothed in a mathematical shape, acquires something of mathematical demonstrative character; and that by applying mathematics to assumed principles of knowledge, we in some measure create a science. I must beg leave very distinctly to repudiate all pretensions of this kind. By stating distinctly our fundamental principles, which such an undertaking as the present requires us to do, we may bring them more clearly under notice and examination than would otherwise be done; but we add nothing whatever to the evidence of the principles. All that we pretend to say is, that if the conclusions be false, the fallacy must be in the principles, if the process of deduction be truly mathematical.

3. The questions which I now intend to consider are some which relate to the connection of demand, supply, and price, whether in the same country, or in different countries. And, first, in the same country. That the price of any commodity depends upon the relation of the demand and the supply is commonly and truly stated. But in order to express this dependance

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[*] *Mathematical Exposition of some Doctrines of Political Economy.* 1829.

in a mathematical manner, we must define with precision the quantities which enter into the relation.

4. Let p be the price of a commodity C, and q the quantity effectually demanded, that is, bought, at that price; p being of course expressed in units of money, and q in terms of some unit which measures the commodity C.

The demand for C depends (*cæteris paribus*) upon the price: q depends upon p. Let p increase in any ratio: the demand, that is, the quantity which buyers are willing to purchase, will commonly diminish: but it may diminish either in the same ratio in which the price increases, or less rapidly. If the price of corn be increased by one-fourth, the consumption may be diminished by one-fifth, one-eighth, or one-tenth. In the first case, the money demand, that is, the money expended on the commodity, would be unaltered: (for it would be $= \text{price} \times \text{quantity} = \frac{5}{4} \times \frac{4}{5} = 1$). In the second case, it would become $= \frac{5}{4} \times \frac{7}{8} = \frac{35}{32} = 1 + \frac{7}{32}$, or would be increased between one-fourth and one-fifth. In the third case, the money demand would become $= \frac{5}{4} \times \frac{9}{10} = \frac{9}{8} = 1 + \frac{1}{8}$, and would be increased by one-eighth.

5. In order to express such a relation generally, let p become $p' = p\,(1 + x)$, and let q become q', such that $p'q' = pq\,(1 + mx)$, where m is a certain coefficient. This expresses the cases just mentioned according as $m = 0$, $m = \frac{3}{8}$, or $m = \frac{1}{2}$.

6. But the price may also be considered as depending on the supply, that is, on the quantity supplied for sale: and the quantity supplied to buyers at a given price is the same as the quantity bought at that price; therefore the equation just given, $p'q' = pq\,(1 + mx)$, expresses the relation between supply (q') and price (p').

If the quantity supplied of any given commodity be ***diminished*** by one-fourth (for example), the price may increase by one-third, or it may increase one-fifth, or it may increase one-half. In these cases the money demand respectively becomes $= \frac{3}{4} \times \frac{4}{3} = 1$, that is, remains unaltered; becomes $= \frac{3}{4} \times \frac{6}{5} = \frac{18}{20} = 1 - \frac{1}{10}$, or diminishes by one-tenth; becomes $= \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1 + \frac{1}{8}$, or increases by one-eighth. And since x is in these cases $= \frac{1}{3}, \frac{1}{5}, \frac{1}{2}$ respectively, and $mx = 0, -\frac{1}{10}, \frac{1}{8}$ respectively; we have in the three cases, $m = 0$, $m = -\frac{1}{2}$, $m = \frac{1}{4}$, respectively.

If the quantity supplied be ***increased*** by one-fourth (for example), the price may be diminished by one-fifth, or one-tenth, or one-half. In these cases, the money demand becomes respectively $= \frac{5}{4} \times \frac{4}{5} = 1$, that is, remains unaltered; becomes $= \frac{5}{4} \times \frac{9}{10} = \frac{9}{8} = 1 + \frac{1}{8}$, that is, increases by one-eighth;—becomes $= \frac{5}{4} \times \frac{1}{2} = \frac{5}{8} = 1 - \frac{3}{8}$, that is, diminishes by three-eighths.

Since x is in these cases $= \frac{1}{5}, \frac{1}{10}$, and $\frac{1}{2}$ respectively, and $mx = 0, \frac{1}{8}, -\frac{3}{8}$, we have $m = 0$, $m = \frac{4}{5}$, $m = -\frac{3}{4}$ in the three cases respectively.

7. Which of these classes of cases is likely to exist in practice?

It will be apparent, on consideration, that one or other will occur according to the nature of commodity, and thus, that m has different values for different commodities. We shall endeavour to indicate certain classes of commodities according to this relation.

8. (*A*) There may be some commodities on which, in a given society, the same sum is expended whatever be the price of the article (per unit); a smaller quantity being bought exactly in proportion as the price is higher. Such would be the case with ornamental attire, for instance, if each person, or if persons on an average, were to spend upon it a determinate sum every year;—an *allowance for dress*, as it might be termed. In this case, when p becomes $p(1+x)$, $p'q'$, the money demand, remains unaltered, whence $pq(1+mx) = pq$. Here $m = 0$.

9. (*B*) There may be other commodities of which the quantity bought is the same whatever be the price: such, for example, may be articles which are looked upon as necessary by rich persons; as, for instance, official dresses, and conventional appendages of persons in office, and the like. Here, when p becomes $p(1+x)$, q remains unaltered. Therefore $pq(1+mx) = pq(1+x)$. Here $m = 1$.

10. (*C*) There are other commodities of which the price increases more rapidly than the quantity supplied diminishes: for instance, the general necessaries of life. It has been supposed that a deficiency of one-fifth in the supply of corn will raise the price four-fifths. Supposing this true, the money demand becomes $= \frac{4}{5} \times \frac{9}{5} = \frac{36}{25} = 1 + \frac{11}{25}$. And since $x = \frac{4}{5}$ and $mx = \frac{11}{25}$, $m = \frac{11}{20}$.

11. The more we suppose prices to rise for a given diminution of supply, the more will m approach to 1. If we suppose that a diminution of one-fifth in the supply will treble the price, we have the money demand $= \frac{4}{5} \times 3 = \frac{12}{5} = 1 + \frac{7}{5}$. And since $mx = \frac{7}{5}$ and $x = 2$, $m = \frac{7}{10} = \frac{14}{20}$.

In all these cases m is between 0 and 1.

12. (*D*) There may be other commodities of which the price increases in a less proportion than the supply diminishes: or, as the case is perhaps more evident, there may be commodities, of which, when the price diminishes, the demand increases, and in so great a proportion that the whole sum expended on them is greater than before. This may be the case with

some luxuries: for instance, when tea or coffee fall to half-price, they may come into use with new and numerous classes of persons, so that the whole sum expended on them may be double what it was. In this case, it is evident that the quantity bought must be four times what it was. In this case, $x = -\frac{1}{2}$, $mx = 1$; hence $m = -2$.

This may be the case in a smaller degree: if we suppose that a fall to half-price increases the money demand by one-half, we have $x = -\frac{1}{2}$, $mx = \frac{1}{2}$; whence $m = -1$.

If we suppose that a fall to half-price increases the money demand by one quarter, we have $x = -\frac{1}{2}$, $mx = \frac{1}{4}$; hence $m = -\frac{1}{2}$.

In the last two cases, the quantity bought is, in the former case trebled, in the latter, increased in the ratio $\frac{5}{2}$.

In all these cases m is negative.

13. We must suppose that the above formulæ apply also in the cases in which x is negative, or the price falls. Applying this remark to the four classes of conditions *A*, *B*, *C*, *D*, we see that

For class *A* of commodities, ($m = 0$,) if the price fall, the money demand remains unaltered, and the quantity sold increases inversely as the price.

For class *B*, ($m = 1$,) if the price fall, the money demand falls in the same proportion; and the residue is saved by the purchasers for other employments.

For class *C*, ($m > 0$ and < 1,) if the price fall, the money demand falls also, but in a less proportion, the quantity sold being increased. Thus, if for any commodity for which $m = \frac{1}{4}$, the price fall one-half, the money demand will fall one-eighth. If when $m = \frac{1}{2}$, the price fall one-half, the money demand will fall one-fourth. To produce the former effect the quantity supply must increase in the ratio $\frac{7}{4}$: to produce the latter the quantity supply must increase in the ratio $\frac{3}{2}$.

14. If m be very nearly $= 1$, for instance if $m = \frac{9}{10}$, and the price fall one-tenth, the money demand will fall $\frac{9}{100}$: to produce this effect, an increase in the quantity of $\frac{1}{90}$ will suffice. If the price fall two-tenths, the money demand will fall $\frac{18}{100}$: to produce this effect, an increase in the quantity of $\frac{1}{40}$ will suffice. If the price fall one-half, the money demand will fall by $\frac{9}{20}$: to produce this effect, an increase in the quantity supplied of $\frac{1}{10}$ will suffice.

Hence it appears that when m is very nearly equal to 1, a small increase in the quantity supplied, $\left(\frac{1}{90}, \frac{1}{40}, \frac{1}{10},\right)$ will produce a large diminution in the price $\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{2}\right)$. In such cases we may say that the price is very susceptible of change, (by alteration of the supply), and since, as m is larger, this *susceptibility* is greater, we may take m to measure the susceptibility of change of price of each commodity.

15. But if we consider the demand as varying with the price, it is evident that for a given increase of price, a greater increase of money demand indicates a stronger effectual demand: and as this is greater according as m is greater, m may measure the strength of demand for each commodity, as shewn when the price changes.

16. We may include these two ways of regarding m, by calling it the *specific rate of change* of each commodity; meaning thereby, both the change of price when the supply varies, and the change of demand when the price varies.

17. For an example of the value of m, we may take data given by Mr. Tooke (*High and Low Prices*, p. 285). He says that to diminutions in the supply of corn we have corresponding augmentations of price, in the following proportion:

Diminution of supply	$\frac{1}{10}$,	$\frac{2}{10}$,	$\frac{3}{10}$,	$\frac{4}{10}$,	$\frac{5}{10}$;
Increase of price	$\frac{3}{10}$,	$\frac{8}{10}$,	$\frac{16}{10}$,	$\frac{28}{10}$,	$\frac{45}{10}$.

Now if we take $m = \frac{1}{2}$, we have the following correspondencies:

Diminution of supply	$\frac{1}{10}$,	$\frac{2}{10}$,	$\frac{3}{10}$,	$\frac{4}{10}$,	$\frac{5}{10}$;
Increase of price	$\frac{2\frac{1}{2}}{10}$,	$\frac{6\frac{2}{3}}{10}$,	$\frac{15}{10}$,	$\frac{40}{10}$,	infinite.

For a deficiency of supply of one-half, the prices would be infinite. This shews that the formula with the value $m = \frac{1}{2}$, is not applicable for so great a deficiency of supply: but upon the whole, $m = \frac{1}{2}$, appears to be near the value of the specific rate of change for corn.

If we take the diminution of supply $= \frac{1}{10}$, and increase of price $= \frac{3}{10}$, we have $m = \frac{17}{30}$.

If we take the diminution of supply $= \frac{5}{10}$, and increase of price $= \frac{45}{10}$, we have $m = \frac{7}{18}$.

The former is greater than $\frac{1}{2}$, the latter is less than $\frac{1}{2}$[*].

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[*] It appears that when the diminution of supply is small, the value $m = \frac{1}{2}$ is too small, and when the diminution of supply is nearly one-half, the value $m = \frac{1}{2}$ is too large. Therefore the function $pq(1 + mx)$ is not really the true formula of the

18. For the class of commodities *D*, (m negative,) if the price rises, the quantity effectually demanded diminishes in a higher ratio.

In the case supposed before, in which a fall of price to one-half produced a fourfold amount of sale, we might at first, perhaps, suppose that the rule applies in the inverse order of change, and that if the price be again doubled, the quantity demanded will again fall to one-fourth, and the money demand to one-half. But this result will not be given by the formula. For in this case m was $= -2$; and if when $p'q' = pq(1 - 2x)$ we suppose q' to be $\frac{1}{4}q$, we have $1 + x = 4(1 - 2x)$, which would give $x = \frac{1}{3}$, denoting an increase of price of one-third, and a money demand diminished to one-third.

19. This apparent inconsistency arises from the formula $p'q' = pq\,(1 + mx)$ being made to rest on a given medium standard price and quantity, p and q, and to express the changes by an increment or decrement x. Hence large changes are not proportionally the same above and below the standard point.

When $m = -\frac{1}{2}$, we have the money demand $= pq\left(1 - \frac{x}{2}\right)$, and hence, since

$$p'q' = pq\left(1 - \frac{x}{2}\right), \quad p' = p\,(1 + x), \quad \text{we have } q' = q\,\frac{1 - \frac{x}{2}}{1 + x}.$$

When the quantity increases by one and one-half times its value, we have this $= \frac{5}{2}$, whence $x = -\frac{1}{2}$, and the price is reduced to one-half, as in the case above stated. When the quantity decreases by one-half, we have $\frac{1 - \frac{x}{2}}{1 + x} = \frac{1}{2}$, which would give $x = \frac{1}{2}$, and the price is increased by one-half.

When the quantity decreases by one-third, we have $\frac{1 - \frac{x}{2}}{1 + x} = \frac{2}{3}$; whence $x = \frac{2}{7}$, the price is increased in the ratio $\frac{9}{7}$: the money demand is diminished in the ratio $\frac{5}{7}$.

20. It appears from what has been said, that we have four classes of commodities, which differ according to different values of m, the susceptibility of change in the price by change

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money demand. It would be easy to devise a formula which should more nearly represent Mr. Tooke's progression; but even if his numbers were derived from facts, they would, of themselves, be insecure grounds for generalization. And the nature of the progression makes it probable that the progression is hypothetical merely: the third difference being constant: as appears thus:

$$\begin{array}{ccccccccccc}
0 & & 3 & & 8 & & 16 & & 28 & & 45 \\
& 3 & & 5 & & 8 & & 12 & & 17 & \\
& & 2 & & 3 & & 4 & & 5 & & \\
& & & 1 & & 1 & & 1 & & &
\end{array}$$

The general term of the series 3, 8, 16, 28, 45 is

$$\frac{n\,(n+1)\,(n+2)}{6} + 2n.$$

of supply, or the rate of change in the money demand for a change of price. The classes occur as m is 1, is between 1 and 0, is 0, or is negative. So far as these classes of commodities are exemplified by the instances above adduced, we may call them *Conventional Necessaries*, *General Necessaries*, *Articles of Fixed Expenditure*, and *Popular Luxuries*. For the first class, the quantity sold is the same whatever be the price. For the second class, when the price rises the quantity sold diminishes, but the money demand increases. For the third class, the money demand is always the same, and therefore the quantity sold is inversely as the price. For the fourth class, when the price falls the quantity sold is augmented, so that the money demand also is augmented.

21. I suppose that there are no commodities of which a greater quantity would be sold if the price were increased, and a less quantity sold if the price were diminished. It is conceivable that this might be, as a matter of caprice or fashion. For instance, we may conceive that diamonds might in some way (by the discovery of abundant mines or the like) become so common as to grow out of use, so that a less quantity might be sold than at present. If there should be such commodities, they would correspond to values of m greater than 1.

22. If it were possible to arrange commodities according to the value of m, the specific rate of change, (as is done hypothetically for the sake of example in the above instances) so that we should for every quantity know the value of m, we might solve a great variety of problems respecting the variations of price, of demand, and of supply, so far as these quantities depend on each other. And so far as the formulæ are applicable, we have the equations

$$p' = p(1 + x), \quad p'q' = pq(1 + mx), \quad q' = q\frac{1 + mx}{1 + x}.$$

23. It is well observed by Mr. J. S. Mill (*Polit. Econ.* i. 529) that instead of saying, as writers have often said, that the price depends upon the *ratio* of demand and supply, we ought rather to say that the price depends upon the *equation* of demand and supply. And we may apply the term, *the equation of demand and supply* to the equation

$$p'q' = pq(1 + mx).$$

24. As examples of the above formulæ, let it be supposed that an increase of $\frac{1}{10}$ in the supply (the whole being sold) produces a fall of $\frac{1}{5}$ in the price.

$$\text{Then } x = -\frac{1}{5}, \quad \text{and } \frac{1 - \frac{m}{5}}{1 - \frac{1}{5}} = \frac{11}{10}, \quad \text{whence } m = \frac{3}{5}.$$

This being the case, what effect on the price would be produced by an increase, and what by a diminution, each of $\frac{1}{4}$, in the supply?

For the increase $\dfrac{1+\dfrac{3x}{5}}{1+x} = \dfrac{5}{4}$; $x = -\dfrac{5}{13}$: the price falls in the ratio $\dfrac{8}{13}$.

For the decrease $\dfrac{1+\dfrac{3x}{5}}{1+x} = \dfrac{3}{4}$; $x = \dfrac{5}{3}$: the price rises in the ratio $\dfrac{8}{3}$.

25. Hitherto we have not considered the manner in which the extent of the supply is determined by its causes. But it is evident that in general there is, between the quantity supplied and the price, a relation depending upon the conditions which govern the supply, as being the produce of agriculture or of manufacture, or of some other agency.

26. So far as this dependance goes, there are three main classes of commodities (J. S. Mill's *Polit. Econ.* i. 524).

(α) Articles of absolutely limited and fixed supply, in which no increase of price can increase the supply, as old pictures, peculiar wines, building ground in a town.

(β) Articles of unlimited supply with proportional labour; as, in general, manufactured articles, cottons, woollens, linens, which might be produced in unlimited abundance by a proportional application of capital and labour. In these, the price or *cost*, in the long run, and independently of temporary fluctuations, is constant; except so far as it may be diminished by improvements in manufacture or cultivation, or increased by the increased cost of the raw material; or raised for a time, and its ultimate value delayed, by want of capital or want of labourers.

(γ) Articles of increased supply at increasing cost, as agricultural produce in a given limited country. In such articles, every addition to the quantity produced increases the cost of some portions, and therefore the price of all. To add a million quarters of corn to the annual produce of a country, would raise the cost-price, it may be, 1 shilling a quarter. To add a second million quarters to the produce, would raise the price still further, it may be a shilling more, or two shillings above the original standard; and so on.

27. But it may be that the increase of cost-price goes on in a higher ratio than the increased amount of produce. Thus the original standard quantity of produce being q millions, and the cost-price of a unit of that quantity being p shillings, let it be supposed that, in order to add to q respectively 1 million, 2 million, 3 million units, the price must rise at each step successively by 1 shilling, 3 shillings, 5 shillings, &c. at the successive steps of extended and more expensive cultivation. Then the whole rise of price at these steps above the original standard will be 1, 4, 9, &c. which are as the squares of 1, 2, 3, &c. the increase of quantity. We may express this relation thus; n being a coefficient hereafter to be determined, we may suppose

$$q' = q(1+y), \quad p' = p(1+ny^2),$$

and we may call this *the equation of increased cost of production.*

28. It is not pretended that the law of the increased cost of increased quantity is exactly represented, for agricultural or any other produce, by the formula just given; but only that such a formula may serve to exemplify a dependance having such a character as has been described.

29. In commodities of the first of the three classes above-mentioned, (α), (commodities of *Fixed Supply*,) the price depends entirely upon the intensity of the demand. As the amount of supply is invariable, the price can vary only from some variation in the demand; and this must arise from changes not expressed in our formulæ.

In commodities of the second class, (β), (commodities of *Fixed Cost*) the quantity supplied will depend upon the amount of capital, labour and skill employed in the production. When this quantity varies for a unit of the commodity, the price will vary. And the relation will be given by the equation of demand and supply (23). If p, the selling price, be greater than the cost price with the usual rate of profits, capital and labour will flow into the employment, till p is reduced down to cost price. If by any cause, for instance, improvement in the process of manufacture, the cost price is diminished, the diminution may be expressed by x, in that equation; and (m being known) the equation will give the quantity which will be sold at the new price.

In commodities of the third class, (γ), (commodities of *Increasing Cost*) the quantity supplied will be determined by the cost price which purchasers are willing to give. The selling price, in the long run and on the large scale, will be the cost price with the ordinary profit. If the purchasers are willing to give a higher price, (as for instance, in consequence of the increase of wealth and population) the quantity produced will be increased by extended or more expensive cultivation. And the amount of increase will be given by the equation of increased cost (27). (If by improved methods of cultivation the cost price p be diminished, this equation no longer applies to that change. The case then comes into the second class and the equation of demand and supply is to be used).

30. For example; let the price which purchasers are willing to pay rise from 9 to 10 shillings, and let $n = 1$. Then $ny^2 = \frac{1}{9}$, and $y = \frac{1}{3}$. Therefore q will be increased in the ratio $\frac{4}{3}$.

An increase of $\frac{1}{3}$ in the population, with a proportional increase in the means of living, would on these suppositions, correspond to an increase of price from 9 to 10.

On the same suppositions, what rise of price would take place on the next equal increase of population, under the like conditions?

Here $y = \frac{2}{3}$, whence $p' = p\left(1 + \frac{4}{9}\right)$. Hence the price would be nearly half as much again as at first.

If the original population be doubled under the like conditions, $y = 1$, $p' = 2p$, and the price is doubled, though for the increase of $\frac{1}{3}$ in the population, the price was only increased $\frac{1}{9}$.

31. But it must be recollected that these suppositions, of a population which, during its increase in numbers, increases in the same proportion in money demand for corn, while the increased production of corn is not accompanied by any improvement of cultivation occasioning a diminution of cost, is an arbitrary hypothesis, made in order to apply the formulæ, and not likely to be realized in fact.

32. And on the other hand, if the quantity of produce of this class has been increased while the price has not been increased, or has been diminished, that circumstance shews that in the country and during the period in question, the commodity has not been one of increasing cost of production. In this case improvements in cultivation must have balanced or more than balanced the increased difficulties which, without them, would have made the cost of every added portion of produce greater than the preceding. To this case our equation does not apply.

33. The preceding formulæ apply to prices as affected by demand and production within the limits of our country. Prices within such a circle are governed ultimately by the cost of production. The prices of a unit of each of two commodities C and D, are as the cost of production of the two; that is, as the labour (including skill estimated in labour) by which each is produced and brought to market. For if the ratio of the prices were different from this, labour would be transferred from the production of the one to that of the other, so as to tend to restore the equality. But between foreign countries there is no such tendency to equilibrium between price and labour, because labour is not transferred from one country to another when the prices are in a different ratio. A pound of tea, if produced in China by the same labour which produces a yard of cloth in England, may nevertheless exchange for two yards, or for half a yard: for there will not be a transfer of tea-producing labour to produce cloth, in the first case, or of cloth-producing labour to tea in the second. Hence then the relation of prices of commodities, native and imported, is not governed by the equations already given. By what then is it governed? What is the principle which regulates *international values?* (Mill, *Polit. Econ.* II. 121.)

34. The principle which regulates such values is (in addition to the principle of supply and demand already spoken of,) this:—that when the international trade has been established, the relative value of all commodities which are exported and imported is the same in the two countries (omitting for the present the cost of carriage). This we may call the *principle of uniformity of international prices.* It is evident that if tea and cloth are exchanged between China and England, the rate of exchange of the two must be the same in the two countries: for if it were not, the current of trade would be determined one way or other, and would, by increasing the import of the one commodity or the other, tend to restore the equilibrium.

35. In order to apply this principle, let there be two commodities (C) and (D) (cloth and linen for example): and let C and D represent the value of a unit (yard) of each in terms of any other commodity.

Suppose that in England p is the price of D in terms of C; and let q be the quantity of (D) consumed (that is, bought) in England at that price.

Then $D = pC$, $C = \frac{D}{p}$ in England.

Suppose that in Germany P is the price of C in terms of D, and let Q be the quantity of (C) consumed in Germany at that price.

Then $C = PD$, $D = \frac{C}{P}$ in Germany.

Now D in England costs pC when there is no international trade; but if obtained from Germany by exporting C, would there cost $\frac{C}{P}$.

Therefore there would be a gain for England in obtaining D by exporting C rather than producing it at home if $pC > \frac{C}{P}$; that is, if $Pp > 1$: for the cost would be less.

In like manner, there would be a gain for Germany in obtaining C by exporting D rather than by producing it at home, if $PD > \frac{D}{p}$; that is, if $Pp > 1$: for the cost would be less.

Hence on the supposition that England exports C and imports D, and Germany exports D and imports C, both countries gain.

36. What will be the amount of the exports and imports, and the prices, when the international trade exists?

In order to solve this problem, we must introduce another principle; namely this:—that in the long run, and in the permanent condition of the trade, the value of the exports of each country must equal the value of its imports. For each country pays for its imports by its exports.

Under the trade let p, the price of D in terms of C, in England, become p'; and q, the quantity of (D) consumed in England, become q', the whole being imported from Germany. And let P, the price of C in terms of D in Germany become P', and let Q, the quantity of (C) consumed in Germany, become Q', the whole being imported from England.

In England $p'C = D$, in Germany, $P'D = C$; and since these equations express prices under the trade, by the principle of uniformity of international prices, the relation of C and D is the same in the two countries. Therefore multiplying together the two equations,

$$P'p' = 1,$$

which is the *equation of uniformity of international prices.*

37. England exports Q' of (C) and imports q' of (D); and of this last the value is $p'q'$ in terms of C: therefore, by (36) $p'q' = Q'$.

In the same manner Germany exports q' of (D) and imports Q' of (C); and of this last the value is $P'Q'$ in terms of D: therefore $P'Q' = q'$.

The equation $p'q' = Q'$, or $P'Q' = q'$, is *the equation of import and export.* The two equations are identical in virtue of the equation $P'p' = 1$.

38. Now to find the quantities of the imports and the prices.

The consumption of (D) in England varies with the price (22). When the price falls from pC to $p'C$, let the quantity consumed be increased from q to q'. Let $p' = p(1-x)$, and let the law of altered money demand be, as before, $p'q' = pq(1-mx)$. Hence $q' = q\dfrac{1-mx}{1-x}$.

In like manner the demand for (C) in Germany varies with the price. Let $P' = P(1-X)$, $P'Q' = PQ(1-MX)$; whence $Q' = Q\dfrac{1-MX}{1-X}$.

Since $p(1-x) = p'$, $P(1-X) = P'$, we have $Pp(1-X)(1-x) = P'p' = 1$, by (36). Hence $(1-X)(1-x) = \dfrac{1}{Pp} = 1-k$, suppose, k being a fraction; since by (36) Pp is greater than 1.

The equation $(1-X)(1-x) = 1-k$, gives $X = \dfrac{k-x}{1-x}$.

The equation $q' = P'Q'$, (37), gives $q\dfrac{1-mx}{1-x} = PQ(1-MX)$.

Put for X its value $\dfrac{k-x}{1-x}$, and solve the equation in x; and we find

$$x = \frac{PQ(1-Mk)-q}{PQ(1-M)-mq}.$$

39. The values of x and X depend upon the ratio existing between PQ and q originally, before the trade: that is, upon the relative value of (C) consumed in Germany and of (D) consumed in England: and also upon m and M, the specific rate of change of each commodity.

In general let $PQ = nq$; and we have

$$x = \frac{n(1-Mk)-1}{n(1-M)-m}, \text{ or } x = \frac{1-n(1-Mk)}{m-n(1-M)}.$$

40. I will apply these formulæ to the numerical examples given by Mr. Mill, (*Polit. Econ.* II. 123).

He supposes that originally, in England, $C = \dfrac{3}{2}D$, and in Germany, $C = 2D$. Hence $p = \dfrac{2}{3}$, $P = 2$, $Pp = \dfrac{4}{3}$ which being > 1, the trade is advantageous;

$$\frac{1}{Pp} \quad \frac{3}{4} = 1-k: \text{ whence } k = \frac{1}{4}.$$

Let m and M be each $= \dfrac{2}{3}$. Hence we have

$$x = \frac{6-5n}{4-2n}.$$

If we suppose n to be $\frac{8}{7}$, we find $x = \frac{1}{6}$.

Hence the price of D in terms of C falls $\frac{1}{6}$ in England; that is, the price of D becomes $\frac{2}{3} \times \frac{5}{6} C = \frac{10}{18} C$, which is one of Mr. Mill's suppositions.

On this supposition, since $q' = q \frac{1 - mx}{1 - x} = q \frac{16}{15}$, the quantity of ($D$) consumed in England is greater by $\frac{1}{15}$ than it was before the trade.

Also $(1 - x)(1 - X) = \frac{3}{4}$. Hence $1 - X = \frac{9}{10}$, $X = \frac{1}{10}$; $Q' = Q \frac{1 - MX}{1 - X} = Q \frac{28}{27}$.

The quantity of (C) consumed in Germany is $\frac{1}{27}$ greater than it was before the trade.

41. If we suppose n to be $\frac{10}{9}$, we find $x = \frac{1}{4}$, $X = 0$.

Hence the price of D in terms of C in England falls $\frac{1}{4}$; that is, the price of D becomes $\frac{3}{4} \times \frac{2}{2} C$ or $\frac{1}{2} C$. The quantity of (D) consumed in England becomes $\frac{10}{9} q$.

In this case the relative price of C and D in Germany is not altered by the trade, and Q is not altered.

If we suppose n to be $\frac{6}{5}$, we find $x = 0$, $X = \frac{1}{4}$, $Q' = \frac{10}{9} Q$.

In this case the relative price of C and D in England is not altered by the trade, and q is not altered.

42. If we suppose n to be greater than $\frac{6}{5}$, for instance, if $x = \frac{5}{4}$, we should find from the formulæ, $x = -\frac{1}{6}$, $X = \frac{5}{14}$.

But this solution, implying that the price of D in England is increased by the trade, is of course inapplicable.

In like manner if n be less than $\frac{10}{9}$ we shall find X negative, and have an inapplicable solution.

43. We can now trace the gain of each country by the trade on the above suppositions.

Let $n = \frac{8}{7}$ and $P = 2$ as above (40). Then $PQ = \frac{8}{7} q$, and since $P = 2$, $Q = \frac{4}{7} q$.

In this case, before the trade, England produces for her own consumption a quantity q of (D) and a certain quantity of (C). During the trade she exports $\frac{28}{27} Q$ of (C) with which she obtains $\frac{16}{15}q$ of (D). The $\frac{28}{27}Q$ of (C) is equal to $\frac{16}{27}q$ of (C) which, in England, is produced by the same labour as $\frac{8}{9}q$ of (D). Therefore, of the labour before employed in producing q of (D), $\frac{1}{9}$ is liberated, and may be employed in producing (C) or any other commodity: and this, along with the $\frac{1}{15}q$ of (D), is the gain resulting to England from the trade.

Before the trade, Germany produces for home consumption a quantity Q of (C) and a certain quantity of (D). During the trade she exports $\frac{16}{15}q$ of (D), with which she obtains $\frac{28}{27}Q$ of (C). The $\frac{16}{15}q$ of (D) is equal to $\frac{28}{15}Q$ of (D) which, in Germany, costs the same labour as $\frac{14}{15}Q$ of (C). Therefore Germany obtains an addition of $\frac{1}{27}$ in the amount of (C), and of so much labour as would produce $\frac{1}{15}$ of the (C) originally consumed, which may be employed in producing (D) or any other commodity.

44. Let $n = \frac{10}{9}$; therefore $Q = \frac{5}{9}q$.

In this case during the trade, England exports Q of (C) with which she obtains in Germany $2Q$ or $\frac{10}{9}q$ of (D). The Q of (C) is equal to $\frac{5}{9}q$ of (C) which in England is produced by the same labour as $\frac{5}{6}q$ of (D). Hence England gains $\frac{1}{9}q$ of (D) and gains as much labour as produced $\frac{1}{6}q$ of (D), which is ready for any new employment. Germany exports $\frac{10}{9}q$ of (D), with which she obtains in England, $\frac{5}{9}q$ of (C) that is, Q of (C). The $\frac{10}{9}q$ of (D) is produced with the same labour as $\frac{5}{9}q$ of (C). Therefore she gains nothing by the trade.

But still the trade will go on, for England offering (C) in Germany at the same price as (C) produced in Germany, this supply of (C) will be sold along with the rest; and if not, a temporary depression of its price in relation to (D) will find it a market; and then, by the ***principle of transferable capital***, there will be a transfer of German capital from (C) to (D) till the whole quantity requisite to produce the $\frac{10}{9}q$ of (D) is transferred.

In the same manner if $n = \frac{6}{5}$, $Q = \frac{3}{5}q$. Germany exports q of (D) with which she purchases in England $\frac{2}{3}q$ of (C), which is $\frac{10}{9}Q$ of (C). Therefore she has $\frac{1}{9}$ more of (C) than she had before. Also the q of (D) cost in Germany the same labour as $\frac{1}{2}q$ of (C), that is, as $\frac{5}{6}Q$ of (C). Therefore $\frac{1}{6}$ of the labour at first employed in producing (C) is set free for any new employment.

England gains nothing, consuming, as before, q of (D) and a certain quantity of (C): but the labour which produced q of (D) is transferred to the production of $\frac{2}{3}q$ of (C) for the German customer.

45. If n be $> \frac{6}{5}$, as, if $n = \frac{5}{4}$, $PQ = \frac{5}{4}q$, $Q = \frac{5}{8}q$. In this case, Germany wants Q of (C) which she can procure in England by exporting $\frac{3}{2}Q$ of (D) instead of producing it at home where it would cost $2Q$ of (D). Hence she gains $\frac{1}{2}Q$ of (D), and this may be expended on (C), (D), or any new commodity.

England exports Q of (C), which requires as much cost as $\frac{2}{3}Q$, that is, $\frac{5}{12}q$, of (D). Hence the labour or cost which produced $\frac{5}{12}q$ of (D) is transferred to produce (C), and produces $\frac{5}{8}q$ of (C), in addition to the quantity required for home consumption. And the production of (D) in England is diminished by $\frac{3}{2}Q$ or $\frac{15}{16}q$ which is brought from Germany.

46. The case in which x or X is negative by the formulæ, is the case in which the inequality of mutual demand is such that no relative value of C and D intermediate between that in Germany and that in England will satisfy the equation of import and export (37). In that case, the gain is all on one side, as in the case last examined.

To determine this gain in general in such cases, we must consider that Germany wants Q of (C) which she can procure in England by exporting $\frac{Q}{p}$ of (D) instead of producing so much (C) at home, where it would cost PQ of (D). Hence her gain is

$$PQ - \frac{Q}{p} = \frac{Pp-1}{p}Q = \frac{kQ}{p} \text{ of } (D), \text{ or } kQ \text{ of } (C),$$

which may be spent in adding to (C) or to (D), or on any other commodity.

47. It appears by what has been said that in the case supposed, where $p = \frac{2}{3}$, $P = 2$, (Mr. Mill's example), there is no possibility of a trade in which both countries shall gain, except n be between $\frac{10}{9}$ and $\frac{6}{5}$. If n be greater than $\frac{6}{5}$, Germany alone gains: if n be less than $\frac{10}{9}$, England alone gains.

This is on the supposition that $m = \frac{2}{3}$. In general, since x must be less than k, and

$$x = \frac{1 - n(1 - Mk)}{m - n(1 - M)},$$

we must have $1 > n(1 - Mk)$ in order that the numerator may be positive. And since $x < k$, $1 - n(1 - Mk) < mk - nk(1 - M)$; whence

$$n > \frac{1 - km}{1 - k}, \text{ and } n < \frac{1}{1 - Mk}.$$

If $k = \frac{1}{4}$, $m = \frac{2}{3}$, this gives $n > \frac{10}{9}$ and $n < \frac{6}{5}$ as above.

48. If $m = \frac{1}{2}$, $M = \frac{1}{2}$. Then $n > \frac{8}{7}$ and $n < \frac{7}{6}$.

In this case $x = \frac{8 - 7n}{4 - 4n}$. Since n must be between $\frac{8}{7}$ and $\frac{7}{6}$, that is, between $\frac{96}{84}$ and $\frac{98}{84}$, let $n = \frac{97}{84}$. Then $x = \frac{7}{52}$, $x = \frac{6}{45}$.

$$\text{Hence } q' = q\,\frac{1 - mx}{1 - x} = \left(1 + \frac{1}{21}\right)q; \; Q' = Q\,\frac{1 - Mx}{1 - x} = \left(1 + \frac{2}{39}\right)Q.$$

Both q and Q are increased by a fraction, and both countries gain.

If n be less than $\frac{8}{7}$, England alone gains: if n be greater than $\frac{7}{6}$, Germany alone gains.

49. If we have to use the formula $x = \dfrac{n(1 - Mk) - 1}{n(1 - M) - 1}$,

$$\text{we must have } n < \frac{1 - mk}{1 - k}, \text{ and } n > \frac{1}{1 - Mk}.$$

50. Let now England produce another exportable commodity (E), and let the price of E, a unit of (E), be $= rC$ in England.

Let Germany produce another exportable commodity (F), and let the price of F be RD in Germany; and let S be the quantity of (E) required in Germany, and s the quantity of (F) required in England at the original prices.

It is required to find the amount and prices of the imports and exports.

51. Let the trade be established: let (E) be exported and (F) imported by England.

In this case, when p becomes $p' = p(1 - x)$, and P becomes $P' = P(1 - X)$, let r become r', and R become R', s become s', and S become S'.

If England be *entirely* supplied with (F) by Germany, and Germany *entirely* supplied with (E) by England, the import of England is now q' of (D) and s' of (F); and the export of England is now Q' of (C) and S' of (E).

52. But we must introduce an additional consideration. The relative price of (C) and (E), both of which are produced in England, will depend on their relative cost in England, and will not be altered by the trade. And, in like manner, the relative price of (D) and (F), which are produced in Germany, will depend upon their relative cost in Germany, and will not be altered by the trade. This we may term the *principle of the relation of cost in the producing country*.

Hence the ratio of r' to p' is the same as the ratio of r to p. Let $r = \gamma p$, then $r' = \gamma p'$. In the same way if $R = \Gamma P$, $R' = \Gamma P'$. And hence x is the same for p' and r'; X, for P' and R'.

53. The quantity s' of (F) is equivalent, in Germany, to $\Gamma s'$ of (D): hence the whole import of England is equivalent to $q' + \Gamma s'$ of (D), which in terms of (C) is $p'q' + \Gamma p's'$. The export of England is Q' of (C) and S' of (F), equivalent to $Q' + \gamma S'$ of (C). Hence the equation of import and export, (37), becomes

$$p'q' + \Gamma p's' = Q' + \gamma S'.$$

In like manner the equation of import and export for Germany is

$$q' + \Gamma s' = P'Q' + \gamma P'S'.$$

Hence, if the equation of supply and demand for (E) and (F) (22) be respectively,

$$r's' = rs\,(1 - nx),\ R'S' = RS\,(1 - NX),$$

we shall have

$$q\frac{1 - mx}{1 - x} + \Gamma s\frac{1 - nx}{1 - x} = PQ(1 - MX) + \gamma PS(1 - NX).$$

But we have, as before, under the trade, by the uniformity of international prices, (36) $P'p' = 1$. And if, as before, $\frac{1}{Pp} = 1 - k$,

$(1 - X)(1 - x) = 1 - k$, whence $X = \frac{k - x}{1 - x}$.

Hence

$$q(1 - mx) + \Gamma s(1 - nx) = PQ\{1 - Mk - (1 - M)x\} + \gamma PS\,\}1 - NK - (1 - N)x\{,$$

$$\text{whence } x = \frac{q + \Gamma s - \{(1 - Mk)PQ + (1 - Nk)\gamma PS\}}{mq + n\Gamma s - \{(1 - M)PQ + \gamma(1 - N)PS\}}.$$

If we suppose that $n = m$ and $N = M$, this becomes

$$x = \frac{q + \Gamma s - (1 - Mk)\,P(Q + \gamma s)}{m(q + \Gamma s) - (1 - M)\,P(Q + \gamma s)}.$$

And if $P(Q + \gamma s) = n'(q + \Gamma s)$,

$$x = \frac{1 - n'(1 - Mk)}{m - n'(1 - M)},$$

as before (39). And the applicable values of x will occur only for values of n' between certain limits, as before. That is, the trade will be profitable to both countries only when the inequality of mutual demand is within certain moderate limits.

54. It is evident that if we had any other commodities (E_1) (E_2) exported by England, and any other commodities (F_1) (F_2) exported by Germany, we should have equations of the same form, to any number of terms.

55. If Pp be considerably greater than 1 (35), the range of values of n within which the trade is possible with mutual advantage will be wider.

Thus, let a yard of cloth (C) exchange for 1 bushel of corn (D) in England; but for 4 bushels in Poland. Here $p = 1$, $P = 4$. $Pp = 4$; $\therefore \frac{1}{Pp} = \frac{1}{4}$, $k = \frac{3}{4}$. Let $m = M = \frac{1}{2}$.

Hence n, (39), which must be between $\frac{1}{1 - Mk}$ and $\frac{1 - mk}{1 - k}$, (48), must be between $\frac{8}{5}$ and $\frac{5}{2}$.

56. For instance, if $n = 2$, $x = \frac{1}{2}$, whence $X = \frac{k - x}{1 - x} = \frac{1}{2}$;

$$\text{and (38) } q' = q\,\frac{1 - mx}{1 - x} = \frac{3}{2}q, \quad Q' = \frac{1 - MX}{1 - x} = \frac{3}{2}Q, \quad p' = \frac{1}{2}p, \quad P' = \frac{1}{2}P.$$

Here, since $PQ = nq$, and $P = 4$, $n = 2$, $2Q = q$.

In this state of the trade, a yard of cloth will exchange for 2 bushels of corn. Poland will produce and export $\frac{3}{2}q$ of corn, besides her own consumption. England will produce and export $\frac{3}{2}Q$ of cloth, which is $\frac{3}{4}q$, besides her own consumption. And in this case England has half as much more corn, Poland half as much more cloth, as before.

57. But this mutual advantage arises from the mutual demand being nearly equal. Poland is supposed to demand, at first, half as many yards of cloth as England demands bushels of corn, for $Q = \frac{1}{2}q$.

If this be otherwise: if, for example, Poland demand at first only one-fourth as many yards of cloth as England demands bushels of corn, or $Q = \frac{1}{4}q$: since $P = 4$, $PQ = q$; therefore $n = 1$.

In this case $\left(\text{supposing } m \text{ and } M \text{ each} = \frac{1}{2}\right)$ by (38) $q' = q\,\frac{1 - mx}{1 - x} = q\,\frac{2 - x}{2 - 2x}$, England's import,

also $P'Q'$ = England's export;

$$\text{and since } X = \frac{k - x}{1 - x} = \frac{3 - 4x}{4 - 4x}, \quad P'Q' = PQ\,(1 - MX) = PQ\,\frac{5 - 4x}{8 - 8x} = q\,\frac{5 - 4x}{8 - 8x}.$$

And the export will be less than the import for every positive value of x; that is, there cannot be any fall of the price of corn, with relation to cloth in England, so small as to make the imports and exports equal.

In this case the relative price of cloth and corn in England will under the trade be ultimately what it was before the trade; and therefore ultimately, the relation of the prices will also be the same in Poland. When this condition has been attained Poland will obtain the $\frac{1}{4}q$ yards of cloth which she requires, by exporting $\frac{1}{4}q$ bushels of corn, which will purchase $\frac{1}{4}q$ yards in England, though the cloth would have cost q bushels in Poland. Poland saves $\frac{3}{4}q$ bushels of corn: England gains nothing.

58. Since England gains nothing by the trade, why does she carry on the trade?

Because otherwise the Polish corn would be sold in England at a lower price than $p = 1$, (a bushel for a yard of cloth). And thus, capital and labour in England would be driven from corn growing to cloth making. When this transfer has begun, it will go on till capital has been transferred enough to produce the $\frac{1}{4}q$ yards of cloth which Poland wants, and then p will again become 1. There is a temporary depression of the price of corn, but the price of cloth is brought down to agree with it by the transfer of producing capital. The depression of the price of corn relative to cloth is only temporary, and has no permanent value which satisfies the equations of international demand and supply.

In this case England imports only part of her supply of corn, namely, $\frac{1}{4}q$, the remaining $\frac{3q}{4}$ are produced at home.

59. Upon the principles here laid down, can a country lose by foreign trade?

Upon the principles here laid down, a country cannot lose by foreign trade.

But among the principles here laid down is (44) the principle of *transferable capital*; that is, the principle that capital and labour can be transferred, without loss, from the production of one commodity to another, when the state of trade produces, or threatens, a diminution of profits in any branch of production.

This can never be exactly the case. In almost all transfer of capital there is loss. But the loss may be temporary: the gain, or saving of further loss is supposed to be enduring; therefore, on the assumption here made, there may still be a general saving in all foreign trade.

60. But if it be not true that capital and labour are transferable without enduring loss, the results here obtained are vitiated by the failure of that part of the foundation. If when a portion of a home-produced commodity is displaced in the home market by importation from a foreign country, the capital and labour thus set free cannot be employed in producing any other commodity for which there is a demand, the assumptions of our investigation fail.

If, for instance, the labourers displaced become paupers, and have to be supported by the country in which the displacement occurs, there may be loss to the country arising from the trade.

61. We have hitherto left out of consideration the cost of carriage of the exported and imported commodities: what will be the result of introducing this element into the calculation?

In this case, the two commodities (C) and (D) (35) will no longer exchange at precisely the same rate in the two countries. The commodity (D) which has to be carried to England, will be dearer there by its cost of carriage; and (C) will be dearer in Germany by the cost of carrying it from England.

Suppose that the cost of carrying a unit C to Germany is a fraction u of the price of C, and the cost of bringing a unit D to England is a fraction U of the price of D. The price of C in Germany is $P'D$, if D be the price of a unit of (D). Therefore the price of a unit C in England is $\frac{P'D}{1+u}$. Also the price of a unit D in England is $D(1+U)$. But the price of C in England is to the price of D as 1 to p'. Therefore

$$\frac{P\,p'D}{1+u} = D(1+U) \text{ and } P'p' = (1+U)(1+u).$$

This equation takes the place of the equation of the uniformity of international prices, $P'p' = 1$, (36) in the former investigation.

62. If, as before, $p' = P(1-x)$, $P' = P(1-X)$, and if we put

$$\frac{(1+U)(1+u)}{Pp} = 1 - k',$$

we shall have, as before, (39) the equations,

$$x = \frac{1 - n(1 - Mk')}{m - n(1 - M)} : \quad (1 - X)(1 - x) = 1 - k'.$$

63. There will be no export of C, and import of D, except

$$\frac{(1+U)(1+u)}{Pp} < 1.$$

Thus, if, as before, (40), $\frac{1}{Pp} = \frac{3}{4}$, but u and U each $= \frac{1}{6}$,

$$\frac{(1+U)(1+u)}{Pp} = \frac{49}{48},$$

and there will be no importation.

64. This may be illustrated by a numerical example. Let the commodities be cloth and linen.

Since $u = \frac{1}{6}$, $U = \frac{1}{6}$, for each 6 yards of C (cloth) manufactured in England, one yard must be manufactured, or the cost of one yard supplied, to provide for the expense of carriage to Germany: that is, the cost of 7 yards must be incurred to take 6 to Germany. In like manner the cost of 7 yards of D (linen) must be provided in order to take 6 to England.

Hence 98 ($= 2 \times 7 \times 7$) yards of cloth in England which is (on our former supposition) equivalent to 147 ($= 3 \times 7 \times 7$) yards of linen, being exported to Germany, is reduced to 84 ($= 2 \times 6 \times 7$) yards by the expense of carriage. And this in Germany exchanges for

168 (= 4 × 6 × 7) yards of linen. But this quantity is reduced to 144 (= 4 × 6 × 6) by the expense of carriage to England. Hence by exporting cloth and importing linen, England would only obtain 144 yards, instead of 147 which she could obtain by the same labour employed on linen at home.

Hence, though the price of linen as compared with cloth is only $\frac{3}{4}$ as great in Germany as in England, the expense of carriage prevents the trade being profitable.

65. In Mr. Mill's example (*Polit. Econ.* II. 125) the cost of carriage of each, C and D, is supposed equal, nearly, to $\frac{1}{17}$ of the price of D; which is equal, nearly, to $\frac{10}{17^2}$ of the price of C, or nearly to $\frac{1}{29}$. Hence $u = \frac{1}{29}$, $U = \frac{1}{17}$.

$$\frac{(1+U)(1+u)}{Pp} = \frac{30 \times 18 \times 3}{29 \times 17 \times 4} = \frac{1620}{1772} = 1 - \frac{1}{5}, \text{ nearly.}$$

And taking, as in (40), $m = M = \frac{2}{3}$, and making $k' = \frac{1}{5}$,

$$x = \frac{1 - n(1 - Mk')}{m - n(1 - M)} = \frac{15 - 13n}{10 - 5n}.$$

In order that there may be an applicable value of x, we must have (47) n between $\frac{15}{13}$ and $\frac{13}{12}$.

If n be $\frac{10}{9}$, which is between those limits,

$$x = \frac{5}{40} = \frac{1}{8}.$$

Also for X, $(1 - X)(1 - x) = 1 - k' = 1 - \frac{1}{5}$;

$$\text{whence } 1 - X = \frac{4}{5} \times \frac{8}{7} = \frac{32}{35}, \text{ and } X = \frac{3}{35}.$$

The relative price of D and C in England is $p' = p\frac{7}{8} = \frac{7}{12}$, for $p = \frac{2}{3}$, (40).

The relative price of C and D in Germany is $P' = P\frac{32}{35} = \frac{64}{35} = \frac{13}{7}$ nearly, for $P = 2$, (40).

In England 7 yards of cloth exchange for 12 of linen; in Germany 7 yards of cloth for 13 of linen nearly; the difference depends on the cost of carriage both ways.

66. In the preceding investigations, I have done little more than put into a general and algebraical form the reasonings which Mr. Mill has presented to his readers in numerical examples. But I think it will have appeared that by this mode of dealing with the subject, the limits of the truth of the theorems of the solutions of the problems are much more easily brought into

view, at least for the mathematical reader, than in the numerical way. It has been seen, for instance, (57), that if there be a trade between any two countries, say Poland and England, Poland exporting, we will suppose, corn, and England, cloth, the case in which the trade is advantageous to both countries is (on the suppositions made) confined within narrow limits, namely, when the mutual demand is very nearly equal: and that if the demand of Poland for our commodity, cloth, be much smaller than our demand for theirs, corn, the advantage, according to the mathematical principles here applied, rests entirely with Poland.

67. But there is another consideration to be attended to. All the principles here applied are *principles of equilibrium:* that is, principles depending on the assumption of an enduring state of equality on the two sides: such are the principles of uniformity of international prices (36), the equality of import and export (37), transferable capital (44). These principles are each expressed by an equation, which enters into the investigation. But in fact, we never can have an absolute equilibrium. Even if the principles just mentioned be true, they exist as tendencies only, not as conditions attained: just as the principle that water finds its level governs the state of fluids, (rivers and oceans) not by producing equilibrium, but by producing motion, the level never being attained. The equilibrium theory of the tides is highly imperfect, and in many respects erroneous: the equilibrium theory of trade is probably no less so. Yet the equilibrium theory of the tides has been very useful in suggesting the form and laws of many of the principal features of tidal phenomena; though it is absolutely necessary to obtain the amount and epoch of the inequalities by local observations. It is possible that, in like manner, the equilibrium theory of trade may suggest propositions respecting the natural progress of trade, which propositions may be afterwards confirmed, limited, and reduced to their due measure by experience.

TRINITY COLLEGE,
April 3, 1850.

VI. *Second Memoir on the Intrinsic Equation of a Curve and its Application.*

By W. WHEWELL, D.D., *Master of Trinity College, Cambridge.*

[Read *April* 15, 1850.]

38. The method of deducing properties of curves which I employed in a former memoir on the Intrinsic Equation of Curves[*] is capable of many other applications. I the more readily proceed to point out some of these, in consequence of remarks which have been made to me upon the former memoir.

Scruples have been entertained by some of my readers as to whether we rightly suppose the portion of curve, added after reaching a cusp, to be negative. It has been said that every cusp may be conceived to be the remnant left by a loop when the breadth of the loop vanishes; and as in a looped curve the increment of the length of the arc could nowhere become negative, it ought not to do so in the ultimate form of the looped curve.

39. In order to see whether this remark is generally true, let the intrinsic equation to a curve be $s = a\phi + b \sin \phi$. Therefore $\frac{ds}{d\phi} = a + b \cos \phi$; and if $b < a$, it is evident that the radius of curvature will vary between the limits $a + b$ and $a - b$, but will never become 0 or negative; and the curve will a looped curve.

But let us begin by considering $a < b$. When $a = 0$, we have $s = b \sin \phi$, the equation to a cycloid. And since $\frac{ds}{d\phi} = a + b \cos \phi$, it is evident that the curve is that which is produced by taking the involute of a cycloid, adding a to the describing radius. In fig. 1, if CD be a semi-cycloid, of which D is the vertex, and b the length, and if the string OQ unwrapping from OD, produce the equal semi-cycloid DB, then if OQ be produced to P, so that $QP = a$, P will describe a curve of which the equation is $\frac{ds}{d\phi} = a + b \cos \phi$, the curve proposed.

It is evident that we have a cusp when $\cos \phi = -\frac{a}{b}$, as at Z; and at Z'. The curve consists of sinuses alternately larger and smaller, as in the figure.

40. The base EC of the semicycloid is $= \frac{2b}{\pi}$. If this be $= a$, it is evident that the

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[*] Namely, an equation between s, the length of the curve, and ϕ, the angle of deviation from the original direction.

points Y, Y' which correspond to the vertex D of the evolute, will coincide; and the form of the curve will be as in figure 2. This is if $\frac{a}{b} = \frac{2}{\pi}$.

If $\frac{a}{b}$ be greater than this, the larger sinuses of the curve intersect, as in figure 3.

41. The cusps Z, Z' approach nearer to each other, as $\frac{a}{b}$ approaches nearer and nearer to 1. When $a = b$, the two cusps disappear, and the curve becomes looped. Yet at the vertex of the loop the radius of curvature is $= 0$.

When $a > b$, the curve is looped, and the radius of curvature never vanishes.

In this example the loop arises from the evanescence of *two* cusps; and if we take the order of changes inversely, the evanescence of the loop gives rise to two cusps. In the curve $s = a\phi + b \sin\phi$, the smaller sinus between two cusps is negative, and when a vanishes, the smaller and larger sinus become equal, which is the case of the cycloid; and hence in that case the alternate sinuses are negative.

42. The cycloid may however be considered as the ultimate form of a looped curve, namely, of the *protracted cycloid*. If while the circle VB, fig. 5, rolls upon the straight line DB, a point P in the radius QC produced describe a curve AP, this curve is the protracted cycloid.

To find the intrinsic equation to the protracted cycloid, let DA be the original position of QP; PM, and CE, perpendicular on AD; $AM = x$, $MP = y$, $VCP = \theta$. And let $CP = a$, $CB = ma$.

$$\text{Then } x = AM = EA - EM = CP - CN = a - a\cos\theta,$$

$$y = MP = MN + NP = DB + NP = BQ + NP = ma\theta + a\sin\theta.$$

Also the curve at P is perpendicular to BP: and the curve at A is perpendicular to DA. Hence the angle between BP and DA, that is, angle $PBN = \phi$.

$$\text{Whence } \tan\phi = \frac{NP}{BN} = \frac{\sin\theta}{m + \cos\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have $dx = a\sin\theta \,.\, d\theta$, $dy = a(\cos\theta + m)\,d\theta$: whence

$$ds = \sqrt{(dx^2 + dy^2)} = a\,d\theta \sqrt{(1 + m^2 + 2m\cos\theta)}. \quad . \quad . \quad . \quad (2)$$

$$\text{Also, from (1),} \qquad \sec^2\phi \,.\, d\phi = \frac{(1 + m\cos\theta)}{(m + \cos\theta)^2} d\theta:$$

$$\text{but } \sec^2\phi = 1 + \tan^2\phi = \frac{1 + m^2 + 2m\cos\theta}{(m + \cos\theta)^2}: \text{ by (1)}$$

$$\text{whence } \frac{d\theta}{d\phi} = \frac{1 + m^2 + 2m\cos\theta}{1 + m\cos\theta}.$$

And before $\dfrac{ds}{d\theta} = a\sqrt{(1 + m^2 + 2m\cos\theta)}$,

whence $\dfrac{ds}{d\phi} = \dfrac{a(1 + m^2 + 2m\cos\theta)^{\frac{3}{2}}}{1 + m\cos\theta}$; . . . . (3)

which is the intrinsic (differential) equation to the protracted cycloid.

To find s in terms of ϕ, we have (1) and (3). The integrations would give an elliptic transcendant. But we may integrate by the following construction.

43. Let ab, fig. 6 be an elliptical quadrant, in which

$$ac = a(1 + m) = DA, \quad bc = a(1 - m)$$

and let ad be a quadrant of a circle with radius ac, and let $dco = \frac{1}{2}\theta$, mpo an ordinate to the ellipse and circle.

Then $$tp^2 = tm^2 + mp^2 = \frac{mo^4}{cm^2} + \frac{bc^2}{ac^2}mo^2,$$

$$to^2 = tm^2 + mo^2 = \frac{mo^4}{cm^2} + mo^2.$$

Hence $$\frac{tp^2}{to^2} = \frac{\dfrac{mo^2}{cm^2} + \dfrac{bc^2}{ac^2}}{\dfrac{mo^2}{cm^2} + 1} = \frac{\dfrac{mo^2}{cm^2} + \dfrac{bc^2}{ac^2}}{\dfrac{ac^2}{cm^2}} = \frac{mo^2}{ac^2} + \frac{bc^2}{ac^2}\,\frac{cm^2}{ac^2},$$

$$= \cos^2\tfrac{1}{2}\theta + \frac{(1 - m)^2}{(1 + m)^2}\sin^2\tfrac{1}{2}\theta,$$

$$= \frac{1 + m^2 + 2m\cos\theta}{(1 + m)^2}.$$

But the differential of the arc bp is to the differential of the arc do as tp to do.

And it is evident that the differential of do is $ac\dfrac{d\theta}{2}$, or $\dfrac{a(1 + m)}{2}d\theta$. Hence

$$d\,.\,bp = \frac{a}{2}d\theta\sqrt{(1 + m^2 + 2m\cos\theta)}.$$

But by (2), $d\,.\,AP = a\,d\theta\sqrt{(1 + m^2 + 2m\cos\theta)}$.

Hence AP, fig. 5, $= 2bp$, fig. 6, since they begin together.

44. Now when m becomes 1, the protracted cycloid becomes the ordinary cycloid, and the loop vanishes. But in this case cb, $= a(1 - m)$, also vanishes, and the arc which was $2bp$ becomes $2cm$, and the increment becomes negative when we go beyond a. Therefore in this case we have a confirmation of the mode in which we have measured s.

The same would be the case if we had begun with the *contracted cycloid* or trochoid. The equations are the same as in the case just examined except that m is > 1. The curve becomes the common cycloid when $m = 1$, and the elliptical integration changes its nature for this case.

45. I will take some other cases of the intrinsic equation.

Let $\frac{ds}{d\phi} = a \sin \phi + b \sin 2\phi = a \sin \phi + 2b \sin \phi \cos \phi$.

It is evident that there is a cusp when $\sin \phi = 0$, and when $\cos \phi = -\frac{a}{2b}$.

Hence the form of the curve is that in figure 7.

We have $dy = ds \sin \phi = a \sin^2 \phi \, d\phi + 2b \sin^2 \phi \cos \phi$.

But $\int \sin^2 \phi \, d\phi = \frac{1}{2}\phi - \frac{1}{2} \sin \phi \cos \phi$.

Therefore $y = \frac{a}{2}\phi - \frac{a}{2} \sin \phi \cos \phi + \frac{2b}{3} \sin^3 \phi$.

From $\phi = 0$ to $\phi = \pi$, this $= \frac{\pi a}{2} = AM$.

Also $x = \int ds \cos \phi = \int (a \sin \phi \cos \phi + b \sin 2\phi \cos \phi) \, d\phi$.

$$= \int (a \sin \phi \cos \phi + 2b \sin \phi \cos^2 \phi) \, d\phi.$$

$$= C - \frac{a}{2} \cos^2 \phi - \frac{2b}{3} \cos^3 \phi$$

$$= \frac{a}{2} (1 - \cos^2 \phi) + \frac{2b}{3} (1 - \cos^3 \phi).$$

When $\phi = \pi$, this $= \frac{4b}{3} = MC$.

Hence AM depends on a alone, and MC on b alone.

46. Again, let $\frac{ds}{d\phi} = a \sin \phi + b \sin 3\phi = a \sin \phi + b\,(3 \sin \phi - 4 \sin^3 \phi)$

$$= (a + 3b) \sin \phi - 4b \sin^3 \phi.$$

There is a cusp when $\sin \phi = 0$, and when $\sin^2 \phi = \frac{a + 3b}{4b}$,

$$dy = ds \sin \phi = (a + 3b) \sin^2 \phi - 4b \sin^4 \phi,$$

$$dx = ds \cos \phi = (a + 3b) \sin \phi \cos \phi - 4b \sin^3 \phi \cos \phi,$$

$$x = \frac{(a + 3b)}{2} \sin^2 \phi - b \sin^4 \phi.$$

When $\phi = \frac{\pi}{2}$, $x = \frac{a + b}{2}$: when $\phi = \pi$, $x = 0$. Form as in figure 8.

47. All such curves as have equations of these forms may be called generally *cycloidal curves*. They have cusps and sinuses which, after going through a certain cycle, recur.

48. If we have $s =$ any series of integral powers of ϕ, we shall still have a curve of this cycloidal kind, as to its finite part.

Thus let $s =$ any integral function of $(n + 1)$ dimensions of ϕ.

Then $\frac{ds}{d\phi} = (\phi - \alpha)(\phi - \beta)(\phi - \gamma)$ &c. to n terms.

And there is a cusp when $\phi = \alpha$, when $\phi = \beta$, when $\phi = \gamma$:

And in the sinuses, s is alternately positive and negative.

With regard to the infinite branches,

If $s = a\phi^{n+1}$, we have $\frac{ds}{d\phi} = (n + 1)\, a\phi^n$

And it is evident that if n be positive, $\frac{ds}{d\phi}$ goes on increasing as ϕ increases, that is, as s increases. Hence there is an infinite diverging spiral

Therefore if $s =$ any integral fraction of ϕ, of $(n + 1)$ dimensions, the curve is a cycloidal formed curve having n cusps, and at each end an infinite diverging spiral. Fig. 9.

49. If n be negative, $\frac{ds}{d\phi}$ goes on diminishing as ϕ increases, that is, as s increases, if $n + 1$ be positive. In this case we have a converging infinite spiral.

If $n + 1$ be negative, (as well as n), $\frac{ds}{d\phi}$ goes on increasing as ϕ decreases, that is, as s increases; wherefore we have a diverging infinite spiral.

When n is negative and $n + 1$ positive, n is a negative proper fraction, and $n + 1$ a positive proper fraction: and if we get rid of fractional indices by involution, we shall have $\phi =$ some integral power of s; which equation, it appears, gives a converging spiral.

Let $\phi =$ any series of integral powers of s. Then we shall have

$$\phi = (s - a)(s - b)(s - c) \text{ \&c.}$$

ϕ will be alternately positive and negative with maxima and minima, and the curve will be a sinuous curve in the neighbourhood of the values $s = a, b, c$ &c., and will have a converging spiral at each end. Fig. 10.

These resemble the running pattern curves $\phi = m \sin s + n \sin 2s$, &c., except that they have not a cycle of recurrence, and have spiral tails.

50. The running pattern curve, noticed Art. 17, &c. of the former memoir, (of which the equation is $\phi = m \sin s$) may be further examined.

As we have remarked, Art. 34, to find the rectangular co-ordinates of this curve, we have

$$x = \int \frac{\cos\phi\, d\phi}{\sqrt{m^2 - \phi^2}}, \qquad y = \int \frac{\sin\phi\, d\phi}{\sqrt{m^2 - \phi^2}}.$$

Let $\int \frac{\cos\phi\, d\phi}{\sqrt{m^2 - \phi^2}} = A \sin^{-1}\frac{\phi}{m} + \sqrt{m^2 - \phi^2}\,(B\phi + C\phi^3 + D\phi^5 + \text{\&c.})$;

$$\therefore\ \frac{\cos\phi}{\sqrt{m^2-\phi^2}} = \frac{A}{\sqrt{m^2-\phi^2}} - \frac{\phi}{\sqrt{m^2-\phi^2}}(B\phi + C\phi^3 + D\phi^5 + \&c.)$$
$$+ \sqrt{m^2-\phi^2}\,(B + 3C\phi^2 + 5D\phi^4 + \&c.)$$

Hence $\cos\phi = A + m^2B - (B\phi^2 + C\phi^4 + D\phi^6 + \&c.)$

$$+ 3Cm^2\phi^2 + 5Dm^2\phi^4 + 7Em^2\phi^6 + \&c.$$
$$- (B\phi^2 + 3C\phi^4 + 5D\phi^6 + \&c.)$$

But $\cos\phi = 1 - \frac{\phi^2}{1.2} + \frac{\phi^4}{1.2.3.4} - \frac{\phi^6}{1.2.3.4.5.6}$.

Hence equating coefficients;

$$A + m^2B = 1,\quad 2B - 3Cm^2 = \frac{1}{1.2},\quad 4C - 5Dm^2 = -\frac{1}{1.2.3.4},\quad 6D - 7Em^2 = \frac{1}{1.2.3.4.5.6}.$$

Hence $D = \frac{1}{1.2.3.4.5.6.6} + \frac{7Em^2}{6}$:

$$C = -\frac{1}{1.2.3.4.4} + \frac{5Dm^2}{4} = -\frac{1}{1.2.3.4.4} + \frac{m^2}{1.2.3.4^2.6^2} + \frac{5.7.Em^4}{4.6}:$$

$$B = \frac{1}{1.2.2} + \frac{3Cm^2}{2} = \frac{1}{1.2.2} - \frac{m^2}{1.2^2.4^2} + \frac{m^4}{1.2^2.4^2.6^2} - \frac{3.5.7Em^6}{2.4.6}:$$

$$A = -m^2B = 1 - \frac{m^2}{2^2} + \frac{m^4}{2^2.4^2} - \frac{m^6}{2^2.4^2.6^2} - \&c.$$

Taking the integral from $\phi = 0$ to $\phi = m$, we have it $= A$.

51. In the same manner

let $\int\frac{\sin\phi\,d\phi}{\sqrt{m^2-\phi^2}} = a\sin^{-1}\frac{\phi}{m} + \sqrt{m^2-\phi^2}\,(b + c\phi^2 + d\phi^4 + e\phi^6 + \&c.)$

$$\frac{\sin\phi}{\sqrt{m^2-\phi^2}} = \frac{a}{\sqrt{m^2-\phi^2}} - \frac{\phi}{\sqrt{m^2-\phi^2}}(b + c\phi^2 + d\phi^4 + \&c.)$$
$$+ \sqrt{m^2-\phi^2}\,(2c\phi + 4d\phi^3 + 6e\phi^5 + \&c.)$$

Hence $\sin\phi = a - (b\phi + c\phi^3 + d\phi^5 + \&c.)$

$$+ 2cm^2\phi + 4dm^2\phi^3 + 6em^2\phi^5 + \&c.$$
$$- (2c\phi^3 + 4d\phi^5 + \&c.)$$

But $\sin\phi = \phi - \frac{\phi^3}{1.2.3} + \frac{\phi^6}{1.2.3.4.5} - \&c.$

Hence equating coefficients

$$a = 0;\ b - 2cm^2 = 1;\ 3c - 4d.m^2 = \frac{1}{1.2.3};\ 5d - 6em^2 = \frac{1}{1.2.3.4.5};\ \&c.$$

Hence $d = -\frac{1}{1.2.3.4.5.5} + \frac{6em^2}{5}$.

$$c = \frac{1}{1.2.3.3} + \frac{4dm^2}{3} = \frac{1}{1.2.3^2} - \frac{m^2}{1.2.3^2.5^2} + \frac{4.6.m^4e^2}{3.5},$$

$$b = -1 + 2cm^2 = -1 + \frac{m^2}{1^2.3^2} - \frac{m^4}{1^2.3^2.5^2} + \&c.$$

Taking the integral from $\phi = 0$ to $\phi = m$, it is $= -mb$.

52. As already remarked, Art. 34, there is a certain value of m for which the whole abscissa x, from $\phi = 0$ to $\phi = m$, is 0. That is

$$0 = 1 - \frac{m^2}{2^2} + \frac{m^4}{2^2.4^2} - \frac{m^6}{2^2.4^2.6^2} + \&c.$$

When $m = 1$, the series is evidently positive.

When $m = 2$, the series is $1 - 1 + \frac{1}{1^2.2^2} - \frac{1}{1^2.2^2.3^2} + \frac{1}{1^2.2^2.3^2.4^2} - \&c.$

This is evidently positive; therefore there is no value which makes the series 0 between $m = 1$ and $m = 2$.

Let $m^2 = 8$; the series is $1 - 2 + 1 - \frac{2}{3^2} + \frac{1}{2^2.3^2} - \frac{1}{2.3^2.5^2} + \&c.$

This is evidently negative; therefore there is a value which makes the series 0 between $m^2 = 4$ and $m^2 = 8$; or between $m = 2$ and $m = 2\sqrt{2}$.

53. In like manner there is a certain value of m which makes $b = 0$, that is

$$0 = 1 - \frac{m^2}{1^2.3^2} + \frac{m^4}{1^2.3^2.5^2} - \&c.$$

W. WHEWELL.

Trinity College,
April 15, 1850.

VII. *On the Knowledge of Body and Space.* *By* H. WEDGWOOD, ESQ., *Christ's College, Cambridge.*

[Read *March* 11, 1850.]

THERE is hardly a question in the entire range of mental philosophy, a satisfactory answer to which would be so influential in dissipating error and in furnishing the ground of solid science, as that which has for its object the origin of our notions of Body and Space.

The problem concerning the evidence for the existence of the material world which has given rise to so much ineffectual controversy and has been made the pivot of so many systems of philosophy can only be solved by a detailed exposition of the intellectual action involved in the apprehension of bodily existence, and the same line of inquiry is the quarter to which we must look for the removal of the difficulties which encumber the outset of Geometrical reasoning and throw so great incertitude on the entire theory of demonstrative knowledge. If every step in the process by which the phenomena of form and magnitude are apprehended in actual existence were clearly understood, it would appear what is the ultimate object of apprehension or of thought in each indivisible element of Surface and of Line (the raw materials of figure) and in what variety of relation the linear and superficial element is intrinsically capable of presenting itself to our imagination. Then by combining indefinite series of such elements in certain relations to each other we should attain the conception and establish the fundamental definition of the simplest species of Geometrical figure, as straight lines and plane surfaces; the laws of whose inmost constitution being thus laid bare should furnish us with adequate grounds for the investigation of all their essential relations, when employed in the construction of more complicated species of figure.

The investigation of the primary elements of figure would carry us too far for the limits of the present paper, in which we shall confine ourselves to the details of the active process by which we become acquainted with the existence of body in space.

It is hardly necessary to premise that we have no knowledge of Body by any of the five senses. What I immediately perceive by sense is the sensible phenomenon itself and not the bodily substance with which it may locally be connected either as proximate cause of the sensation, or as the organ by or in which it is felt. When I suffer under toothache, or when a pin is run into me unawares, the thing of which I have actual apprehension is the pain I suffer and not the bodily substance of the pin or of the tooth. When a gun goes off before my windows what I *hear* or perceive by the ear is neither the bodily gun nor the vibrations of the air by which the material action is conveyed to my ear, but the sound itself. When I gaze upon the stars the visible image before my eyes affords a substantive object of contemplation apart from all speculation as to the bodily nature of the object seen. Thus the exercise of the senses displays to us five elementary modes of being logically unconnected with the notion of bodily substance—five kinds of being upon which we may think independent of all intellectual reference to a bodily support. We have now to investigate the source of our knowledge of that material and extended universe in connexion with which all the phenomena of sense are exhibited to our apprehension, and of which the centre, to each individual, is his own living body.

The complexity of the problem was hardly suspected by Locke. "There is no idea" he says "which we receive more constantly from sensation than *solidity*" (or *bodily substance*). "The idea of solidity we receive by our touch," by the resistance, as he subsequently explains it, which body opposes to manual pressure. The idea of space he supposes to be acquired independently by touch and by sight, but how the same simple idea could be derived from the exercise of two distinct faculties, or how far touch, as understood by him, could be considered as an elementary faculty, it did not occur to him to inquire.

The relation between visible and tangible extension was investigated by Berkeley, who shewed that the fundamental standard of form and magnitude is that which is measured by the hand; our visual perceptions being mainly of importance as enabling us to judge of the corresponding phenomena of touch.

On closer examination the co-operation of two distinct powers became evident even in the tactual apprehension of extended things, viz. first, the sense of Touch properly so called in which the sentient being is no otherwise active than in adverting to the phenomena substantively displayed by organic action; and secondly, the muscular power exerted in moving our hand over the surface, or in pressing against the solid body of which we acquire sensible experience.

It was conclusively established by Reid and Thomas Brown that the phenomena of extension form no part of the original information of touch independent of muscular exertion: while a logical difficulty was supposed to forbid the obvious and apparently the only remaining supposition, that these conceptions are acquired by the motion of the hand or other organ. The knowledge of the movement of the hand, it was argued, plainly involves the knowledge of the object in motion, that is, the knowledge of a body of certain size, and form, and place. It would seem then that in trying to account for the knowledge of space from the motion of the hand we are compelled at the outset to give credit for the very notions which ought originally to arise out of the experience indicated in our reasoning, and Stewart expresses his belief in the impossibility of escape from the apparent paralogism.

The difficulty occurred to the modern school of French philosophy under a somewhat different aspect. "The utmost" says Cousin "that can be attained by touch (including the power of muscular action) and sight is the knowledge of body." But all bodies are endowed with a certain form and size, and occupy a certain place. The idea of space then (in which those of form, and size, and place, are comprehended) is logically involved in the idea of body. In the order of actual experience, on the contrary, the idea of body is antecedent to that of space. We have no conception of space until we are made acquainted with body, but simultaneous with the first apprehension of body the idea of space enters the mind by a principle or law of the understanding in virtue of which we necessarily judge that every body is comprehended in space. *Chronologically*, in the language of Cousin, the idea of space *supposes* or is preceded by that of body, while *logically* the idea of body comprehends, or essentially involves the idea of space. The same contrast between the logical and chronological order of dependence of correlative ideas is pointed out by Cousin in several other cases besides that of body and space, the conception of the logical antecedent being accounted for in every case by a separate principle of necessary judgment.

The value of such an explanation must depend upon the knowledge we may have of this supposed capacity in the understanding for the conception of relations and qualities logically involved in the notion of things made known by experience, but forming no part of that which experience itself reveals. If we have a clear conception or independent knowledge of such a mode of intellectual developement the theory may be admitted (as far as it goes) as a sound explanation of the several conceptions for which it professes to account; but if the principle of judgment to which the appeal is made in each of these cases, be known only by inference from the single fact which forms the subject of inquiry, it is hard to see what useful end can be promoted by the hypothesis. Now it cannot be pretended that we have any knowledge of the necessary principle assumed in accounting for the notion of space, except in as far as it may be inferred from finding that the first apprehension of body involves notions of Form and Extension, which do not appear to be the proper object of any recognized faculty, and which therefore must apparently be derived from some other principle than that of direct observation in actual existence. It is certain, moreover, that the theory of Cousin leaves the matter open after all to questions, a satisfactory answer to which would afford an effective solution of the problem on purely empirical principles.

The idea of space, it is said, arises spontaneously in the mind on the occasion of the first apprehension of body. But the apprehension of a finite body is not a momentary act. The knowledge of the form and size of the body apprehended is only acquired by degrees as we pass our hand over the solid surface. At what precise moment then in this operation does the body become clothed in the relations of space under the influence of the necessary judgment? And what is the exact purport of the judgment supplied on this occasion by the constitution of the understanding independent of experience?

Does the infant, the moment he first presses his hand against a bodily obstacle, before he has an opportunity of forming any judgment of its shape or magnitude, suddenly become inspired with the conception of space reaching away from the object made known by touch to an infinite distance in an infinite variety of directions? Or is the idea of space superinduced only after the infant has some experience of distance and direction as exemplified in bodily surface? In the latter alternative it is plainly supposed that the phenomena of distance and direction exemplified in bodily surface may be the object of positive experience, and it will accordingly be incumbent on the metaphysician to shew how that experience can be acquired by the exercise of the faculties employed in the apprehension of body. But if we succeed in tracing the notions of distance and direction to an empirical origin it may perhaps be found that not much of the notion of space will remain to be accounted for by a mysterious necessity.

On the other hand, if it be held, not only that the general notion of space is given by necessary judgment, but also the knowledge of the specific dimensions ascertained in passing our hand over a particular body, it must at least be admitted that our conception of the shape and dimensions of the body in question is practically dependant on the tactual sensation and muscular exertion of which we are conscious in the experiment, just as the particular colour or taste apprehended in the act of sight or of eating is dependant upon the physiological affections simultaneously taking place in the eye or the palate. It will then in the first place be incumbent on the supporter of the necessary theory to explain in what essential particular a revelation by necessary judgment so depending upon simultaneous modifications of the organic system differs from the ordinary experience of sense; but whether he succeed in establishing such a distinction or no; whether the muscular and sentient action practically efficient in the apprehension of a given body be considered as the *occasion* merely on which certain conceptions of form and magnitude are supplied from the inherent constitution of the understanding itself, or whether they be considered as elements of a complex operation in which the form and magnitude as well as the substance of the body under examination are made the object of direct experience; the investigation of the conditions of tactual sensation and muscular action in dependence on which the elementary notions of form and magnitude are originally revealed to the understanding, will remain a problem of equal interest.

The stumbling-block which has so long stood in the way of a successful theory of the notion of extension is the assumption that the intelligence can only derive information from the exercise of muscular power through a conception of the outward manifestation in the bodily member employed as special organ of the exertion. It is taken for granted that nothing can be learned from the action of the finger but what can be gathered from the knowledge of the motion effected in that member of our body, or prevented from taking place by some external obstacle.

In this assumption there is an entire oversight of the cardinal fact of instinctive action. The rational exercise of a power, or exertion to which the agent is led from regarding it as a means of obtaining some real or supposed gratification, obviously supposes a previous acquaintance with the power to be exerted, and as the knowledge of each elementary faculty can only be acquired by actual experience of the function it fulfils, it is plain that the rational (or as Cousin calls it the *voluntary*) exercise of each of our faculties must be preceded by what he terms the *spontaneous* exercise of the same power; that is to say, by action *spontaneous* as it were in the organic system unguided by the intention of the thinking faculty.

The knowledge of muscular action, or capacity of thinking on, and consequently of intending a definite exertion of muscular power can only be acquired by previous experience of that kind of action. It is plain therefore that there must be some provision in our sentient frame by means of which we may be excited to muscular action in total ignorance, as well of the power to be exerted, as of every object of thought the knowledge of which essentially depends upon the exercise of the same power—in ignorance accordingly of the existence of body of any kind, and especially of the bodily member employed as organ of the exertion. Such a provision may be found in the principle of Instinctive Action, the more complicated instances of which are so striking to the imagination where an animal is seen laboriously compassing a purpose of which he can have no previous conception. The extent to which we are indebted to the same principle in executing the ordinary functions of active life may easily escape observation.

The immediate incentive in instinctive action seems to consist in simple impressions of one or other of the senses, the influence of the sensation in inciting to action being sometimes matter of direct intuition, while at other times it can only be inferred from seeing that the sensation is the immediate antecedent of the instinctive action; but in either case the connexion between the sensation and the corresponding exertion is as profound a mystery as that between the physical affection of a special organ in sensation and the sensible experience of the organised being. When I draw my breath after holding it as long as I conveniently can, I am conscious of being immediately actuated by the extreme and rapidly increasing uneasiness I suffer, and not from any conviction of danger to be apprehended from a further continuance of the experiment. We observe that the infant begins to suck the moment the breast is placed between his lips, before he can have any knowledge of the mechanism of his mouth or any notion of the gratification to be attained by the exertion; but how the sensation of touch upon his lips (and probably also the accompanying smell) can induce the exertion of muscular power is a question beyond the reach of human investigation.

Again, if we place our finger within the palm of the hand of a very young infant unseen by him, even when his attention is engaged in another quarter, we shall find that he will immediately grasp our finger, of which his only intimation must arise from the sensation of touch produced by contact of the foreign body. On the other hand, we see him instinctively withdraw his limb from a burn or irritation of a painful nature at a period when, as remarked by Sir C. Bell, he makes no attempt with his hand to ward off the most painful operation on any other part of his body, shewing plainly that the sense of feeling alone does not originally involve any conception of the bodily member in which the pain is organically seated.

In like manner when the horse twitches the skin of his back in the place on which a fly has settled, he can have no objective knowledge of that particular portion of his skin, or of the muscles by the action of which the fly is shaken off, but he must be directly actuated by the tickling sensation or sting of the fly.

From these and similar facts it would appear that there is a constitutional connexion between the sense of touch and the muscular system, in virtue of which the sensations of touch operate as motives to muscular exertion, instinctively inciting the sentient being to make use of the member in which the sensation is felt in pursuit or avoidance of the external cause of the sensation, and subsequently guiding him in the voluntary execution of the like purposes.

Now let us endeavour to place ourselves in the intellectual position of the infant at the moment when he begins instinctively to close his hand upon the unseen finger. The first effect of the muscular action will be an increased experience of tactual sensation, arising from the larger surface of his hand brought into closer contact with our finger. But in addition to this increased activity of the sense of touch, it cannot be doubted that in some shape or another the infant will be cognizant of the resistance of the finger, the consciousness of which would form so large a portion of our own experience if we were in his position. He cannot be altogether without sense of the muscular exertion; otherwise the contraction of his hand on our finger would no more be entitled to the designation of *his act* than the beating of his heart or a cramp in his leg. The final test of agency is the consciousness of the agent himself; and this it is that leads me to say '*I breathe*,' but '*my heart beats*;' because, although the function of breathing goes on as well whether I think of it or no, yet whenever my attention is directed to the question, I am internally conscious of the exertion, while the action of my heart lies wholly without the sphere of my effort. But in our experiment the muscular exertion though instinctive is undoubtedly the act of the infant himself, and therefore must under some aspect or another come within the grasp of his intelligence. The experience however of the resisted exertion cannot at once give rise to the notion of resistance to motion, because the infant has as yet no knowledge of the bodily existence of his hand, and no intention of setting such an organ in motion. It seems to me that the effect of the entire action in his intelligence will be the direct apprehension of that which we understand by the term *body*, a complex object, consisting of *surface* (undeveloped as yet in form and magnitude), apprehensible by touch, and *substance*, revealed by resistance to muscular exertion, furnishing a new type of being essentially different from any of the five fundamental kinds with which the infant is hitherto acquainted.

The muscular power exerted by the infant in closing his hand on our finger, being instinctively guided by the sense of touch, his attention will be directed by the joint exercise of the faculties to a single object, which will appear as *solid* in virtue of the resistance it opposes to muscular exertion, while the impression it makes on the organ of external Touch will give it a definite *place* in the material world.

The apprehension of bodily substance as a direct and elementary operation of the intelligence was correctly referred by Locke to the exertion of muscular power, although he failed to distinguish the latter as a perceptive faculty from the passive sense of touch. "If any one ask me," he says, [II. 4. § 6.] "what this solidity" [or bodily substance] "is, I send him to his senses to inform him. Let him put a flint or a football between his hands and then endeavour to join them, and he will know. If he thinks this not a sufficient explication of solidity, what it is and wherein it consists; I promise to tell him when he tells me what thinking is or wherein it consists."

When I press my hand against the table I feel the solid substance of which it consists as a positive object of thought, as completely independent of any logical reference to the motion of my hand prevented from taking place by the interposition of the table, as is the thought of colour or of sound of any logical reference to the mechanism of the eye or the ear. In the latter case the affection of the eye or the ear by the luminous or aerial undulations is *physiologically* instrumental in the act of sight or of hearing, and precisely in the same way it appears to me that the resisted exertion of muscular power under the guidance of external touch is physiologically instrumental in the apprehension of bodily substance. The difference is that in the former case the organic condition instrumental in the display of the phenomenon is dependent upon forces external to the percipient being, of the action of which he has no immediate consciousness, while it consists in the latter case in the exertion of his own muscular power by the percipient himself. Seeing then that the percipient is necessarily cognisant of the physical means employed in the apprehension of body, it was an easy error to suppose that the thought of the means was an essential element in the thought of the object apprehended—the thought of the resisted movement of the hand in the conception of the resisting object. But if it be true that the connexion between the pressure of my hand against the table and the information of a solid obstacle thereby obtained is analogous to the connexion between the physical modification of the organic system in sight or hearing and the dependent sensation, there will be no greater difficulty in supposing the first apprehension of body (independent as yet of any reference to the notions of Form or Magnitude) to take place antecedent to all objective knowledge of the hand, than there is in supposing a sensation of sight or of hearing antecedent to the earliest knowledge of the eye or the ear.

The relation between body and space may be illustrated by comparison with the case of light and darkness; the second of the two correlatives in both instances belonging to the class called by Locke *positive ideas from negative causes.* It is certain that a man born blind can have as little knowledge of black or darkness as he has of colour or of light; but as soon as his eye has once been opened to the visible world he will be able to look in a direction from whence no light reaches the eye, or even to try to see in the total absence of all light, when the effect of the effort will be to direct his attention to a phenomenon, which, under the name of black or darkness, will constitute as positive an object of thought as that which arises from the actual impact of light upon the retina.

In like manner it is generally admitted that the infant antecedent to the first apprehension of body is as ignorant of the space which surrounds him, as the born-blind is of darkness. But as soon as the infant has once apprehended *body*, without as yet having any thought of his own corporeal frame, he will be able (like the blind man, who after recovering his sight, tries to see in the absence of light) to seek with the muscular organ after the phenomenon with which he has newly become acquainted; that is, to feel for body in a direction in which there is none within his reach, when of course he will move his hand freely through space. The question is, what effect will such an action produce on the knowledge of the agent? Under what aspect can such an action be contemplated in the mind of a being who has no thought of himself and no conception of a bodily instrument in muscular exertion? It cannot be supposed that the infant in stretching out his hand in the search after body will be sensible of none of those fundamental impressions by which, if the action were ours, we should appreciate the extent of the exertion. He cannot be wholly unconscious of an effort not induced by mere organic excitement but aimed at an object of distinct thought, viz. the body represented to his imagination. It seems to me that the case is precisely analogous to that of the born-blind who is trying to see when first replaced in darkness after the recovery of vision; and as, in the latter case the attempt to exercise the visual faculty has the effect of directing the attention of the sentient being to the darkness which meets his efforts, and thereby becomes manifest to his intelligence; so it appears to me the attempt to apprehend body with the organ of external touch will direct the attention of the infant to the *space* traversed by his hand, as a positive phenomenon, exhibiting an elementary kind of being, the idea of which will thus be acquired in the same empirical manner by the exercise of the same faculties with the idea of Body.

When once we have learned to consider Space as the object revealed by muscular effort in the absence of Body, the resistance of a rigid body to the pressure of our hand will be felt as the action of the bodily substance opposing our effort and preventing us from obtaining experience of space lying beyond the resisting surface, and pervading the solid substance underneath.

In process of time circumstances will arise which will lead us to pass our hand over the surface of bodies in a lateral direction, when the absence of resistance to muscular exertion will give experience of space, while the sense of touch will give witness to the continued presence of bodily surface. The coexistence of Space and bodily Surface, or lateral extension of Surface, will thus be known by actual experience, and the notion of Body will be completed as occupying space both in breadth and depth.

Having thus succeeded in tracing step by step the course of experience terminating in the actual apprehension of space by the exercise of faculties whose operation is well understood, it might seem superfluous to turn back to *à priori* arguments against the possibility of an empirical origin of the idea. There are, however, two objections taken on so high a ground and on which so much stress has been laid, that it will be worth while to meet them with a direct answer instead of trusting to the practical refutation of the foregoing investigation.

It is said that experience is conversant only with the limited and the finite, and is therefore necessarily incapable of giving rise to the knowledge of that infinitude which is an essential attribute of space. It should, however, be borne in mind that the knowledge derived from the exercise of the senses is not confined within the limits of actual or even possible experience. We have seen that the sense of resistance gives rise to the conception of the space pervading the substance of the resisting body, which can by no possibility be known by actual apprehension.

When once the infant has attained to the conception of space, the mode in which he obtains the knowledge of distances beyond the immediate reach of his limbs may easily be understood. He learns by degrees his power of transferring his entire body from place to place and gradually forms a more and more extensive notion of the world in which he lives. He has only to imagine the continuance of action with which he is familiar in order to conceive the extension of space in any given direction to an indefinite distance, and when we seek for the utmost limits to which such extension can be carried, we find that the only positive obstacle to the continued experience of space of which we can conceive the possibility is the occurrence of a bodily obstacle. But the imagination, as we have seen, on the occurrence of bodily resistance supplies us with the notion of ulterior space in the direction in which we are prevented from pursuing our experience, and thus we are driven to the conception of space stretching away in all directions around us without limit, and further conception of infinity than this can be entertained by no one.

Again it is said that we can conceive the destruction of all the bodies in the universe but not the destruction of a single particle of space, and so great a difference is this distinction supposed to imply between the ideas of body and space, as to give rise to the assumption that they must be derived from fundamentally different principles in the understanding. The idea of body it is said is *contingent;* it involves an object which may or may not be given in actual existence, while space, being a thing of which it is impossible to conceive the non-existence constitutes the subject of a necessary idea; and hence it is concluded the notion of space must have a deeper-seated origin in the constitution of the mind than simple experience. I confess I see no connexion whatever between the premises and the conclusion in the foregoing syllogism. For how does it appear *à priori* that such an origin as that attributed by Cousin to the idea of space should render it impossible to conceive the destruction of the thing conceived?

Moreover the necessity of the idea of space (understanding thereby the impossibility of divesting ourselves of the notion, or conceiving the non-existence of the thing itself) may, as far as it can truly be admitted, be satisfactorily accounted for from the negative foundation of the idea as expounded in the preceding pages. The sole external condition necessary for the apprehension of space by the exercise of muscular power being the absence of body, the manifestation of space does not depend upon the positive action of any external principle the removal of which would be felt as the destruction of the phenomenon. So long therefore as we retain the consciousness of touch and muscular activity we find it impossible to escape from the notion of the space required for the exercise of those faculties. But if we suppose a being whose whole experience was confined to sound, and smell, and taste, and passive touch, he could have no conception of space, and it is hard to say that an extended universe would in the nature of things be necessary for his existence; though of course we can form no positive conception of the mode in which an unembodied life could actually be carried on.

VIII. *On the numerical Calculation of a Class of Definite Integrals and Infinite Series.* *By* G. G. STOKES, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *March* 11, 1850.]

IN a paper "On the Intensity of Light in the neighbourhood of a Caustic*," Mr. Airy the Astronomer Royal has shown that the undulatory theory leads to an expression for the illumination involving the square of the definite integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$, where m is proportional to the perpendicular distance of the point considered from the caustic, and is reckoned positive towards the illuminated side. Mr. Airy has also given a table of the numerical values of the above integral extending from $m = -4$ to $m = +4$, at intervals of 0.2, which was calculated by the method of quadratures. In a Supplement to the same paper† the table has been re-calculated by means of a series according to ascending powers of m, and extended to $m = \pm 5.6$. The series is convergent for all values of m, however great, but when m is at all large the calculation becomes exceedingly laborious. Thus, for the latter part of the table Mr. Airy was obliged to employ 10-figure logarithms, and even these were not sufficient for carrying the table further. Yet this table gives only the first two roots of the equation $W = 0$, W denoting the definite integral, which answer to the theoretical places of the first two dark bands in a system of spurious rainbows, whereas Professor Miller was able to observe 30 of these bands. To attempt the computation of 30 roots of the equation $W = 0$ by means of the ascending series would be quite out of the question, on account of the enormous length to which the numerical calculation would run.

After many trials I at last succeeded in putting Mr. Airy's integral under a form from which its numerical value can be calculated with extreme facility when m is large, whether positive or negative, or even moderately large. Moreover the form of the expression points out, without any numerical calculation, the law of the progress of the function when m is large. It is very easy to deduce from this expression a formula which gives the i^{th} root of the equation $W = 0$ with hardly any numerical calculation, except what arises from merely passing from $\left(\frac{m}{3}\right)^{\frac{3}{2}}$, the quantity given immediately, to m itself.

The ascending series in which W may be developed belongs to a class of series which are of constant occurrence in physical questions. These series, like the expansions of ϵ^{-x}, $\sin x$, $\cos x$, are convergent for all values of the variable x, however great, and are easily calculated numerically when x is small, but are extremely inconvenient for calculation when x is large, give no indication of the law of progress of the function, and do not even make known what the function becomes when $x = \infty$. These series present themselves, sometimes as developements of definite integrals to which we are led in the first instance in the solution of physical problems, sometimes as the integrals of linear differential equations which do not admit of integration in finite terms. Now the method which I have employed in the case of the integral W appears to be of very general application to series of this class. I shall attempt here to give some sort of idea of it, but it does not well admit of being described in general terms, and it will be best understood from examples.

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* *Camb. Phil. Trans.* Vol. VI. p. 379.

† Vol. VIII. p. 595.

Suppose then that we have got a series of this class, and let the series be denoted by y or $f(x)$, the variable according to ascending powers of which it proceeds being denoted by x. It will generally be easy to eliminate the transcendental function $f(x)$ between the equation $y = f(x)$ and its derivatives, and so form a linear differential equation in y, the coefficients in which involve powers of x. This step is of course unnecessary if the differential equation is what presented itself in the first instance, the series being only an integral of it. Now by taking the terms of this differential equation in pairs, much as in Lagrange's method of expanding implicit functions which is given by Lacroix[*], we shall easily find what terms are of most importance when x is large: but this step will be best understood from examples. In this way we shall be led to assume for the integral a circular or exponential function multiplied by a series according to descending powers of x, in which the coefficients and indices are both arbitrary. The differential equation will determine the indices, and likewise the coefficients in terms of the first, which remains arbitrary. We shall thus have the complete integral of the differential equation, expressed in a form which admits of ready computation when x is large, but containing a certain number of arbitrary constants, according to the order of the equation, which have yet to be determined.

For this purpose it appears to be generally requisite to put the infinite series under the form of a definite integral, if the series be not itself the developement of such an integral which presented itself in the first instance. We must now endeavour to determine by means of this integral the leading term in $f(x)$ for indefinitely large values of x, a process which will be rendered more easy by our previous knowledge of the *form* of the term in question, which is given by the integral of the differential equation. The arbitrary constants will then be determined by comparing the integral just mentioned with the leading term in $f(x)$.

There are two steps of the process in which the mode of proceeding must depend on the particular example to which the method is applied. These are, first, the expression of the ascending series by means of a definite integral, and secondly, the determination thereby of the leading term in $f(x)$ for indefinitely large values of x. Should either of these steps be found impracticable, the method does not on that account fall to the ground. The arbitrary constants may still be determined, though with more trouble and far less elegance, by calculating the numerical value of $f(x)$ for one or more values of x, according to the number of arbitrary constants to be determined, from the ascending and descending series separately, and equating the results.

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[*] *Traité du Calcul*, &c. Tom. I. p. 104.

In this paper I have given three examples of the method just described. The first relates to the integral W, the second to an infinite series which occurs in a great many physical investigations, the third to the integral which occurs in the case of diffraction with a circular aperture in front of a lens. The first example is a good deal the most difficult. Should the reader wish to see an application of the method without involving himself in the difficulties of the first example, he is requested to turn to the second and third examples.

FIRST EXAMPLE.

1. Let it be required to calculate the integral

$$W = \int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw \quad . \quad . \quad . \quad (1)$$

for different values of m, especially for large values, whether positive or negative, and in particular to calculate the roots of the equation $W = 0$.

2. Consider the integral

$$u = \int_0^\infty \epsilon^{-(\cos 3\theta + \sqrt{-1}\sin 3\theta)(x^3 - nx)}\,dx, \quad . \quad . \quad (2)$$

where θ is supposed to lie between $-\frac{\pi}{6}$ and $+\frac{\pi}{6}$, in order that the integral may be convergent.

Putting

$$x = (\cos\theta - \sqrt{-1}\sin\theta)\,z,$$

we get $dx = (\cos\theta - \sqrt{-1}\sin\theta)\,dz$, and the limits of z are 0 and ∞; whence, writing for shortness

$$p = (\cos 2\theta + \sqrt{-1}\sin 2\theta)\,n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we get

$$u = (\cos\theta - \sqrt{-1}\sin\theta)\int_0^\infty \epsilon^{-(z^3 - pz)}\,dz^{*}. \quad . \quad . \quad . \quad . \quad (4)$$

3. Let now θ, which hitherto has been supposed less than $\frac{\pi}{6}$, become equal to $\frac{\pi}{6}$. The integral obtained from (2) by putting $\theta = \frac{\pi}{6}$ under the integral sign may readily be proved to

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* The legitimacy of this transformation rests on the theorem that if $f(x)$ be a continuous function of x, which does not become infinite for any real or imaginary, but finite, value of x, we shall obtain the same result for the integral of $f(x)\,dx$ between two given real or imaginary limits through whatever series of real or imaginary values we make x pass from the inferior to the superior limit. It is unnecessary here to enunciate the theorem which applies to the case in which $f(x)$ becomes infinite for one or more real or imaginary values of x. In the present case the limits of x are 0 and real infinity, and accordingly we may first integrate with respect to z from 0 to a large real quantity z_1, θ' (which is supposed to be written for θ in the expression for x) being constant, then leave z equal to z_1, make θ' vary, and integrate from θ to 0, and lastly make z_1 infinite. But it may be proved without difficulty, (and the proof may be put in a formal shape as in Art. 8,) that the second integral vanishes when z_1 becomes infinite, and consequently we have only to integrate with respect to z from 0 to real infinity.

be convergent. But this is not sufficient in order that we may be at liberty to assert the equality of the results obtained from (2), (4) by putting $\theta = \frac{\pi}{6}$ before integration. It is moreover necessary that the convergency of the integral (2) should not become infinitely slow when θ approaches indefinitely to $\frac{\pi}{6}$, in other words, that if X be the superior limit to which we must integrate in order to render the remainder, or rather its modulus, less than a given quantity which may be as small as we please, X should not become infinite when θ becomes equal to $\frac{\pi}{6}$[*]. This may be readily proved in the present case, since the integral (2) is even more convergent than the integral

$$\int_0^\infty \epsilon^{-\sqrt{-1}\sin 3\theta\,(x^3 - n x)}\,dx,$$

which may be readily proved to be convergent.

Putting then $\theta = \frac{\pi}{6}$ in (2) and (4), we get

$$u = \int_0^\infty \cos(x^3 - n x)\,dx - \sqrt{-1}\int_0^\infty \sin(x^3 - n x)\,dx, \quad . \quad . \quad (5)$$

$$u = \left(\cos\frac{\pi}{6} - \sqrt{-1}\sin\frac{\pi}{6}\right)\int_0^\infty \epsilon^{-(x^3 - p x)}\,dx, \quad . \quad . \quad . \quad (6)$$

where

$$p = \left(\cos\frac{\pi}{3} + \sqrt{-1}\sin\frac{\pi}{3}\right)n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let

$$u = U - \sqrt{-1}\,U',$$

and in the expression for U got from (5) put

$$x = \left(\frac{\pi}{2}\right)^{\frac{1}{3}} w, \qquad n = \left(\frac{\pi}{2}\right)^{\frac{2}{3}} m; \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

then we get

$$W = \left(\frac{\pi}{2}\right)^{-\frac{1}{3}} U. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

4. By the transformation of u from the form (5) to the form (6), we are enabled to differentiate it as often as we please with respect to n by merely differentiating under the integral sign. By expanding the exponential $\epsilon^{p x}$ in (6) we should obtain u, and therefore U, in a series according to ascending powers of n. This series is already given in Mr. Airy's Supplement. It is always convergent, but is not convenient for numerical calculation when n is large.

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[*] See Section III. of a paper, "On the Critical Values of the sums of Periodic Series." *Camb. Phil. Trans.* Vol. VIII. p. 561.

We get from (6)

$$\frac{d^2u}{dp^2}-\frac{pu}{3}=\left(\cos\frac{\pi}{6}-\sqrt{-1}\sin\frac{\pi}{6}\right)\int_0^\infty \epsilon^{-(x^3-px)}\left(x^2-\frac{p}{3}\right)dx=\frac{1}{3}\left(\cos\frac{\pi}{6}-\sqrt{-1}\sin\frac{\pi}{6}\right),$$

which becomes by (7)

$$\frac{d^2u}{dn^2}+\frac{n}{3}u=\tfrac{1}{3}\sqrt{-1}. \quad \dots\dots\dots \quad (10)$$

Equating to zero the real part of the first member of this equation, we get

$$\frac{d^2U}{dn^2}+\frac{n}{3}U=0. \quad \dots\dots\dots \quad (11)$$

5. We might integrate this equation by series according to ascending powers of n, and we should thus get, after determining the arbitrary constants, the series which have been already mentioned. What is required at present is, to obtain for U an expression which shall be convenient when n is large.

The form of the differential equation (11) already indicates the general form of U for large values of n. For, suppose n large and positive, and let it receive a small increment δn. Then the proportionate increment of the coefficient $\frac{n}{3}$ will be very small; and if we regard this coefficient as constant, and δn as variable, we shall get for the integral of (11)

$$U=N\cos\left\{\sqrt{\left(\frac{n}{3}\right)}\,.\,\delta n\right\}+N'\sin\left\{\sqrt{\left(\frac{n}{3}\right)}\,.\,\delta n\right\}, \quad \dots \quad (12)$$

where N, N' are regarded as constants, δn being small, which does not prevent them from being in the true integral of (11) slowly varying functions of n. The approximate integral (12) points out the existence of circular functions such as $\cos f(n)$, $\sin f(n)$ in the true integral; and since $\sqrt{\left(\frac{n}{3}\right)}\,.\,\delta n$ must be the small increment of $f(n)$, we get $f(n)=\frac{2}{3}\sqrt{\frac{n^3}{3}}$, omitting the constant, which it is unnecessary to add. When n is negative, and equal to $-n'$, the same reasoning would point to the existence of exponentials with $\pm\frac{2}{3}\sqrt{\frac{n'^3}{3}}$ in the index. Of course the exponential with a positive index will not appear in the particular integral of (11) with which we are concerned, but both exponentials would occur in the complete integral. Whether n be positive or negative, we may, if we please, employ exponentials, which will be real or imaginary as the case may be.

6. Assume then to satisfy (11)

$$U=\epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left\{An^\alpha+Bn^\beta+Cn^\gamma+\dots\right\}^*, \quad \dots \quad (13)$$

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* The idea of multiplying the circular functions by a series according to descending powers of n was suggested to me by

where $A, B, C \ldots \alpha, \beta, \gamma \ldots$ are constants which have to be determined. Differentiating, and substituting in (11), we get

$$\alpha(\alpha-1)An^{\alpha-2}+\beta(\beta-1)Bn^{\beta-2}+\ldots$$
$$+\frac{\sqrt{-1}}{2\sqrt{3}}\left\{(4\alpha+1)An^{\alpha-\frac{1}{2}}+(4\beta+1)Bn^{\beta-\frac{1}{2}}+\ldots\right\}=0.$$

As we want a series according to descending powers of n, we must put

$$4\alpha+1=0,\quad \beta=\alpha-\tfrac{3}{2},\quad \gamma=\beta-\tfrac{3}{2}\ldots$$

$$B=2\sqrt{-3}\;\frac{\alpha(\alpha-1)}{4\beta+1}A,\quad C=2\sqrt{-3}\;\frac{\beta(\beta-1)}{4\gamma+1}B\ldots\ldots$$

whence

$$U=An^{-\frac{1}{4}}\epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left\{1-\frac{1.5}{1}\,\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}+\frac{1.5.7.11}{1.2}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^2\right.$$
$$\left.-\frac{1.5.7.11.13.17}{1.2.3}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^3+\ldots\right\}.\quad .\quad .\quad (14)$$

By changing the sign of $\sqrt{-1}$ both in the index of ϵ and in the series, writing B for A, and adding together the results, we shall obtain the complete integral of (11) with its two arbitrary constants. The integral will have different forms according as n is positive or negative.

First, suppose n positive. Putting the function of n of which A is the coefficient at the second side of (14) under the form $P+\sqrt{-1}\,Q$, and observing that an expression of the form

$$A(P+\sqrt{-1}\,Q)+B(P-\sqrt{-1}\,Q),$$

where A and B are imaginary arbitrary constants, and which is supposed to be real, is equivalent to $AP+BQ$, where A and B are real arbitrary constants, we get

$$U=An^{-\frac{1}{4}}\left(R\cos\frac{2}{3}\sqrt{\frac{n^3}{3}}+S\sin\frac{2}{3}\sqrt{\frac{n^3}{3}}\right)+Bn^{-\frac{1}{4}}\left(R\sin\frac{2}{3}\sqrt{\frac{n^3}{3}}-S\cos\frac{2}{3}\sqrt{\frac{n^3}{3}}\right),\quad .\ (15)$$

where

$$\left.\begin{aligned}R&=1-\frac{1.5.7.11}{1.2.16^2.3n^3}+\frac{1.5.7.11.13.17.19.23}{1.2.3.4.16^4.3^2n^6}\ \ldots\ldots\\ S&=\frac{1.5}{1.16(3n^3)^{\frac{1}{2}}}-\frac{1.5.7.11.13.17}{1.2.3.16^3(3n^3)^{\frac{3}{2}}}+\ldots\ldots\end{aligned}\right\}\quad .\ .\ .\ .\ .\ (16)$$

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seeing in Moigno's *Repertoire d'optique moderne*, p. 189, the following formulæ which M. Cauchy has given for the calculation of Fresnel's integrals for large, or moderately large, values of the superior limit:

$$\int_0^\infty\cos\frac{\pi}{2}x^2dx=\frac{1}{2}-N\cos\frac{\pi}{2}m^2+M\sin\frac{\pi}{2}m^2;$$
$$\int_0^\infty\sin\frac{\pi}{2}x^2dx=\frac{1}{2}-M\cos\frac{\pi}{2}m^2-N\sin\frac{\pi}{2}m^2;$$

where

$$M=\frac{1}{m\pi}-\frac{1.3}{m^5\pi^3}+\frac{1.3.5.7}{m^9\pi^5}-\ldots;\quad N=\frac{1}{m^3\pi^2}-\frac{1.3.5}{m^7\pi^4}+\ldots.$$

The demonstration of these formulæ will be found in the 15th Volume of the *Comptes Rendus*, pp. 554 and 573. They may be readily obtained by putting $\pi x^2=2x$, and integrating by parts between the limits $\frac{1}{2}\pi m^2$ and ∞ of x.

Secondly, suppose n negative, and equal to $-n'$. Then, writing $-n'$ for n in (14), and changing the arbitrary constant, and the sign of the radical, we get

$$U = Cn'^{-\frac{1}{4}}\epsilon^{-\frac{2}{3}\sqrt{\frac{n'^3}{3}}}\left\{1 - \frac{1.5}{1.16\,(3n^3)^{\frac{1}{2}}} + \frac{1.5.7.11}{1.2.16.3n^3} - \dots\dots\right\}. \quad . \quad . \quad . \quad (17)$$

It is needless to write down the part of the complete integral of (11) which involves an exponential with a positive index, because, as has been already remarked, it does not appear in the particular integral with which we are concerned.

7. When n or n' is at all large, the series (16) or (17) are at first rapidly convergent, but they are ultimately in all cases hypergeometrically divergent. Notwithstanding this divergence, we may employ the series in numerical calculation, provided we do not take in the divergent terms. The employment of the series may be justified by the following considerations.

Suppose that we stop after taking a finite number of terms of the series (16) or (17), the terms about where we stop being so small that we may regard them as insensible; and let U_1 be the result so obtained. From the mode in which the constants A, B, C,... α, β, γ... in (13) were determined, it is evident that if we form the expression

$$\frac{d^2 U_1}{dn^2} + \frac{n}{3}U_1, \text{ or } \frac{d^2 U_1}{dn'^2} - \frac{n'}{3}U_1,$$

according as n is positive or negative, the terms will destroy each other, except one or two at the end, which remain undestroyed. These terms will be of the same order of magnitude as the terms at the part of the series (16) or (17) where we stopped, and therefore will be insensible for the value of n or n' for which we are calculating the series numerically, and, much more, for all superior values. Suppose the arbitrary constants A, B in (16) determined by means of the ultimate form of U for $n = \infty$, and C in (17) by means of the ultimate form of U for $n' = \infty$. Then U_1 satisfies exactly a differential equation which differs from (11) by having the zero at the second side replaced by a quantity which is insensible for the value of n or n' with which we are at work, and which is still smaller for values comprised between that and the particular value, (namely ∞,) by means of which the arbitrary constants were determined so as to make U_1 and U agree. Hence U_1 will be a near approximation to U. But if we went too far in the series (16) or (17), so as, after having gone through the insensible terms, to take in some terms which were not insensible, the differential equation which U_1 would satisfy exactly would differ sensibly from (11), and the value of U_1 obtained would be faulty.

8. It remains to determine the arbitrary constants A, B, C. For this purpose consider the integral

$$Q = \int_0^\infty \epsilon^{-x^3 + 3q^2 x}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where q is any imaginary quantity whose amplitude does not lie beyond the limits $-\frac{\pi}{6}$ and

$+\frac{\pi}{6}$. Since the quantity under the integral sign is finite and continuous for all finite values of x, we may, without affecting the result, make x pass from its initial value 0 to its final value ∞ through a series of imaginary values. Let then $x = q + y$, and we get

$$Q = \epsilon^{2q^3} \int_{-q}^{\infty} \epsilon^{-y^3 - 3qy^2} dy,$$

where the values through which y passes in the integration are not restricted to be such as to render x real. Putting $y = (3q)^{-\frac{1}{2}} t$, where that value of the radical is supposed to be taken which has the smallest amplitude, we get

$$Q = (3q)^{-\frac{1}{2}} \epsilon^{2q^3} \int \epsilon^{-(3q)^{-\frac{1}{2}} t^3 - t^2} dt. \quad . \quad . \quad . \quad . \quad (19)$$

The limits of t are $-3^{\frac{1}{2}} q^{\frac{3}{2}}$ and an imaginary quantity with an infinite modulus and an amplitude equal to $\frac{1}{2}a$, where a denotes the amplitude of q. But we may if we please integrate up to a real quantity ρ, and then, putting $t = \rho \epsilon^{\theta \sqrt{-1}}$, and leaving ρ constant, integrate with respect to θ from 0 to $\frac{1}{2}a$, and lastly put $\rho = \infty$. The first part of the integral will be evidently convergent at the limit ∞, since the amplitude of the coefficient of t^3 in the index does not lie beyond the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; and calling the two parts of the integral with respect to t in (19) T, T_4, we get

$$T = \int_{-3^{\frac{1}{2}} q^{\frac{3}{2}}}^{\infty} \epsilon^{-(3q)^{-\frac{1}{2}} t^3 - t^2} dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$T_4 = \text{limit}_{(\rho = \infty)}\, \rho \sqrt{-1} \int_0^{\frac{1}{2}a} \epsilon^{-(3q)^{-\frac{1}{2}} \rho^3 \epsilon^{3\theta\sqrt{-1}} - \rho^2 \epsilon^{2\theta\sqrt{-1}} + \theta\sqrt{-1}} d\theta. \quad . \quad (21)$$

We shall evidently obtain a superior limit to either the real or the imaginary part of T_4 by reducing the expression under the integral sign to its modulus. The modulus is $\epsilon^{-\Theta}$ where

$$\Theta = (3c)^{-\frac{1}{2}} \rho^3 \cos(3\theta - \tfrac{3}{2}a) + \rho^2 \cos 2\theta,$$

c being the modulus of q. The first term in this expression is never negative, being only reduced to zero in the particular case in which $\theta = 0$ and $a = \pm\frac{\pi}{6}$. The second term is never less than $\rho^2 \cos\frac{\pi}{3}$ or $\frac{1}{2}\rho^2$, and is in general greater. Hence both the real and the imaginary parts of the expression of which T_4 is the limit are numerically less that $\frac{1}{2} a \rho \epsilon^{-\frac{1}{2}\rho^2}$, which vanishes when $\rho = \infty$, and therefore $T_4 = 0$. Hence we have rigorously

$$Q = (3q)^{-\frac{1}{2}} \epsilon^{2q^3} T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Let us now seek the limit to which T tends when c becomes infinite. For this purpose divide the integral T into three parts T_1, T_2, T_3, where T_1 is the integral taken from $-3^{\frac{1}{2}} q^{\frac{3}{2}}$ to a real negative quantity $-a$, T_2 from $-a$ to a real positive quantity $+b$, and T_3 from b to ∞; and suppose c first to become infinite, a and b remaining constant, and lastly make a and b infinite.

Changing the sign of t in T_1, and the order of the limits, we get

$$T_1 = \int_a^{3^{\frac{1}{2}}c^{\frac{1}{3}}} \epsilon^{(3q)^{-\frac{1}{2}}t^2 - t^3}dt.$$

Put $t = \rho\epsilon^{\theta\sqrt{-1}}$. Then we may integrate first from $\rho = a$ to $\rho = 3^{\frac{1}{2}}c^{\frac{1}{3}}$ while θ remains equal to 0, and afterwards from $\theta = 0$ to $\theta = \frac{3}{2}\alpha$ while ρ remains equal to $3^{\frac{1}{2}}c^{\frac{1}{3}}$. Let the two parts of the integral be denoted by T', T''. We shall evidently obtain a superior limit to T' by making the following changes in the integral: first, replacing t^3 in the index by the product of t^2 and the greatest value (namely $3^{\frac{1}{2}}c^{\frac{1}{3}}$) which t receives in the integration; thirdly, replacing a by the smallest quantity (namely 0) to which it can be equal, and, fourthly, extending the superior limit to ∞. Hence the real and imaginary parts of T' are both numerically less than $\int_a^\infty \epsilon^{-\frac{1}{3}t^3}dt$, a quantity which vanishes in the limit, when a becomes infinite.

We shall obtain a superior limit to the real or imaginary part of T'' by reducing the quantity under the integral sign to its modulus, and omitting $\sqrt{-1}$ in the coefficient. Hence L will be such a limit if

$$L = 3^{\frac{1}{2}}c^{\frac{1}{3}}\int_0^{\frac{3}{2}\alpha} \epsilon^{-c^3 f(\theta)}d\theta, \text{ where } f(\theta) = 3\cos 2\theta - \cos(3\theta - \tfrac{3}{2}\alpha).$$

We may evidently suppose α to be positive, if not equal to zero, since the case in which it is negative may be reduced to the case in which it is positive by changing the signs of α and θ. When $\theta = \frac{\pi}{6}$, the first term in $f(\theta)$ is equal to $\frac{3}{2}$, which, being greater than 1, determines the sign of the whole, and therefore $f(\theta)$ is positive; and $f(\theta)$ is evidently positive from $\theta = 0$ to $\theta = \frac{\pi}{6}$, since for such values $\cos 2\theta > \frac{1}{2}$. Also in general $f'(\theta) = -6\sin 2\theta + 3\sin(3\theta - \frac{3}{2}\alpha)$, which is evidently positive from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{4}$, and the latter is the largest value we need consider, being the extreme value of θ when α has its extreme value $\frac{\pi}{6}$. When θ has its extreme value $\frac{3}{2}\alpha$, $f(\theta) = 2\cos 3\alpha$, which is positive when $\alpha < \frac{\pi}{6}$, and vanishes when $\alpha = \frac{\pi}{6}$. Hence $f(\theta)$ is positive when $\theta < \frac{3}{2}\alpha$; for it has been shewn to be positive when $\theta < \frac{\pi}{6}$, which meets the case in which $\alpha < \frac{\pi}{9}$ or $= \frac{\pi}{9}$, and to be constantly decreasing from $\theta = \frac{\pi}{6}$ to $\theta = \frac{3}{2}\alpha$, which meets the case in which $\theta > \frac{\pi}{9}$. Hence when $\alpha < \frac{\pi}{6}$ the limit of L for $c = \infty$ is zero, inasmuch as the coefficient of c^3 in the index of ϵ is negative and finite; and when $\alpha = \frac{\pi}{6}$ the same is true, for the same reason, if it be not for a range of integration lying as near as we please to the superior limit. In this case put for shortness $f(\theta) = \delta$, regard $\frac{3}{2}\alpha - \theta$ as a function of δ, $F(\delta)$, and integrate from $\delta = 0$ to $\delta = \beta$, where β is a constant which may be as small as we please. By what precedes, $F'(\delta)$ will be finite in the integration, and may be made as nearly as we please equal to the constant $F'(0)$ by diminishing β. Hence the integral ultimately becomes $3^{\frac{1}{2}}F'(0)c^{\frac{3}{2}}\int_0^\beta \epsilon^{-c^3\delta}d\delta$, which vanishes when c becomes infinite. Hence the limit of T_1 is zero.

We have evidently

$$T_3 < \int_b^\infty \epsilon^{-t^2}dt,$$

which vanishes when b becomes infinite. Hence the limit of T is equal to that of T_2. Now making c first infinite and afterwards a and b, we get

$$\text{limit of } T_2 = \text{limit of } \int_{-a}^{b} \epsilon^{-t^2}dt = \int_{-\infty}^{+\infty} \epsilon^{-t^2}dt = \sqrt{\pi},$$

and therefore we have ultimately, for very large values of c,

$$Q = \left(\frac{\pi}{3q}\right)^{\frac{1}{2}}\epsilon^{2q^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In order to apply this expression to the integral u given by (6) we must put

$$3q^2 = n\epsilon^{\frac{\pi}{3}\sqrt{-1}}, \text{ whence } q = \left(\frac{n}{3}\right)^{\frac{1}{2}}\epsilon^{\frac{\pi}{6}\sqrt{-1}},\ \epsilon^{-\frac{\pi}{6}\sqrt{-1}}\left(\frac{\pi}{3q}\right)^{\frac{1}{2}} = \frac{\pi^{\frac{1}{2}}}{(3n)^{\frac{1}{4}}}\epsilon^{-\frac{\pi}{4}\sqrt{-1}},\ 2q^3 = 2\left(\frac{n}{3}\right)^{\frac{3}{2}}\sqrt{-1},$$

whence we get ultimately

$$u = \frac{\pi^{\frac{1}{2}}}{(3n)^{\frac{1}{4}}}\epsilon^{\left\{2\left(\frac{n}{3}\right)^{\frac{3}{2}} - \frac{\pi}{4}\right\}\sqrt{-1}}, \qquad U = \frac{\pi^{\frac{1}{2}}}{(3n)^{\frac{1}{4}}}\cos\left\{2\left(\frac{n}{3}\right)^{\frac{3}{2}} - \frac{\pi}{4}\right\}^{*} \quad . \quad (23)$$

Comparing with (15) we get

$$A = B = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}3^{\frac{1}{4}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

9. We cannot make n pass from positive to negative through a series of real values, so long as we employ the series according to descending powers, because these series become illusory when n is small. When n is imaginary we cannot speak of the integrals which appear at the right hand side of (5), because the exponential with a positive index which would appear under the integral signs would render each of these integrals divergent. If however we take equation (6) as the definition of u, and suppose U always derived from u by changing the sign of $\sqrt{-1}$ in the coefficient of the integral and in the value of p, but not in the expression for n, and taking half the sum of the results, we may regard u and U as certain functions of n whether n be real or imaginary. According to this definition, the series involving ascending

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* This result might also have been obtained from the integral U in its original shape, namely, $\int_0^\infty \cos(x^3 - nx)\,dx$, by a method similar to that employed in Art. 21. If x_1 be the positive value of x which renders $x^3 - nx$ a minimum, we have $x_1 = 3^{-\frac{1}{2}}n^{\frac{1}{2}}$. Let the integral U be divided into three parts, by integrating separately from $x = 0$ to $x = x_1 - a$, from $x = x_1 - a$ to $x = x_1 + b$, and from $x = x_1 + b$ to $x = \infty$; then make n infinite while a and b remain finite, and lastly, let a and b vanish. In this manner the second of equations (23) will be obtained, by the assistance of the known formulæ

$$\int_{-\infty}^{\infty}\cos x^2\,dx = \int_{-\infty}^{\infty}\sin x^2\,dx = 2^{-\frac{1}{2}}\pi^{\frac{1}{2}}.$$

integral powers of n, which is convergent for all values of n, real or imaginary, however great be the modulus, will continue to represent u when n is imaginary. The differential equation (11), and consequently the descending series derived from it, will also hold good when n is imaginary; but since this series contains radicals, while U is itself a rational function of n, we might expect beforehand that in passing from one imaginary value of n to another it should sometimes be necessary to change the sign of a radical, or make some equivalent change in the coefficients A, B. Let $n = n_1 \epsilon^{\nu\sqrt{-1}}$, where n_1 is positive. Since both values of $2\left(\frac{n}{3}\right)^{\frac{3}{2}}$ are employed in the series, with different arbitrary constants, we may without loss of generality suppose that value of $n^{\frac{3}{2}}$ which has $\frac{3}{2}\nu$ for its amplitude to be employed in the circular functions or exponentials, as well as in the expression for S. In the multiplier we may always take $-\frac{\nu}{4}$ for the amplitude of $n^{-\frac{1}{4}}$ by including in the constant coefficients the factor by which one fourth root of n differs from another; but then we must expect to find the arbitrary constants discontinuous. In fact, if we observe the forms of R and S, and suppose the circular functions in (15) expanded in ascending series, it is evident that the expression for U will be of the form

$$An^{-\frac{1}{4}}N + Bn^{\frac{1}{4}}N', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where N and N' are rational functions of n. At least, this will be the case if we regard as a rational function a series involving descending integral powers of n, and which is at first rapidly convergent, though ultimately divergent, or rather, if we regard as such the function to which the convergent part of the series is a very close approximation when the modulus of n is at all large. Now, if A and B retained the same values throughout, the above expression would not recur till ν was increased by 8π, whereas U recurs when ν is increased by 2π. If we write $\nu + 2\pi$ for ν, and observe that N and N' recur, the expression (25) will become

$$-\sqrt{-1}\,An^{-\frac{1}{4}}N + \sqrt{-1}\,Bn^{\frac{1}{4}}N';$$

and since U recurs it appears that A, B become $\sqrt{-1}\,A$, $-\sqrt{-1}\,B$, respectively, when ν is increased by 2π. Also the imaginary part of the expression (25) changes sign with ν, as it ought; so that, in order to know what A and B are generally, it would be sufficient to know what they are from $\nu = 0$ to $\nu = \pi$.

If we put $n_1\epsilon^{\pi\sqrt{-1}}$ for n in the second member of equation (15), and write β for $2 \cdot 3^{-\frac{3}{2}} n_1^{\frac{3}{2}}$, and R_1, S_1 for what R, S become when n_1 is put for n in the second members of equations (16) and all the terms are taken positively, we shall get as our result

$$\tfrac{1}{2}\epsilon^{-\frac{\pi}{4}\sqrt{-1}}\,n_1^{-\frac{1}{4}}\left\{(A - \sqrt{-1}\,B)(R_1 + S_1)\,\epsilon^{\beta} + (A + \sqrt{-1}\,B)(R_1 - S_1)\,\epsilon^{-\beta}\right\}.$$

Now the part of this expression which contains $(R_1 + S_1)\epsilon^{\beta}$ ought to disappear, as appears from (17). If we omit the first part of the expression, and in the second part put for A and B their values given by (24), we shall obtain an expression which will be identical with the second member of (17) provided

$$C = \frac{\pi^{\frac{1}{2}}}{2 \cdot 3^{\frac{1}{4}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This mode of determining the constant C is anything but satisfactory. I have endeavoured in vain to deduce the leading term in U for n negative from the integral itself, whether in the original form in which it appears in (5), or in the altered form in which it is obtained from (6). The correctness of the above value of C will however be verified further on.

10. Expressing n, U in terms of m, W by means of (8) and (9), putting for shortness

$$\phi = 2\left(\frac{n}{3}\right)^{\frac{3}{2}} = \pi\left(\frac{m}{3}\right)^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where the numerical values of m and n are supposed to be taken when these quantities are negative, observing that $16\sqrt{(3n^3)} = 72\phi$, and reducing, we get when m is positive

$$W = 2^{\frac{1}{2}}(3m)^{-\frac{1}{4}}\left\{R\cos\left(\phi - \frac{\pi}{4}\right) + S\sin\left(\phi - \frac{\pi}{4}\right)\right\}, \quad . \quad . \quad . \quad (28)$$

where

$$\left.\begin{aligned} R &= 1 - \frac{1.5.7.11}{1.2\,(72\phi)^2} + \frac{1.5.7.11.13.17.19.23}{1.2.3.4\,(72\phi)^4} - \ldots, \\ S &= \frac{1.5}{1.72\phi} - \frac{1.5.7.11.13.17}{1.2.3\,(72\phi)^3} + \ldots. \end{aligned}\right\} \quad . \quad . \quad (29)$$

When m is negative, so that W is the integral expressed by writing $-m$ for m in (1), we get

$$W = 2^{-\frac{1}{2}}(3m)^{-\frac{1}{4}}\epsilon^{-\phi}\left\{1 - \frac{1.5}{1.72\phi} + \frac{1.5.7.11}{1.2\,(72\phi)^2} - \ldots\right\}. \quad . \quad . \quad . \quad (30)$$

11. Reducing the coefficients of ϕ^{-1}, ϕ^{-2}... in the series (29) for numerical calculation, we have, not regarding the signs,

order	i	ii	iii	iv	v	vi
logarithm	$\bar{2}.841638$;	$\bar{2}.569766$;	$\bar{2}.579704$;	$\bar{2}.760793$;	$\bar{1}.064829$;	$\bar{1}.464775$
coefficient	.0694444;	.0371335;	.0379930;	.0576490;	.116099;	.291592.

Thus, for $m = 3$, in which case $\phi = \pi$, we get for the successive terms after the first, which is 1,

.022105, .003762, .001225, .000592, .000379, .000303.

We thus get for the value of the series in (30), by taking half the last term but one and a quarter of its first difference, .980816; whence for $m = 3$, $W = 6^{-\frac{1}{2}} \times .980816\,\epsilon^{-\pi} = .0173038$, of which the last figure cannot be trusted. Now the number given by Mr. Airy to 5 decimal places, and calculated from the ascending series and by quadratures separately, is .01730, so that the correctness of the value of C given by (26) is verified.

For $m = +3$ we have from (28)

$$W = -3^{-\frac{1}{2}}(R - S) = -3^{-\frac{1}{2}}(.9965 - .0213) = -.5632,$$

which agrees with Mr. Airy's result $-.56322$ or $-.56323$. As m increases, the convergency of the series (29) or (30) increases rapidly.

12. The expression (28) will be rendered more easy of numerical calculation by assuming $R = M\cos\psi$, $S = M\sin\psi$, and expanding M and $\tan\psi$ in series to a few terms. These series will evidently proceed, the first according to even, and the second according to odd inverse powers of ϕ. Putting the several terms, taken positively, under the form 1, $a\phi^{-1}$, $ab\phi^{-2}$, $abc\phi^{-3}$, $abcd\phi^{-4}$ &c., and proceeding to three terms in each series, we get

$$M = 1 - a\left(b - \frac{a}{2}\right)\phi^{-2} + a\left\{bc(d-a) + \frac{a^2}{2}\left(b - \frac{a}{4}\right)\right\}\phi^{-4}, \quad . \quad . \quad (31)$$

$$\tan\psi = a\phi^{-1} - ab(c-a)\phi^{-3} + ab\{cd(e-a) - ab(c-a)\}\phi^{-5}. \quad . \quad . \quad (32)$$

The roots of the equation $W = 0$ are required for the physical problem to which the integral W relates. Now equations (28), (29) shew that when m is at all large the roots of this equation are given very nearly by the formula $\phi = (i - \frac{1}{4})\pi$, where i is an integer. From the definition of ψ it follows that the root satisfies exactly the equation

$$\phi = (i - \tfrac{1}{4})\pi + \psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

By means of this equation we may expand ϕ in a series according to descending powers of Φ, where $\Phi = (i - \frac{1}{4})\pi$. For this purpose it will be convenient first to expand ψ in a series according to descending powers of ϕ, by means of the expansion of $\tan^{-1}x$ and the equation (32), and having substituted the result in (33) to expand by Lagrange's theorem. The result of the expansion carried as far as to Φ^{-5} is

$$\begin{aligned}\phi = \Phi + a\Phi^{-1} - \{ab(c-a) + \tfrac{1}{3}a^3 + a^2\}\Phi^{-3}\\ + \{ab[cd(e-a) - ab(c-a)] + a^3b(c-a) + \tfrac{1}{5}a^5 + 4a[ab(c-a) + \tfrac{1}{3}a^3] + 2a^3\}\Phi^{-5}. \ \ldots (34)\end{aligned}$$

13. To facilitate the numerical calculation of the coefficients let

$$a = \frac{a'}{1 \,.\, D}; \quad b = \frac{b'}{2 \,.\, D}; \quad c = \frac{c'}{3 \,.\, D}; \ \text{\&c.},$$

and let the coefficients of ϕ^{-2}, ϕ^{-4} in (31) be put under the forms $-\dfrac{A_2}{1 \,.\, 2\,D^2}$, $\dfrac{A_4}{1 \,.\, 2 \,.\, 3 \,.\, 4\,D^4}$, and similarly with respect to (32), (34). Then to calculate W for a given value of m, we have

$$W = 2^{\frac{1}{2}}(3m)^{-\frac{1}{4}} M\cos\left(\phi - \frac{\pi}{4} - \psi\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where

$$M = 1 - \frac{A_2}{1 \,.\, 2\,D^2}\phi^{-2} + \frac{A_4}{1 \,.\, 2 \,.\, 3 \,.\, 4\,D^4}\phi^{-4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$\tan\psi = \frac{C_1}{1 \,.\, D}\phi^{-1} - \frac{C_3}{1 \,.\, 2 \,.\, 3\,D^3}\phi^{-3} + \frac{C_5}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5\,D^5}\phi^{-5}, \quad . \quad . \quad (37)$$

and for calculating the roots of the equation $W = 0$, we have

$$\phi = \Phi + \frac{E_1}{1 \,.\, D}\Phi^{-1} - \frac{E_3}{1 \,.\, 2 \,.\, 3\,D^3}\Phi^{-3} + \frac{E_5}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5\,D^5}\Phi^{-5}. \quad . \quad . \quad . \quad (38)$$

The coefficients in these formulæ are given by the equations

$$\left.\begin{array}{r} A_2 = a'(b'-a')\,; \quad A_4 = a'\left\{b'c'(d'-4a') + 3a'^2(2b'-a')\right\}; \\ C_1 = a'\,; \quad C_3 = a'b'(c'-3a')\,; \quad C_5 = a'b'\left\{c'd'(e'-5a') - 10C_3\right\}; \\ E_1 = a'\,; \quad E_3 = C_3 + 2a'^2(3D+a')\,; \\ E_5 = C_5 + 20a'(4D+a')\,C_3 + 24a'^3 + 80a'^3D(3D+2a'). \end{array}\right\} \quad . \qquad (39)$$

14. Putting in these formulæ

$$a' = 1\,.\,5\,; \quad b' = 7\,.\,11\,; \quad c' = 13\,.\,17\,; \quad d' = 19\,.\,23\,; \quad e' = 25\,.\,29\,; \quad D = 72\,;$$

we get

$$A_2 = 5\,.\,72\,; \quad A_4 = 3\,.\,5\,.\,72^2\,.\,457\,; \quad C_1 = 5\,; \quad C_3 = 2\,.\,5\,.\,7\,.\,11\,.\,103\,;$$

$$C_5 = 4^2\,.\,5^3\,.\,7^2\,.\,11\,.\,23861\,; \quad E_1 = 5\,; \quad E_3 = 72\,.\,1255\,; \quad E_5 = 4\,.\,5^3\,.\,72^2\,.\,10883\,;$$

whence we obtain, on substituting in (36), (37), (38),

$$M = 1 - \frac{5}{144}\phi^{-2} + \frac{2285}{41472}\phi^{-4},$$

$$\tan\psi = \frac{5}{72}\phi^{-1} - \frac{39655}{1119744}\phi^{-3} + \frac{321526975}{2902376448}\phi^{-5},$$

$$\phi = \Phi + \frac{5}{72}\Phi^{-1} - \frac{1255}{31104}\Phi^{-3} + \frac{272075}{2239488}\Phi^{-5}.$$

Reducing to decimals, having previously divided the last equation by π, and put for Φ its value $(i - \frac{1}{4})\,\pi$, we get

$$M = 1 - .034722\,\phi^{-2} + .055097\,\phi^{-4}, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\tan\psi = .069444\,\phi^{-1} - .035414\,\phi^{-3} + .110781\,\phi^{-5}, \quad . \quad . \quad (41)$$

$$\frac{\phi}{\pi} = i - .25 + \frac{.028145}{4i-1} - \frac{.026510}{(4i-1)^3} + \frac{.129402}{(4i-1)^5} \quad . \quad . \quad . \quad (42)$$

15. Supposing $i = 1$ in (42), we get

$$\frac{\phi}{\pi} = .75 + .0094 - .0010 + 0005 = .7589\,;$$

whence $m = 3\left(\frac{\phi}{\pi}\right)^{\frac{2}{3}} = 2.496$. The descending series obtained in this paper fail for small values of m; but it appears from Mr. Airy's table that for such values the function W is positive, the first change of sign occurring between $m = 2.4$ and $m = 2.6$. Hence the integer i in (42) is that which marks the order of the root. A more exact value of the first root, obtained by interpolation from Mr. Airy's table, is 2.4955. For $i = 1$ the series (42) is not convergent enough to give the root to more than three places of decimals, but the succeeding roots are given by this series with great accuracy. Thus, even in the case of the second root the value of the last term in (42) is only .000007698. It appears then that this term might have been left out altogether.

16. To determine when W is a maximum or minimum we must put $\frac{dW}{dm} = 0$. We might get $\frac{dW}{dm}$ by direct differentiation, but the law of the series will be more easily obtained from the differential equation. Resuming equation (11), and putting V for $\frac{dU}{dn}$, we get by dividing by n and then differentiating

$$\frac{d^2V}{dn^2} - \frac{1}{n}\frac{dV}{dn} + \frac{n}{3}V = 0.$$

This equation may be integrated by descending series just as before, and the arbitrary constants will be determined at once by comparing the result with the derivative of the second member of (15), in which A, B are given by (24). As the process cannot fail to be understood from what precedes, it will be sufficient to give the result, which is

$$V = 3^{-\frac{3}{4}}\pi^{\frac{1}{2}}n^{\frac{1}{4}}\left\{R'\cos\left(\phi + \frac{\pi}{4}\right) + S'\sin\left(\phi + \frac{\pi}{4}\right)\right\}, \quad . \quad . \quad . \quad . \quad (43)$$

where

$$\left.\begin{aligned} R' &= 1 - \frac{-1\,.\,7\,.\,5\,.\,13}{1\,.\,2\,(72\phi)^2} + \frac{-1\,.\,7\,.\,5\,.\,13\,.\,11\,.\,19\,.\,17\,.\,25}{1\,.\,2\,.\,3\,.\,4\,(72\phi)^4} - \ldots, \\ S' &= \frac{-1\,.\,7}{1\,.\,72\phi} - \frac{-1\,.\,7\,.\,5\,.\,13\,.\,11\,.\,19}{1\,.\,2\,.\,3\,(72\phi)^3} + \ldots\,. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (44)$$

17. The expression within brackets in (43) may be reduced to the form $M\cos\left(\phi + \frac{\pi}{4} - \psi\right)$ just as before, and the formulæ of Art. 13 will apply to this case if we put

$$a' = -1\,.\,7: \quad b' = 5\,.\,13; \quad c' = 11\,.\,19; \quad \&c., \quad D = 72.$$

The roots of the equation $\frac{dW}{dm} = 0$ are evidently the same as those of $V = 0$. They are given approximately by the formula $\phi = (i - \frac{3}{4})\,\pi$, and satisfy exactly the equation $\phi = (i - \frac{3}{4})\,\pi + \psi$. The root corresponding to any integer i may be expanded in a series according to the inverse odd powers of $4i - 3$ by the formulæ of Art. 13. Putting $(i - \frac{3}{4})\,\pi$ for Φ, and taking the series to three terms only, we get

$$E_1 = -7; \quad E_3 = -84168;$$

whence

$$\phi = \Phi - \frac{7}{72}\Phi^{-1} + \frac{1169}{31104}\Phi^{-3};$$

or, reducing as before,

$$\frac{\phi}{\pi} = i - .75 - \frac{.039403}{4i - 3} + \frac{.024693}{(4i - 3)^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

This series will give only a rough approximation to the first root, but will answer very well for the others.

For $i = 1$ the series gives $\pi^{-1}\phi = .25 - .039 + .025$, which becomes on taking half the second term and a quarter of its first difference $.25 - .019 - .004 = .227$, whence $m = 1.12$.

The value of the first root got by interpolation from Mr. Airy's table is 1.0845. For the 2nd and 3rd roots we get from (45)

$$\text{for } i = 2,\ \pi^{-1}\phi = 1.25 - .00788 + .00020 = 1.24232\,;$$
$$\text{for } i = 3,\ \pi^{-1}\phi = 2.25 - .00438 + .00003 = 2.24565.$$

For higher values of i the last term in (45) may be left out altogether.

18. The following table contains the first 50 roots of the equation $W = 0$, and the first 10 roots of the derived equation. The first root in each case was obtained by interpolation from Mr. Airy's table; the series (42) and (45) were sufficiently convergent for the other roots. In calculating the 2nd root of the derived equation, a rough value of the first term left out in (45) was calculated, and its half taken since the next term would be of opposite sign. The result was only $-.000025$, so that the series (45) may be used even when i is as small as 2. By far

i	m	diff.	i	m	diff.
1	2.4955	1.8676	26	26.1602	·6730
2	4.3631	1.5291	27	26.8332	.6647
3	5.8922	1.3514	28	27.4979	.6567
4	7.2436	1.2352	29	28.1546	.6491
5	8.4788	1.1512	30	28.8037	.6419
6	9.6300	1.0861	31	29.4456	.6349
7	10.7161	1.0335	32	30.0805	.6284
8	11.7496	.9899	33	30.7089	.6219
9	12.7395	.9529	34	31.3308	.6159
10	13.6924	.9208	35	31.9467	.6100
11	14.6132	.8927	36	32.5567	.6043
12	15.5059	.8676	37	33.1610	.5989
13	16.3735	.8452	38	33.7599	.5936
14	17.2187	.8250	39	34.3535	.5885
15	18.0437	.8065	40	34.9420	.5836
16	18.8502	.7897	41	35.5256	.5788
17	19.6399	.7740	42	36.1044	.5742
18	20.4139	.7597	43	36.6786	.5698
19	21.1736	.7463	44	37.2484	.5655
20	21.9199	.7337	45	37.8139	.5612
21	22.6536	.7221	46	38.3751	.5572
22	23.3757	.7111	47	38.9323	.5532
23	24.0868	.7008	48	39.4855	.5494
24	24.7876	.6909	49	40.0349	.5456
25	25.4785	.6817	50	40.5805	
1	1.0845	2.3824	6	9.0599	1.1175
2	3.4669	1.6777	7	10.1774	1.0590
3	5.1446	1.4336	8	11.2364	1.0111
4	6.5782	1.2903	9	12.2475	.9710
5	7.8685	1.1914	10	13.2185	

the greater part of the calculation consisted in passing from the values of $\pi^{-1}\phi$ to the corresponding values of m. In this part of the calculation 7-figure logarithms were used in obtaining the value of $\frac{1}{3}m$, and the result was then multiplied by 3.

A table of differences is added, for the sake of exhibiting the decrease indicated by theory in the interval between the consecutive dark bands seen in artificial rainbows. This decrease will be readily perceived in the tables which contain the results of Professor Miller's observations*. The table of the roots of the derived equation, which gives the maxima of W^2, is calculated for the sake of meeting any observations which may be made on the supernumerary bows accompanying a natural rainbow, since in that case the maximum of the red appears to be what best admits of observation.

SECOND EXAMPLE.

19. Let us take the integral

$$u = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(x\cos\theta)\,d\theta = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots \dagger \quad . \quad (46)$$

which occurs in a great many physical investigations. If we perform the operation $x\dfrac{d}{dx}$ twice in succession on the series we get the original series multiplied by $-x^2$, whence

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + u = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

20. The form of this equation shews that when x is very large, and receives an increment δx, which, though not necessarily a very small fraction itself, is very small compared with x, u is expressed by $A\cos\delta x + B\sin\delta x$, where under the restrictions specified A and B are sensibly constant‡. Assume then, according to the plan of Art. 5,

$$u = \epsilon^{x\sqrt{-1}}\{Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + \dots\}. \quad . \quad . \quad . \quad . \quad (48)$$

On substituting in (47) we get

$$\sqrt{-1}\,\{(2\alpha + 1)Ax^{\alpha-1} + (2\beta + 1)Bx^{\beta-1} + \dots\}$$
$$+ \alpha^2 Ax^{\alpha-2} + \beta^2 Bx^{\beta-2} + \dots = 0.$$

Since we want a descending series, we must put

$$2\alpha + 1 = 0;\ \beta = \alpha - 1;\ \gamma = \beta - 1 \dots;$$
$$(2\beta + 1)B = \sqrt{-1}\,\alpha^2 A;\ (2\gamma + 1)C = \sqrt{-1}\,\beta^2 B \dots;$$

whence

$$\alpha = -\tfrac{1}{2};\ \beta = -\tfrac{3}{2};\ \gamma = -\tfrac{5}{2} \dots;$$

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* *Cambridge Philosophical Transactions*, Vol. VII. p. 277.

† This integral has been tabulated by Mr. Airy from $x = 0$ to $x = 10$, at intervals of 0.2. The table will be found in the 18th Volume of the *Philosophical Magazine*, page 1.

‡ That the 1st and 3rd terms in (47) are ultimately the important terms, may readily be seen by trying the terms two and two in the way mentioned in the introduction. Thus, if we suppose the first two to be the important terms, we get ultimately $U = A$ or $U = B\log x$, either of which would render the last term more important than the 1st or 2nd, and if we suppose the 2nd and 3rd to be the important terms, we get ultimately $u = A\epsilon^{-\frac{x^2}{4}}$, which would render the 1st term more important than either of the others.

$$B = -\frac{1^2}{1\,.\,8}\sqrt{-1}\,A\,;\quad C = +\frac{1^2\,.\,3^2}{1\,.\,2\,.\,8^2}(\sqrt{-1})^2 A\,;\quad D = -\frac{1^2\,.\,3^2\,.\,5^2}{1\,.\,2\,.\,3\,.\,8^3}(\sqrt{-1})^3 A\,\ldots.$$

Substituting in (48), reducing the result to the form $A(P + \sqrt{-1}\,Q)$, adding another solution of the form $B(P - \sqrt{-1}\,Q)$, and changing the arbitrary constants, we get

$$u = Ax^{-\frac{1}{2}}(R\cos x + S\sin x) + Bx^{-\frac{1}{2}}(R\sin x - S\cos x), \quad . \quad (49)$$

where

$$\left.\begin{aligned} R &= 1 - \frac{1^2\,.\,3^2}{1\,.\,2\,(8x)^2} + \frac{1^2\,.\,3^2\,.\,5^2\,.\,7^2}{1\,.\,2\,.\,3\,.\,4\,(8x)^4}\ldots, \\ S &= \frac{1^2}{1\,.\,8x} - \frac{1^2\,.\,3^2\,.\,5^2}{1\,.\,2\,.\,3\,(8x)^3} + \ldots\,. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (50)$$

21. It remains to determine the arbitrary constants A, B. In equation (46) let $\cos\theta = 1 - \mu$, whence

$$d\theta = \frac{d\mu}{\sin\theta} = \frac{d\mu}{(2\mu - \mu^2)^{\frac{1}{2}}} = \frac{d\mu}{(2\mu)^{\frac{1}{2}}} + M\,d\mu,$$

where

$$M = (2\mu - \mu^2)^{-\frac{1}{2}} - (2\mu)^{-\frac{1}{2}},$$

a quantity which does not become infinite between the limits of μ. Substituting in (46) we get

$$u = \frac{\sqrt{2}}{\pi}\int_0^1 \cos\{(1-\mu)\,x\}\,\mu^{-\frac{1}{2}}d\mu + \frac{2}{\pi}\int_0^1 \cos\{(1-\mu)\,x\}\,M\,d\mu. \quad . \quad (51)$$

By considering the series whose n^{th} term is the part of the latter integral for which the limits of μ are $n\pi x^{-1}$ and $(n+1)\pi x^{-1}$ respectively, it would be very easy to prove that the integral has a superior limit of the form Hx^{-1}, where H is a finite constant, and therefore this integral does not furnish any part of the leading terms in u. Putting $\mu x = \nu$ in the first integral in (51), so that

$$\mu^{-\frac{1}{2}}d\mu = x^{-\frac{1}{2}}\nu^{-\frac{1}{2}}d\nu,$$

observing that the limits of ν are 0 and x, of which the latter ultimately becomes ∞, and that

$$\int_0^\infty \cos\nu\,.\,\nu^{-\frac{1}{2}}d\mu = 2\int_0^\infty \cos\lambda^2 d\lambda = \sqrt{\frac{\pi}{2}} = 2\int_0^\infty \sin\lambda^2 d\lambda = \int_0^\infty \sin\nu\,.\,\nu^{-\frac{1}{2}}d\nu,$$

we get ultimately for very large values of x

$$u = (\pi x)^{-\frac{1}{2}}(\cos x + \sin x).$$

Comparing with (49) we get

$$A = B = \pi^{-\frac{1}{2}},$$

whence

$$u = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} R\cos\left(x - \frac{\pi}{4}\right) + \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} S\sin\left(x - \frac{\pi}{4}\right).^{*} \quad . \quad . \quad (52)$$

---

[*] This expression for u, or rather an expression differing from it in nothing but notation and arrangement, has been already obtained in a different manner by Sir William R. Hamilton, in a memoir *On Fluctuating Functions*. See *Transactions of the Royal Irish Academy*, Vol. XIX. p. 313.

For example, when $x = 10$ we have, retaining 5 decimal places in the series,

$$R = 1 - .00070 + .00001 = .99931\,; \quad S = .01250 - .00010 = .01240$$

$$\text{Angle } x - \frac{\pi}{4} = 527^\circ.95780 = 3 \times 180^\circ - 12^\circ 2' 32''\,; \quad \text{whence } u = -.24594,$$

which agrees with the number (– .2460) obtained by Mr. Airy by a far more laborious process, namely, by calculating from the original series.

22. The second member of equation (52) may be reduced to the same form as that of (28), and a series obtained for calculating the roots of the equation $u = 0$ just as before. The formulæ of Art. 13 may be used for this purpose on putting

$$a' = 1^2\,; \ b' = 3^2\,; \ c' = 5^2\,; \ \&c.\,; \ D = 8,$$

and writing x, X for ϕ, Φ, where $X = (i - \frac{1}{4})\pi$. We obtain

$$A_2 = 8\,; \ A_4 = 3\,.\,8^2.\,53\,; \ C_1 = 1\,; \ C_3 = 2\,.\,3^2.\,11\,; \ C_5 = 3^2.\,4^2.\,5\,.\,1139\,;$$

$$E_1 = 1\,; \ E_3 = 8\,.\,31\,; \ E_5 = 4^4.\,3779\,;$$

whence we get for calculating u for a given value of x

$$M = 1 - \frac{1}{16}x^{-2} + \frac{53}{512}x^{-4},$$

$$\tan\psi = \frac{1}{8}x^{-1} - \frac{33}{512}x^{-3} + \frac{3417}{16384}x^{-5},$$

$$u = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} M \cos\left(x - \frac{\pi}{4} - \psi\right) . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

For calculating the roots of the equation $u = 0$ we have

$$x = X + \frac{1}{8}X^{-1} - \frac{31}{384}X^{-3} + \frac{3779}{15360}X^{-5}.$$

Reducing to decimals as before, we get

$$M = 1 - .0625\,x^{-2} + .103516\,x^{-4}, \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\tan\psi = .125\,x^{-1} - .064453\,x^{-3} + .208557\,x^{-5}, \quad . \quad . \quad (55)$$

$$\frac{x}{\pi} = i - \cdot 25 + \frac{\cdot 050661}{4i - 1} - \frac{\cdot 053041}{(4i-1)^3} + \frac{\cdot 262051}{(4i-1)^5} . \quad . \quad (56)$$

As before, the series (56) is not sufficiently convergent when $i = 1$ to give a very accurate result. In this case we get

$$\pi^{-1}x = .75 + .017 - .002 + .001 = \cdot 766,$$

whence $x = 2.41$. Mr. Airy's table gives $u = + \cdot 0025$ for $x = 2 \cdot 4$, and $u = - \cdot 0968$ for $x = 2 \cdot 6$, whence the value of the root is 2.4050 nearly.

The value of the last term in (56) is .0000156 for $i = 2$, and .00000163 for $i = 3$, so that all the roots after the first may be calculated very accurately from this series.

THIRD EXAMPLE.

23. Consider the integral

$$v = \frac{2}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} \cos(x\cos\theta)\, x\, dx\, d\theta = \int_0^x u x\, dx = \frac{x^2}{2} - \frac{x^4}{2^2 . 4} + \frac{x^6}{2^2 . 4^2 . 6} - \ldots ^* \quad (57)$$

which occurs in investigating the diffraction of an object-glass with a circular aperture.

By performing on the series the operation denoted by $x\dfrac{d}{dx}x^{-1}\dfrac{d}{dx}$, we get the original series with the sign changed, whence

$$\frac{d^2 v}{dx^2} - \frac{1}{x}\frac{dv}{dx} + v = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

We may obtain the integral of this equation in a form similar to (49). As the process is exactly the same as before, it will be sufficient to write down the result, which is

$$v = A' x^{\frac{1}{2}}(R\cos x + S\sin x) + B' x^{\frac{1}{2}}(R\sin x - S\cos x), \quad . \quad (59)$$

where

$$\left.\begin{aligned} R &= 1 - \frac{-1.3.1.5}{1.2(8x)^2} + \frac{-1.3.1.5.3.7.5.9}{1.2.3.4(8x)^4} - \ldots, \\ S &= \frac{-1.3}{1.8x} - \frac{-1.3.1.5.3.7}{1.2.3(8x)^3} + \ldots, \end{aligned}\right\} \quad . \quad . \quad (60)$$

the last two factors in the numerator of any term being formed by adding 2 to the last two factors respectively in the numerator of the term of the preceding order.

The arbitrary constants may be easily determined by means of the equation

$$\frac{dv}{dx} = ux. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Writing down the leading terms only in this equation, we have

$$x^{\frac{1}{2}}(-A'\sin x + B'\cos x) = \pi^{-\frac{1}{2}} x^{\frac{1}{2}}(\cos x + \sin x),$$

whence

$$-A' = B' = \pi^{-\frac{1}{2}},$$

$$v = \left(\frac{2x}{\pi}\right)^{\frac{1}{2}}\left\{R\cos\left(x - \frac{3\pi}{4}\right) + S\sin\left(x - \frac{3\pi}{4}\right)\right\}. \quad . \quad . \quad (62)$$

24. Putting in the formulæ of Art. 13,

$$a' = -1.3; \quad b' = 1.5; \quad c' = 3.7; \quad d' = 5.9; \quad e' = 7.11; \quad D = 8;$$

we get

$$A_2 = -3.8; \quad A_4 = -3^3 . 8^2 . 11; \quad C_1 = -3; \quad C_3 = -2.3^2 . 5^2; \quad C_5 = -3^3 . 4^2 . 5^2 . 127;$$

$$E_1 = -3; \quad E_3 = -3^2 . 8; \quad E_5 = -3^3 . 4 . 8^2 . 131;$$

---

[*] The series $1 - \dfrac{x^2}{2.4} + \dfrac{x^4}{2.4^2.6} \ldots$ or $\dfrac{2v}{x^2}$ has been tabulated by Mr. Airy from $x=0$ to $x=12$ at intervals of 0.2. See *Camb. Phil. Trans.* Vol. v. p. 291. The same function has also been calculated in a different manner and tabulated by M. Schwerd in his work on diffraction. The argument in the latter table is the angle $\dfrac{180^\circ}{\pi}x$, and the table extends from 0° to 1125° at intervals of 15°, that is, from $x=0$ to $x=19.63$ at intervals of 0.262 nearly.

whence we get for the formulæ answering to those of Art. 22,

$$M = 1 + \frac{3}{16}x^{-2} - \frac{99}{512}x^{-4},$$

$$\tan\psi = -\frac{3}{8}x^{-1} + \frac{75}{512}x^{-3} - \frac{5715}{16384}x^{-5},$$

$$x = X - \frac{3}{8}X^{-1} + \frac{3}{128}X^{-3} + \frac{1179}{5120}X^{-5},$$

X being in this case equal to $(i + \frac{1}{4})\pi$.

Reducing to decimals as before, we get for the calculation of v for a given value of x,

$$M = 1 + .1875x^{-2} + .193359x^{-4}, \quad . \quad . \quad . \quad (63)$$

$$\tan\psi = -.375x^{-1} + .146484x^{-3} - .348817x^{-5}, \quad . \quad . \quad (64)$$

$$v = \left(\frac{2x}{\pi}\right)^{\frac{1}{2}} M\cos\left(x - \frac{3\pi}{4} - \psi\right); \quad . \quad . \quad . \quad . \quad (65)$$

and for calculating the roots of the equation $v = 0$,

$$\frac{x}{\pi} = i + .25 - \frac{.151982}{4i+1} + \frac{.015399}{(4i+1)^3} - \frac{.245835}{(4i+1)^5}. \quad . \quad . \quad (66)$$

25. The following table contains the first 12 roots of each of the equations $u = 0$, and $x^{-2}v = 0$. The first root of the former was got by interpolation from Mr. Airy's table, the others were calculated from the series (56). The roots of the latter equation were all calculated from the series (66), which is convergent enough even in the case of the first root. The columns which contain the roots are followed by columns which contain the differences between consecutive roots, which are added for the purpose of shewing how nearly equal these differences are to 1, which is what they ultimately become when the order of the root is indefinitely increased.

i	$\frac{x}{\pi}$ for $u = 0$	diff.	$\frac{x}{\pi}$ for $v = 0$	diff.
1	.7655	.9916	1.2197	1.0133
2	1.7571	.9975	2.2330	1.0053
3	2.7546	.9988	3.2383	1.0028
4	3.7534	.9993	4.2411	1.0017
5	4.7527	.9995	5.2428	1.0011
6	5.7522	.9997	6.2439	1.0009
7	6.7519	.9997	7.2448	1.0006
8	7.7516	.9998	8.2454	1.0005
9	8.7514	.9999	9.2459	1.0004
10	9.7513	.9999	10.2463	1.0003
11	10.7512	.9999	11.2466	1.0003
12	11.7511		12.2469	

26. The preceding examples will be sufficient to illustrate the general method. I will remark in conclusion that the process of integration applied to the equations (11), (47), and (58) leads very readily to the complete integral in finite terms of the equation

$$\frac{d^2y}{dx^2} - \left\{q^2 + \frac{i(i+1)}{x^2}\right\} y = 0, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

where i is an integer, which without loss of generality may be supposed positive. The form under which the integral immediately comes out is

$$y = A\epsilon^{qx}\left\{1 - \frac{i(i+1)}{1.2qx} + \frac{(i-1)\,i\,(i+1)(i+2)}{1.2\,(2qx)^2} - \ldots\right\},$$

$$+ B\epsilon^{-qx}\left\{1 + \frac{i(i+1)}{1.2qx} + \frac{(i-1)\,i\,(i+1)(i+2)}{1.2\,(2qx)^2} + \ldots\right\},$$

where each series will evidently contain $i + 1$ terms. It is well known that (67) is a general integrable form which includes as a particular case the equation which occurs in the theory of the figure of the earth, for q in (67) is any quantity real or imaginary, and therefore the equation formed from (67) by writing $+q^2y$ for $-q^2y$ may be supposed included in the form (67).

It may be remarked that the differential equations discussed in this paper can all be reduced to particular cases of the equation obtained by replacing $i(i+1)$ in (67) by a general constant. By taking $gn^{\frac{1}{2}}$, where g is any constant, for the independent variable in place of n in the differential equations which U, V in the first example satisfy, these equations are reduced to the form

$$\frac{d^2y}{dx^2} + \frac{2a}{x}\frac{dy}{dx} + \left(\frac{b}{x^2} + c\right) y = 0,$$

and (47), (58) are in this form already. Putting now $y = x^{-a}z$, we shall reduce the last equation to the form required.

G. G. STOKES.

Pembroke College,
Feb. 4, 1850.

D^r WHEWELL ON THE INTRINSIC EQUATION OF CURVES, MEM. II.

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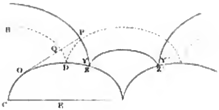

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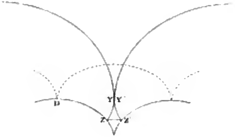

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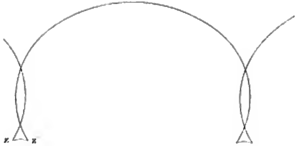

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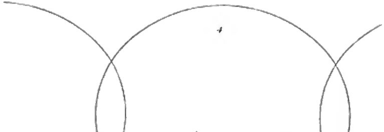

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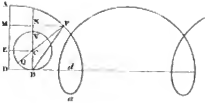

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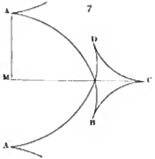

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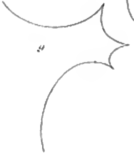

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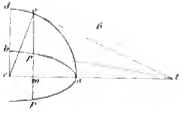

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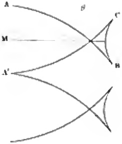

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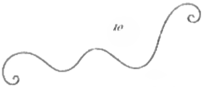

J W Lowry fc

IX. *Mathematical Exposition of Certain Doctrines of Political Economy. Third Memoir.* *By* W. WHEWELL, D.D. *Master of Trinity College.*

[Read Nov. 11, 1850.]

68. IN my former memoirs on this subject, I have remarked that the mathematical mode of treating questions of Political Economy may be very useful in deducing the results of fundamental principles, assumed or independently established, although of course, it cannot add anything to the certainty of the principles themselves. It was further remarked, that this mode of treatment might be expected to shew, more clearly than any other, within what limits, and under what conditions, propositions in Political Economy are true. This remark was illustrated by the mathematical exposition of certain doctrines concerning the results of international trade; from which exposition it appeared, that assertions which have been made in a general form, respecting the advantages of such trade to the countries concerned in it, are true only within very narrow limits, even on the principles which have been put forward as their basis.

In the last memoir the international exchange of commodities was considered without reference to money, as the medium of exchange, or means of paying balances. In the present memoir I shall consider money, that is, gold and silver, as among the things exchanged, and shall endeavour to trace the effect produced by this element in the different countries, which are parties to the exchange.

69. As has already been said, we must, in such mathematical expositions, assume our fundamental principles, as being supplied to the science of Political Economy by means of other modes of reasoning. The principle according to which prices in different countries are affected by the exports and imports of gold and silver may be, in special cases, obscure and complex; but there is one general principle affecting these changes, in the long run, and on the large scale, which writers are in the habit of asserting, and which we shall assume as our basis; leaving it to others to illustrate and confirm, or to correct it.

We shall assume that if the quantity of metallic currency, that is, of the gold and silver, used as money in any country, be augmented, other things remaining the same, (for example, the quantity of commodities to be circulated, and the activity of such transactions) the prices of all commodities will be augmented in the same ratio. Thus, if the metallic currency of England be 30 millions of pounds sterling, and if there be added to it, by extraneous causes, 10 millions more, prices will be increased in the ratio of 3 to 4. And in the same manner if the currency be diminished by 10 millions, so as to be only 20 millions, prices will be diminished in the ratio of 3 to 2.

70. In every country which does not produce its own supply of gold and silver, there

must be a perpetual import of those metals, to supply the wear and tear, both of coin, and of the other uses of gold and silver, all which we may include in the term *plate*. According to Mr. Jacob (*Of the Precious Metals*, chap. XXIII.), the loss by friction annually, is on gold coin $\frac{1}{800}$, on silver coin, $\frac{1}{200}$, of the value. If we suppose that the compound loss on the gold and silver is $\frac{1}{500}$, we should require, in order to keep up a metallic currency of 30 millions, an import of gold and silver to the amount of £60,000 annually. If the import of the precious metals were greater than this, the amount of the currency would be increased, and prices would rise. If the amount of the precious metals imported were less than £60,000, prices would fall: and according to the principle just stated, if the excess or defect of metallic import were permanent, the rise or fall of prices would be permanent, and would not cease till a new position of equilibrium was attained. If, for instance, the annual import of gold and silver, which went to supply the currency, were continually £80,000 instead of £60,000, prices would go on rising, till the currency was 40 millions instead of 30 millions; and then there would be an equilibrium, because £80,000 exactly supplies the wear and tear of a currency of 40 millions. Prices would, in this case, have risen by one third, as we have already stated. If the regular annual import of gold and silver were £40,000, instead of £60,000, prices would fall till the currency was reduced to 20 millions, which amount of currency is exactly kept up by an annual supply of £40,000; and prices would be reduced in the ratio of 3 to 2, as above stated.

71. In the mode of treating the subject here adopted, we consider only states of equilibrium, like those just spoken of. For although, as I observed at the close of the last memoir (67), a state of equilibrium is never attained, the state of equilibrium is that to which all the motions tend; and an *equilibrium theory* is the best attempt at a theory of trade which we can at present make. We must, further, point out some other principles of equilibrium which we here adopt.

We assume, as in the former memoir (37), the equality of import and export. But in the former memoir, we spoke only of commodities in general, as exchanged for one another, without considering the precious metals in particular. It is plain that when we say that in a state of equilibrium the total exchange values of the imports and of the exports are equal, we must include, in the imports, the gold and silver which are annually imported. But as the metals which are used as currency require peculiar consideration in the present discussion, we shall separate them from the other imports. Let I be *the whole of the annual imports*, exclusive of gold and silver, P the *quantity* of those metals annually required to supply the wear and tear of *plate*; Q the *quantity* annually required to supply the wear and tear of the *currency*; E the whole annual value of the exports; then we have

$$E = I + P + Q,$$

and this is now the equation of imports and exports.

72. Let us now consider what the effect upon international trade would be, if the equation of imports and exports were deranged by some considerable addition to the imports. Suppose, for instance, that some commodity (K), which had hitherto been principally pro-

duced at home, were henceforth to be imported in great quantity from abroad. In what manner will each of the above terms be modified?

Perhaps it may be said that in the new resulting condition of equilibrium the imports and exports must still be equal, and that thus, the derangement will remedy itself. But this remark does not suffice. It does not shew how the new position of equilibrium is attained, nor what are its consequences. There is nothing in the direct effect of the newly added import to produce any new export, except the export of money. If the imports be increased, the exports of commodities remaining the same, the balance must be paid, so far as that transaction alone indicates, in gold and silver; and thus, the importing country will have the total import of gold and silver diminished, or turned into an export. It is true, that a country not producing much gold or silver cannot go on exporting gold or silver: but still the question remains, what is it which stops this drain of the precious metals, and gives rise to a new condition of equilibrium? And to this, the reply, upon the principles here assumed, is, that the efflux or diminished influx of gold and silver reduces the currency; this reduction of the currency diminishes prices in the country in question; and prices being diminished, some commodities may be exported now, which could not be exported before; and thus, the exports are increased, and the increase may balance the increase of imports, so as to give rise to a new condition of equilibrium. And the mathematical problem now to be solved is, to find at what point, the necessary assumptions being made, this new equilibrium will take place.

73. Besides the effect of diminished prices in producing an increase of exports, there may be an effect of the same kind produced by the diminution of wages in the country in question, arising from the competition of the labour previously employed in producing the commodity (K) which is henceforth to be imported. Of course, the price of labour will be diminished along with all other prices: but, by the competition of the labourers thrown out of work by the new importation, the price of labour may be diminished in a still greater ratio. This effect however we shall not at present consider; but shall investigate the alteration of exports and imports arising from the change of prices only.

74. We had the equation $E = I + P + Q$. Let K be the value of the new imported commodity (K) which deranges the equilibrium, and produces a new state of things: then I becomes $I + K$; and we have $E = I + K + P + Q$. Let P, the import of gold and silver, annually required for plate, remain unaltered. Let the quantity of gold and silver required to pay for the new import, when the increased export has been deducted, be a fraction of the whole annual import of gold and silver; that is, let it $= mQ$. Therefore $(1 - m)Q$ is the new import of gold and silver. Let F be the increase of the exports: then the equation of imports and exports under the new conditions is

$$E + F = I + K + P + (1 - m)Q;$$

and we had $E = I + K + P + Q$,

whence, $F = K - mQ$.

Since the quantity of gold and silver imported, in order to keep up the currency, is not now Q, but $(1 - m)\,Q$, the quantity of currency will be annually diminished; and there will be no equilibrium till the currency is diminished in the same ratio, since the amount of the currency is a certain multiple, (500 in our supposition) of the annual supply. Hence, prices will be reduced in the ratio $1 : 1 - m$. This diminution of prices will produce an increase, F, of the exports; and we have to consider in what way F may be supposed to depend upon m.

75. The quantity F consists of commodities sent to the general market of the world as supply. It will include all commodities which our country can export at such a rate as to compete with other countries. The lower are prices in our country, the greater will be F. As a plausible hypothesis, made for the sake of reducing the relation to a formula, we may suppose that F, the value added to the exports, is proportional to the fall of price: for instance, that a general fall of our prices of one tenth would add £100,000 to our exports; that a fall of prices of two tenths would add £200,000 to the exports; and so on. On this supposition, we may put $F = mG$, where G is a quantity depending on the manufacturing skill and activity of our country, as shewn in providing articles of export. Then our equation becomes

$$mG = K - mQ\,;\ \text{whence } m = \frac{K}{G + Q}.$$

76. We thus find the fall of price produced by the importation of K. For instance, let K, the value of the new import, = £100,000, and, as before, let Q, the original annual import of money, = £60,000. Also let mG = £100,000, when $m = \frac{1}{10}$, as above (75); therefore G = £1,000,000.

$$\text{Hence } m = \frac{100{,}000}{1{,}000{,}000 + 60{,}000} = \frac{5}{53}.$$

In this case, the prices in our country are lowered by less than one tenth. In the new condition of equilibrium, the annual import of gold and silver to keep up the currency will be $£54{,}340 = \frac{48}{53} \times 60{,}000$. The new exports, F, will be $\frac{5}{53} \times 1{,}000{,}000 = £94{,}340$, which compared with the new imports, £100,000, leaves a new balance against us of £5660. On the old exports and imports, the balance in our favour was £60,000. Therefore in the new state, it is £54,340, which just keeps up the currency.

77. If, the other data being the same, K be 1,000,000 instead of 100,000, we shall find

$$m = \frac{100}{106}.$$

$$\text{Hence } 1 - m = \frac{6}{106} = \frac{1}{18} \text{ nearly.}$$

In this case, according to the suppositions which we have made, the prices in our country would be reduced to *one eighteenth* of what they were before, in consequence of the scarcity of the precious metals arising from the exportation of them to pay for the new import K. But it is evident that such a reduction of prices would produce so great a difference between neighbouring countries, that it could not long subsist, while there was trade between the countries. A difference so great (if it did not produce a great diminution of our internal trade,) would produce an export of commodities beyond what our formula supposes. The additional export F would not be merely mG, but would, for values of m approaching to 1, be a greater quantity. It would be easy to propose a function of m which should satisfy these conditions; but we do not need to do so, in order to shew the nature of the mutual dependence of the quantities which enter into our formula.

78. The quantity mG, the value of the new export, is always less than K, the value of the new import, (for $mG + mQ = K$;) therefore if the labour displaced by the import of K be all employed in producing mG, it will obtain a smaller return than it did before.

But this smaller return may be of equal, less or greater value, as commanding commodities in our country, in consequence of the reduction of prices.

79. Thus, taking the first example above given, (76), let $K = 100{,}000$, $Q = 60{,}000$; and let m be $\frac{5}{53}$. Then $mG = \frac{5}{53} \times 1{,}000{,}000 = 94{,}340$; therefore the labour which before earned 100,000, (or more, for the import K costs less than the home-produced article did which it replaces) now earns 94,340, or is diminished in the ratio $\frac{50}{53}$ (at least). But prices (of all things, and therefore of articles consumed by the labourers), are diminished in the ratio $\frac{48}{53}$; and therefore, on these suppositions, the labourers who pass from the production of K to that of mG may be gainers.

80. But let $Q = 100{,}000$, $K = 100{,}000$ as before, $G = 1{,}000{,}000$: then we have

$$m = \frac{K}{G+Q} = \frac{1}{10+1} = \frac{1}{11}.$$

Hence $mG = \frac{10}{11} \times 100{,}000$, while $K = 100{,}000$; and the payment of the labour which K displaces is reduced in the ratio $\frac{10}{11}$. But prices are also reduced in the ratio $\frac{10}{11}$. Therefore on those suppositions, the labour which is displaced by the importation of K may be as well paid as before.

But this supposition that K, the sum paid for the new import, is as great as the sum paid, before the import of K, to the home-producers of the same article, is evidently erroneous. The occasion of the import of (K) is, that the importers' price is less than the home-producers' price; and hence, even on the supposition made, the previous home-producer of the commodity (K), when driven by the import to produce new exportable commodities (F), may be worse paid than before.

81. If K be smaller than Q, the displaced labour cannot be so well paid as before.

Thus let $K = 10{,}000$, $G = 1{,}000{,}000$, $Q = 60{,}000$; then $m = \frac{K}{G+Q} = \frac{1}{106}$: $1 - m = \frac{105}{106}$.

Hence $mG = \frac{100}{106} \times 10{,}000$, which is less than K in the ratio $\frac{100}{106}$.

But prices are only lessened in the ratio $\frac{105}{106}$: hence the former producers of (K) who are driven to produce (F), lose.

82. The formula $F = mG$, can perhaps hardly be trusted for actual results when the differences are small. It may however still serve to shew the possible results of our suppositions.

83. It appears, by the previous reasoning, that the equality of the imports and exports of a country is not true, even as a permanent or average state, except we include among the imports, gold and silver; and that there may be and must be a constant annual current of gold and silver into every country which does not itself produce those metals in sufficient abundance to supply its own wants. It appears also that the metallic current which thus flows constantly into any country, may change its amount from year to year, in consequence of the alteration of the amount of other imports and exports; and further, that with such a change of the amount of the metallic current into any country, there arises, when the change has a permanent character, a general change of prices in the country. When the imports into a country become permanently greater, all prices in that country undergo a permanent fall. Those countries have the highest prices, which produce the greatest abundance of exportable articles, compared with the amount of the imports. The excess of other commodity exports over imports, produces a constant mercantile pressure, which keeps the metallic currency at a higher level in that country than in surrounding countries. The different levels of the general scale of prices in different neighbouring countries depend upon the continual metallic currents, which, in virtue of their mutual trade, set in each to or from the others. The different levels at which the precious metals stand in different countries are no more at variance with the general tendency of those commodities to find their level, than the fact, that a series of ponds, connected by pumps constantly working, may be at different levels, is at variance with the tendency of water to find its level; the pumping of international trade determines a metallic current between each two countries, which current affects the metallic level in both countries, and in turn, has its velocity determined by the difference of level.

84. It has generally been supposed that, for many years, prices in general have been higher in England than in most other countries; and various ways of accounting for this fact have been proposed by various writers. Mr. Senior, in his *Lectures on the Cost of the Precious Metals*, ascribes the difference to the superior efficiency of English as compared with foreign labour. But Mr. J. S. Mill (*Polit. Econ.* II. p. 142) rightly, as I conceive, dissents from this view, and gives an account of the conditions on which the general scale of prices depends, agreeing, in the main, with the account given here.

85. I have not thought it necessary to consider the subject of the *exchanges* between different countries, although the state of the exchanges may often affect the amount of exportation. For this effect of the exchanges is not an independent and distinct effect, but

is a derivative and exponent of the effect of the tendency of the metallic current from one country to another. The rate of exchange may be looked upon as an instrument which measures the force of that current, and does not add anything to the force, or produce any effect of its own, except, it may be, to regulate and reduce to steadiness the casual and transient impulses.

86. It may serve to illustrate the bearing of the above hypothetical discussions upon the actual facts, if I state some of the numbers which Mr. Jacob has given in his *Historical Inquiry into the Production and Consumption of the Precious Metals*. The result of his investigations is, that the annual produce of all the gold and silver mines in the world, from 1700 to 1809, was 8 millions sterling a year: that of this sum, two-fifths went regularly to Asia, to pay for tea, silk, spices, and the like; that of the remaining three-fifths, two-thirds, or two-fifths of the whole, were employed for other purposes than coin; and thus there is, for the coin of the whole of Europe, an annual supply of £1,600,000. The annual wear of the coin he estimates at about one-400dth. And the annual supply being much more than the annual wear, the amount of coin has gone on increasing. He reckons that the amount of coin in Europe in 1700 was 226 millions, which was in 1809 increased to 380 millions.

87. In the former memoir, Art. 20, I divided commodities into four classes, according to the relations of supply, demand and price. In three of these classes, when the price falls, the money demand either falls, or at least does not rise. These three classes included most cases; and therefore in most cases, the money demand for our new export (F), even if that commodity be offered by us somewhat cheaper than it was offered by other sellers, will only be so much taken from the demand for the same commodity at the hands of others. In these cases, it does not appear likely that the extension of demand, F, will increase with greater rapidity than the price falls. But in the fourth class of commodities, *popular luxuries*, the money demand increases with a fall of price. In these cases, the extension of demand for the new commodity may increase much more rapidly than the fall of the price, and a different formula instead of $F = mG$ may be needed.

W. W.

TRINITY LODGE,
Nov. 11, 1850.

X. *On the Effect of the Internal Friction of Fluids on the Motion of Pendulums.*

By G. G. Stokes, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *December* 9, 1850.]

The great importance of the results obtained by means of the pendulum has induced philosophers to devote so much attention to the subject, and to perform the experiments with such a scrupulous regard to accuracy in every particular, that pendulum observations may justly be ranked among those most distinguished by modern exactness. It is unnecessary here to enumerate the different methods which have been employed, and the several corrections which must be made, in order to deduce from the actual observations the result which would correspond to the ideal case of a simple pendulum performing indefinitely small oscillations in vacuum. There is only one of these corrections which bears on the subject of the present paper, namely, the correction usually termed the *reduction to a vacuum*. On account of the inconvenience and expense attending experiments in a vacuum apparatus, the observations are usually made in air, and it then becomes necessary to apply a small correction, in order to reduce the observed result to what would have been observed had the pendulum been swung in a vacuum. The most obvious effect of the air consists in a diminution of the moving force, and consequent increase in the time of vibration, arising from the buoyancy of the fluid. The correction for buoyancy is easily calculated from the first principles of hydrostatics, and formed for a considerable time the only correction which it was thought necessary to make for reduction to a vacuum. But in the year 1828 Bessel, in a very important memoir in which he determined by a new method the length of the seconds' pendulum, pointed out from theoretical considerations the necessity of taking account of the inertia of the air as well as of its buoyancy. The numerical calculation of the effect of the inertia forms a problem of hydrodynamics which Bessel did not attack; but he concluded from general principles that a fluid, or at any rate a fluid of small density, has no other effect on the time of very small vibrations of a pendulum than that it diminishes its gravity and increases its moment of inertia. In the case of a body of which the dimensions are small compared with the length of the suspending wire, Bessel represented the increase of inertia by that of a mass equal to k times the mass of the fluid displaced, which must be supposed to be added to the inertia of the body itself. This factor k he determined experimentally for a sphere a little more than two inches in diameter, swung in air and in water. The result for air, obtained in a rather indirect way, was $k = 0{\cdot}9459$, which value Bessel in a subsequent paper increased to 0·956. A brass sphere of the above size having been swung in water with two different lengths of wire in succession gave two values of k, differing a little from each other, and equal to only about two-thirds of the value obtained for air.

The attention of the scientific world having been called to the subject by the publication of Bessel's memoir, fresh researches both theoretical and experimental soon appeared. In order to examine the effect of the air by a more direct method than that employed by Bessel, a large vacuum apparatus was erected at the expense of the Board of Longitude, and by means of this apparatus Captain (now Colonel) Sabine determined the effect of the air on the time of vibration of a particular invariable pendulum. The results of the experiments are contained in a memoir read before the Royal Society in March 1829, and printed in the *Philosophical Transactions* for that year. The mean of eight very consistent experiments gave 1·655 as the factor by which for that pendulum the old correction for buoyancy must be multiplied in order to give the whole correction on account of the air. A very remarkable fact was discovered in the course of these experiments. While the effects of air at the atmospheric pressure and under a pressure of about half an atmosphere were found to be as nearly as possible proportional to the densities, it was found that the effect of hydrogen at the atmospheric pressure was much greater, compared with the effect of air, than corresponded with its density. In fact, it appeared that the ratio of the effects of hydrogen and air on the times of vibration was about 1 to $5\frac{1}{4}$, while the ratio of the densities is only about 1 to 13. In speaking of this result Colonel Sabine remarks, "The difference of this ratio from that shewn by experiment is greater than can well be ascribed to accidental error in the experiment, particularly as repetition produced results almost identical. May it not indicate an inherent property in the elastic fluids, analogous to that of viscidity in liquids, of resistance to the motion of bodies passing through them, independently of their density? a property, in such case, possessed by air and hydrogen gas in very different degrees; since it would appear from the experiments that the ratio of the resistance of hydrogen gas to that of air is more than double the ratio following from their densities. Should the existence of such a distinct property of resistance, varying in the different elastic fluids, be confirmed by experiments now in progress with other gases, an apparatus more suitable than the present to investigate the ratio in which it is possessed by them, could scarcely be devised: and the pendulum, in addition to its many important and useful purposes in general physics, may find an application for its very delicate, but, with due precaution, not more delicate than certain, determinations, in the domain of chemistry." Colonel Sabine has informed me that the experiments here alluded to were interrupted by a cause which need not now be mentioned, but that as far as they went they confirmed the result of the experiments with hydrogen, and pointed out the existence of a specific action in different gases, quite distinct from mere variations of density.

Our knowledge on the subject of the effect of air on the time of vibration of pendulums has received a most valuable addition from the labours of the late Mr Baily, who erected a vacuum apparatus at his own house, with which he performed many hundreds of careful experiments on a great variety of pendulums. The experiments are described in a paper read before the Royal Society on the 31st of May 1832. The result for each pendulum is expressed by the value of n, the factor by which the old correction for buoyancy must be multiplied in order to give the whole effect of the air as deduced from observation. Four spheres, not quite $1\frac{1}{2}$ inch in diameter, gave as a mean $n = 1·864$, while three spheres, a little more than 2 inches in diameter, gave only 1·748. The latter were nearly of the same size as those with which Bessel, by a different method, had obtained $k = 0{\cdot}946$ or 0·956, which corresponds to $n = 1{\cdot}946$ or 1·956. Among the "Additional Experiments" in the latter part of Baily's paper, is a set in which the pendulums consisted of plain cylindrical rods. With these pendulums it was found that n regularly increased, though according to an unknown law, as the diameter of the rod decreased. While a brass tube $1\frac{1}{2}$ inch in diameter gave n equal to about 2·3, a thin rod or thick wire only 0·072 inch in diameter gave for n a value as great as 7·530.

Mathematicians in the meanwhile were not idle, and several memoirs appeared about this time, of which the object was to determine from hydrodynamics the effect of a fluid on the motion of a pendulum. The first of these came from the pen of the celebrated Poisson. It was read before the French Academy on the 22nd of August 1831, and is printed in the 11th Volume of the Memoirs. In this paper, Poisson considers the case of a sphere suspended by a fine wire, and oscillating in the air, or in any gas. He employs the ordinary equations of motion of an elastic fluid, simplified by neglecting the terms which involve the square of the velocity; but in the end, in adapting his solution to practice, he neglects, as insensible, the terms by which alone the action of an elastic differs from that of an incompressible fluid, so that the result thus simplified is equally applicable to fluids of both classes. He finds that when insensible quantities are neglected $n = 1{\cdot}5$, so that the mass which we must suppose added to that of the pendulum is equal to half the mass of the fluid displaced. This result does not greatly differ from the results obtained experimentally by Bessel in the case of spheres oscillating in water, but differs materially from the result he had obtained for air. It agrees pretty closely with some experiments which had been performed about fifty years before by Dubuat, who had in fact anticipated Bessel in shewing that the time of vibration of a pendulum vibrating in a fluid would be affected by the inertia of the fluid as well as by its density. Dubuat's labours on this subject had been altogether overlooked by those who were engaged in pendulum experiments; probably because such persons were not likely to seek in a treatise on hydraulics for information connected with the subject of their researches. Dubuat had, in fact, rather applied the pendulum to hydrodynamics than hydrodynamics to the pendulum.

In the *Philosophical Magazine* for September 1833, p. 185, is a short paper by Professor Challis, on the subject of the resistance to a ball pendulum. After referring to a former paper, in which he had shewn that no sensible error would be committed in a problem of this nature by neglecting the compressibility of the fluid even if it be elastic, Professor Challis, adopting a particular hypothesis respecting the motion, obtains 2 for the value of the factor n for such a pendulum. This mode of solution, which is adopted in several subsequent papers, has given rise to a controversy between Professor Challis and the Astronomer Royal, who maintains the justice of Poisson's result.

In a paper read before the Royal Society of Edinburgh on the 16th of December 1833, and printed in the 13th Volume of the Society's *Transactions*, Green has determined from the common equations of fluid motion the resistance to an ellipsoid performing small oscillations without rotation. The result is expressed by a definite integral; but when two of the principal axes of the ellipsoid become equal, the integral admits of expression in finite terms, by means of circular or logarithmic functions. When the ellipsoid becomes a sphere, Green's result reduces itself to Poisson's.

In a memoir read before the Royal Academy of Turin on the 18th of January 1835, and printed in the 37th Volume of the memoirs of the Academy, M. Plana has entered at great length into the theory of the resistance of fluids to pendulums. This memoir contains, however, rather a detailed examination of various points connected with the theory, than the determination of the resistance for any new form of pendulum. The author first treats the case of an incompressible fluid, and then shews that the result would be sensibly the same in the case of an elastic fluid. In the case of a ball pendulum, the only one in which a complete solution of the problem is effected, M. Plana's result agrees with Poisson's.

In a paper read before the Cambridge Philosophical Society on the 29th of May 1843, and printed in the 8th Volume of the *Transactions*, p. 105, I have determined the resistance to a ball pendulum oscillating within a concentric spherical envelope, and have pointed out the source of an error into which Poisson had fallen, in concluding that such an envelope would have no effect. When the radius of the envelope becomes infinite, the solution agrees with that which Poisson had obtained for the case of an unlimited mass of fluid. I have also investigated the increase of resistance due to the confinement of the fluid by a distant rigid plane. The same paper contains likewise the calculation of the resistance to a long cylinder oscillating in a mass of fluid either unlimited, or confined by a cylindrical envelope, having the same axis as the cylinder in its position of equilibrium. In the case of an unconfined mass of fluid, it appeared that the effect of inertia was the same as if a mass equal to that of the fluid displaced were distributed along the axis of the cylinder, so that $n = 2$ in the case of a pendulum consisting of a long cylindrical rod. This nearly agrees with Baily's result for the long $1\frac{1}{2}$ inch tube; but, on comparing it with the results obtained with the cylindrical rods, we observe the same sort of discrepancy between theory and observation as was noticed in the case of spheres. The discrepancy is, however, far more striking in the present case, as might naturally have been expected, after what had been observed with spheres, on account of the far smaller diameter of the solids employed.

A few years ago Professor Thomson communicated to me a very beautiful and powerful method which he had applied to the theory of electricity, which depended on the consideration of what he called *electrical images*. The same method, I found, applied, with a certain modification, to some interesting problems relating to ball pendulums. It enabled me to calculate the resistance to a sphere oscillating in presence of a fixed sphere, or within a spherical envelope, or the resistance to a pair of spheres either in contact, or connected by a narrow rod, the direction of oscillation being, in all these cases, that of the line joining the centres of the spheres. The effect of a rigid plane perpendicular to the direction of motion is of course included as a particular case. The method even applies, as Professor Thomson pointed out to me, to the uncouth solid bounded by the exterior segments of two intersecting spheres, provided the exterior angle of intersection be a submultiple of two right angles. A set of corresponding problems, in which the spheres are replaced by long cylinders, may be solved in a similar manner. These results were mentioned at the meeting of the British Association at Oxford in 1847, and are noticed in the volume of reports for that year, but they have not yet been published in detail.

The preceding are all the investigations that have fallen under my notice, of which the object was to calculate from hydrodynamics the resistance to a body of given form oscillating as a pendulum. They all proceed on the ordinary equations of the motion of fluids. They all fail to account for one leading feature of the experimental results, namely, the increase of the factor n with a decrease in the dimensions of the body. They recognize no distinction between the action of different fluids, except what arises from their difference of density.

In a conversation with Dr Robinson about seven or eight years ago on the subject of the application of theory to pendulums, he noticed the discrepancy which existed between the results of theory and experiment relating to a ball pendulum, and expressed to me his conviction that the discrepancy in question arose from the adoption of the ordinary theory of fluid motion, in which the pressure is supposed to be equal in all directions. He also described to me a remarkable experiment of Sir James South's which he had witnessed. This experiment has not been published, but Sir James South has kindly allowed me to mention it. When a pendulum is in motion, one would naturally have supposed that the air near the moving body glided past the surface, or the surface past it, which comes to the same thing if the relative motion only be considered, with a velocity comparable with the absolute velocity of the surface itself. But on attaching a piece of gold leaf to the bottom of a pendulum, so as to stick out in a direction perpendicular to the surface, and then setting the pendulum in motion, Sir James South found that the gold leaf retained its perpendicular position just as if the pendulum had been at rest; and it was not till the gold leaf carried by the pendulum had been removed to some distance from the surface, that it began to lag behind. This experiment shews clearly the existence of a tangential action between the pendulum and the air, and between one layer of air and another. The existence of a similar action in water is clearly exhibited in some experiments of Coulomb's which will be mentioned in the second part of this paper, and indeed might be concluded from several very ordinary phenomena. Moreover Dubuat, in discussing the results of his experiments on the oscillations of spheres in water, notices a slight increase in the effect of the water corresponding to an increase in the time of vibration, and expressly attributes it to the *viscosity* of the fluid.

Having afterwards occupied myself with the theory of the friction of fluids, and arrived at general equations of motion, the same in essential points as those which had been previously obtained in a totally different manner by others, of which, however, I was not at the time aware, I was desirous of applying, if possible, these equations to the calculation of the motion of some kind of pendulum. The difficulty of the problem is of course materially increased by the introduction of internal friction, but as I felt great confidence in the essential parts of the theory, I thought that labour would not be ill-bestowed on the subject. I first tried a long cylinder, because the solution of the problem appeared likely to be simpler than in the case of a sphere. But after having proceeded a good way towards the result, I was stopped by a difficulty relating to the determination of the arbitrary constants, which appeared as the coefficients of certain infinite series by which the integral of a certain differential equation was expressed. Having failed in the case of a cylinder, I tried a sphere, and presently found that the corresponding differential equation admitted of integration in finite terms, so that the solution of the problem could be completely effected. The result, I found, agreed very well with Baily's experiments, when the numerical value of a certain constant was properly assumed; but the subject was laid aside for some time. Having afterwards attacked a definite integral to which Mr Airy had been led in considering the theory of the illumination in the neighbourhood of a caustic, I found that the method which I had employed in the case of this integral would apply to the problem of the resistance to a cylinder, and it enabled me to get over the difficulty with which I had before beeen baffled. I immediately completed the numerical calculation, so far as was requisite to compare the formulæ with Baily's experiments on cylindrical rods, and found a remarkably close agreement between theory and observation. These results were mentioned at the meeting of the British Association at Swansea in 1848, and are briefly described in the volume of reports for that year.

The present paper is chiefly devoted to the solution of the problem in the two cases of a sphere and of a long cylinder, and to a comparison of the results with the experiments of Baily and others. Expressions are deduced for the effect of a fluid both on the time and on the arc of vibration of a pendulum consisting either of a sphere, or of a cylindrical rod, or of a combination of a sphere and a rod. These expressions contain only one disposable constant, which has a very simple physical meaning, and which I propose to call the *index of friction* of the fluid. This constant we may conceive determined by one observation, giving the effect of the fluid either on the time or on the arc of vibration of any one pendulum of one of the above forms, and then the theory ought to predict the effect both on the time and on the arc of vibration of all such pendulums. The agreement of theory with the experiments of Baily on the time of vibration is remarkably close. Even the rate of decrease of the arc of vibration, which it formed no part of Baily's object to observe, except so far as was necessary for making the small correction for reduction to indefinitely small vibrations, agrees with the result calculated from theory as nearly as could reasonably be expected under the circumstances.

It follows from theory that with a given sphere or cylindrical rod the factor n increases with the time of vibration. This accounts in a good measure for the circumstance that Bessel obtained so large a value of k for air, as is shewn at length in the present paper; though it unquestionably arose in a great degree from the increase of resistance due to the close proximity of a rigid plane to the swinging ball.

I have deduced the value of the index of friction of water from some experiments of Coulomb's on the decrement of the arc of oscillation of disks, oscillating in water in their own plane by the torsion of a wire. When the numerical value thus obtained is substituted in the expression for the time of vibration of a sphere, the result agrees almost exactly with Bessel's experiments with a sphere swung in water.

The present paper contains one or two applications of the theory of internal friction to problems which are of some interest, but which do not relate to pendulums. The resistance to a sphere moving uniformly in a fluid may be obtained as a limiting case of the resistance to a ball pendulum, provided the circumstances be such that the square of the velocity may be neglected. The resistance thus determined proves to be proportional, for a given fluid and a given velocity, not to the surface, but to the radius of the sphere; and therefore the accelerating force of the resistance increases much more rapidly, as the radius of the sphere decreases, than if the resistance varied as the surface, as would follow from the common theory. Accordingly, the resistance to a minute globule of water falling through the air with its terminal velocity depends almost wholly on the internal friction of air. Since the index of friction of air is known from pendulum experiments, we may easily calculate the terminal velocity of a globule of given size, neglecting the part of the resistance which depends upon the square of the velocity. The terminal velocity thus obtained is so small in the case of small globules such as those of which we may conceive a cloud to be composed, that the apparent suspension of the clouds does not seem to present any difficulty. Had the resistance been determined from the common theory, it would have been necessary to suppose the globules much more minute, in order to account in this way for the phenomenon. Since in the case of minute globules falling with their terminal velocity the part of the resistance depending upon the square of the velocity, as determined by the common theory, is quite insignificant compared with the part which depends on the internal friction of the air, it follows that were the pressure equal in all directions in air in the state of motion, the quantity of water which would remain suspended in the state of cloud would be enormously diminished. The pendulum thus, in addition to its other uses, affords us some interesting information relating to the department of meteorology.

The fifth section of the first part of the present paper contains an investigation of the effect of the internal friction of water in causing a series of oscillatory waves to subside. It appears from the result that in the case of the long swells of the ocean the effect of friction is insignificant, while in the case of the ripples raised by the wind on a small pool, the motion subsides very rapidly when the disturbing force ceases to act.

PART I.

ANALYTICAL INVESTIGATION.

SECTION I.

Adaptation of the general equations to the case of the fluid surrounding a body which oscillates as a pendulum. General laws which follow from the form of the equations. Solution of the equations in the case of an oscillating plane.

1. In a paper "*On the Theories of the Internal Friction of Fluids in Motion, &c.,*" which the Society did me the honour to publish in the 8th Volume of their *Transactions*, I have arrived at the following equations for calculating the motion of a fluid when the internal friction of the fluid itself is taken into account, and consequently the pressure not supposed equal in all directions:

$$\frac{dp}{dx} = \rho\left(X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}\right) + \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)$$

$$+ \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with two more equations which may be written down from symmetry. In these equations u, v, w are the components of the velocity along the rectangular axes of x, y, z; X, Y, Z are the components of the accelerating force; p is the pressure, t the time, ρ the density, and μ a certain constant depending on the nature of the fluid.

The three equations of which (1) is the type are not the general equations of motion which apply to a heterogeneous fluid when internal friction is taken into account, which are those numbered 10 in my former paper, but are applicable to a homogeneous incompressible fluid, or to a homogeneous elastic fluid subject to small variations of density, such as those which accompany sonorous vibrations. It must be understood to be included in the term *homogeneous* that the temperature is uniform throughout the mass, except so far as it may be raised or lowered by sudden condensation or rarefaction in the case of an elastic fluid. The general equations contain the differential coefficients of the quantity μ with respect to x, y, and z; but the equations of the form (1) are in their present shape even more general than is required for the purposes of the present paper.

These equations agree in the main with those which had been previously obtained, on different principles, by Navier, by Poisson, and by M. de Saint-Venant, as I have elsewhere observed*. The differences depend only on the coefficient of the last term, and this term vanishes in the case of an incompressible fluid, to which Navier had confined his investigations.

The equations such as (1) in their present shape are rather complicated, but in applying them to the case of a pendulum they may be a good deal simplified without the neglect of any quantities which it would be important to retain. In the first place the motion is supposed very small, on which account it will be allowable to neglect the terms which involve the square of the velocity. In the second place, the nature of the motion that we have got to deal with is such that the compressibility of the fluid has very little influence on the result, so that we may treat the fluid as incompressible, and consequently omit the last terms in the equations. Lastly, the forces X, Y, Z are in the present case the components of the force of gravity, and if we write

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* Report on recent researches in Hydrodynamics. Report of the British Association for 1846, p. 16.

$$p + \Pi + \rho \int (X dx + Y dy + Z dz)$$

for p, we may omit the terms X, Y, Z.

If z' be measured vertically downwards from a horizontal plane drawn in the neighbourhood of the pendulum, and if g be the force of gravity, $\int (X dx + Y dy + Z dz) = gz'$, the arbitrary constant, or arbitrary function of the time if it should be found necessary to suppose it to be such, being included in Π. The part of the whole force acting on the pendulum which depends on the terms $\Pi + g\rho z'$ is simply a force equal to the weight of the fluid displaced, and acting vertically upwards through the centre of gravity of the volume.

When simplified in the manner just explained, the equations such as (1) become

$$\left.\begin{aligned} \frac{dp}{dx} &= \mu \left(\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2}\right) - \rho \frac{du}{dt}, \\ \frac{dp}{dy} &= \mu \left(\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2}\right) - \rho \frac{dv}{dt}, \\ \frac{dp}{dz} &= \mu \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \frac{d^2 w}{dz^2}\right) - \rho \frac{dw}{dt}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

which, with the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \quad \ldots\ldots\ldots\ldots \quad (3)$$

are the only equations which have to be satisfied at all points of the fluid, and at all instants of time.

In applying equations (2) to a particular pendulum experiment, we may suppose μ constant; but in order to compare experiments made in summer with experiments made in winter, or experiments made under a high barometer with experiments made under a low, it will be requisite to regard μ as a quantity which may vary with the temperature and pressure of the fluid. As far as the result of a single experiment*, which has been already mentioned, performed with a single elastic fluid, namely air, justifies us in drawing such a general conclusion, we may assert that for a given fluid at a given temperature μ varies as ρ.

2. For the formation of the equations such as (1), I must refer to my former paper;

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* The first of the experiments described in Col. Sabine's paper, in which the gauge stood as high as 7 inches, leads to the same conclusion; but as the vacuum apparatus had not yet been made stanch it is perhaps hardly safe to trust this experiment in a question of such delicacy.

but it will be possible, in a few words, to enable the reader to form a clear idea of the meaning of the constant μ.

Conceive the fluid to move in planes parallel to the plane of xy, the motion taking place in a direction parallel to the axis of y. The motion will evidently consist of a sort of continuous sliding, and the differential coefficient $\frac{dv}{dx}$ may be taken as a measure of the rate of sliding. In the theory it is supposed that in general the pressure about a given point is compounded of a normal pressure, corresponding to the density, which being normal is necessarily equal in all directions, and of an oblique pressure or tension, altering from one direction to another, which is expressed by means of linear functions of the nine differential coefficients of the first order of u, v, w with respect to x, y, z, which define the state of relative motion at any point of the fluid. Now in the special case considered above, if we confine our attention to one direction, that of the plane of xy, the total pressure referred to a unit of surface is compounded of a normal pressure corresponding to the density, and a tangential pressure expressed by $\mu \frac{dv}{dx}$, which tends to reduce the relative motion.

In the solution of equations (2), μ always appears divided by ρ. Let $\mu = \mu' \rho$. The constant μ' may conveniently be called the *index of friction* of the fluid, whether liquid or gas, to which it relates. As regards its dimensions, it expresses a moving force divided by the product of a surface, a density, and the differential coefficient of a velocity with respect to a line. Hence μ' is the square of a line divided by a time, whence it will be easy to adapt the numerical value of μ' to a new unit of length or of time.

3. Besides the general equations (2) and (3), it will be requisite to consider the equations of condition at the boundaries of the fluid. For the purposes of the present paper there will be no occasion to consider the case of a free surface, but only that of the common surface of the fluid and a solid. Now, if the fluid immediately in contact with a solid could flow past it with a finite velocity, it would follow that the solid was infinitely smoother with respect to its action on the fluid than the fluid with respect to its action on itself. For, conceive the elementary layer of fluid comprised between the surface of the solid and a parallel surface at a distance h, and then regard only so much of this layer as corresponds to an elementary portion dS of the surface of the solid. The impressed forces acting on the fluid element must be in equilibrium with the effective forces reversed. Now conceive h to vanish compared with the linear dimensions of dS, and lastly let dS vanish[*]. It is evident that the conditions of equilibrium will ultimately reduce themselves to this, that the oblique pressure which the fluid element experiences on the side of the solid must be equal and opposite to the pressure which it experiences on the side of the fluid. Now if the fluid could flow past the solid with a finite velocity, it would follow that the tangential pressure

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[*] To be quite precise it would be necessary to say, Conceive h and dS to vanish together, h vanishing compared with the linear dimensions of dS; for so long as dS remains finite we cannot suppose h to vanish altogether, on account of the curvature of the elementary surface. Such extreme precision in unimportant matters tends, I think, only to perplex the reader, and prevent him from entering so readily into the spirit of an investigation.

called into play by the continuous sliding of the fluid over itself was no more than counteracted by the abrupt sliding of the fluid over the solid. As this appears exceedingly improbable *a priori*, it seems reasonable in the first instance to examine the consequences of supposing that no such abrupt sliding takes place, more especially as the mathematical difficulties of the problem will thus be materially diminished. I shall assume, therefore, as the conditions to be satisfied at the boundaries of the fluid, that the velocity of a fluid particle shall be the same, both in magnitude and direction, as that of the solid particle with which it is in contact. The agreement of the results thus obtained with observation will presently appear to be highly satisfactory. When the fluid, instead of being confined within a rigid envelope, extends indefinitely around the oscillating body, we must introduce into the solution the condition that the motion shall vanish at an infinite distance, which takes the place of the condition to be satisfied at the surface of the envelope.

To complete the determination of the arbitrary functions which would be contained in the integrals of (2) and (3), it would be requisite to put $t = 0$ in the general expressions for u, v, w, obtained by integrating those equations, and equate the results to the initial velocities supposed to be given. But it would be introducing a most needless degree of complexity into the solution to take account of the initial circumstances, nor is it at all necessary to do so for the sake of comparison of theory with experiment. For in a pendulum experiment the pendulum is set swinging and then left to itself, and the first observation is not taken till several oscillations have been completed, during which any irregularities attending the initial motion would have had time to subside. It will be quite sufficient to regard the motion as already going on, and limit the calculation to the determination of the simultaneous periodic movements of the pendulum and the surrounding fluid. The arc of oscillation will go on slowly decreasing, but it will be so nearly constant for several successive oscillations that it may be regarded as strictly such in calculating the motion of the fluid; and having thus determined the resultant action of the fluid on the solid we may employ the result in calculating the decrement of the arc of oscillation, as well as in calculating the time of oscillation. Thus the assumption of periodic functions of the time in the expressions for u, v, w will take the place of the determination of certain arbitrary functions by means of the initial circumstances.

4. Imagine a plane drawn perpendicular to the axis of x through the point in the fluid whose co-ordinates are x, y, z. Let the oblique pressure in the direction of this plane be decomposed into three pressures, a normal pressure, which will be in the direction of x, and two tangential pressures in the directions of y, z, respectively. Let P_1 be the normal pressure, and T_3 the tangential pressure in the direction of y, which will be equal to the component in the direction of x of the oblique pressure on a plane drawn perpendicular to the axis of y. Then by the formulæ (7), (8) of my former paper, and (3) of the present,

$$P_1 = p - 2\mu \frac{du}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$T_3 = -\mu \left(\frac{du}{dy} + \frac{dv}{dx}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These formulæ will be required in finding the resultant force of the fluid on the pendulum, after the motion of the fluid has been determined in terms of the quantities by which the motion of the pendulum is expressed.

5. Before proceeding to the solution of the equations (2) and (3) in particular cases, it will be well to examine the general laws which follow merely from the dimensions of the several terms which appear in the equations.

Consider any number of similar systems, composed of similar solids oscillating in a similar manner in different fluids or in the same fluid. Let a, a', a''... be homologous lines in the different systems; T, T', T''... corresponding times, such for example as the times of oscillation from rest to rest. Let x, y, z be measured from similarly situated origins, and in corresponding directions, and t from corresponding epochs, such for example as the commencements of oscillations when the systems are beginning to move from a given side of the mean position.

The form of equations (2), (3) shews that the equations being satisfied for one system will be satisfied for all the systems provided

$$u \propto v \propto w, \quad x \propto y \propto z, \text{ and } p \propto \frac{\mu u}{x} \propto \frac{\rho u x}{t}.$$

The variations $x \propto y \propto z$ merely signify that we must compare similarly situated points in inferring from the circumstance that (2), (3) are satisfied for one system that they will be satisfied for all the systems. If c, c', c''... be the maximum excursions of similarly situated points of the fluids

$$u \propto \frac{c}{T}, \quad x \propto a, \quad t \propto T,$$

and the sole condition to be satisfied, in addition to that of geometrical similarity, in order that the systems should be dynamically similar, becomes

$$\frac{a^2}{T} \propto \frac{\mu}{\rho} \text{ or } \propto \mu'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This condition being satisfied, similar motions will take place in the different systems, and we shall have

$$p \propto \frac{\rho a c}{T^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It follows from the equations (4), (5), and the other equations which might be written down from symmetry, that the pressures such as P_1, T_3 vary in the same manner as p, whence it appears from (7) that the resultant or resultants of the pressures of the fluids on the solids, acting along similarly situated lines, which vary as pa^2, vary as ρa^3 and $c\,T^{-2}$ conjointly. In other words, these resultants in two similar systems are to one another in a ratio compounded of the ratio of the masses of fluid displaced, and of the ratio of the maximum accelerating effective forces belonging to similarly situated points in the solids.

6. In order that two systems should be similar in which the fluids are confined by envelopes that are sufficiently narrow to influence the motion of the fluids, it is necessary that

the envelopes should be similar and similarly situated with respect to the solids oscillating within them, and that their linear dimensions should be in the same ratio as those of the oscillating bodies. In strictness, it is likewise necessary that the solids should be similarly situated with respect to the axis of rotation. If however two similar solids, such as two spheres, are attached to two fine wires, and made to perform small oscillations in two unlimited masses of fluid, and if we agree to neglect the effect of the suspending wires, and likewise the effect of the rotation of the spheres on the motion of the fluid, which last will in truth be exceedingly small, we may regard the two systems as geometrically similar, and they will be dynamically similar provided the condition (6) be satisfied. When the two fluids are of the same nature, as for example when both spheres oscillate in air, the condition of dynamical similarity reduces itself to this, that the times of oscillation shall be as the squares of the diameters of the spheres.

If, with Bessel, we represent the effect of the inertia of the fluid on the time of oscillation of the sphere by supposing a mass equal to k times that of the fluid displaced added to the mass of the sphere, which increases its inertia without increasing its weight, we must expect to find k dependant on the nature of the fluid, and likewise on the diameter of the sphere. Bessel, in fact, obtained very different values of k for water and for air. Baily's experiments on spheres of different diameters, oscillating once in a second nearly, shew that the value of k increases when the diameter of the sphere decreases. Taking this for the present as the result of experiment, we are led from theory to assert that the value of k increases with the time of oscillation; in fact, k ought to be as much increased as if we had left the time of oscillation unchanged, and diminished the diameter in the ratio in which the square root of the time is increased. It may readily be shewn that the value of k obtained by Bessel's method, by means of a long and short pendulum, is greater than what belongs to the long pendulum, much more, greater than what belongs to the shorter pendulum, which oscillated once in a second nearly. The value of k given by Bessel is in fact considerably larger than that obtained by Baily, by a direct method, from a sphere of nearly the same size as those employed by Bessel, oscillating once in a second nearly.

The discussion of the experiments of Baily and Bessel belongs to Part II. of this paper. They are merely briefly noticed here to shew that some results of considerable importance follow readily from the general equations, even without obtaining any solution of them.

7. Before proceeding to the problems which mainly occupy this paper, it may be well to exhibit the solution of equations (2) and (3) in the extremely simple case of an oscillating plane.

Conceive a physical plane, which is regarded as infinite, to be situated in an unlimited mass of fluid, and to be performing small oscillations in the direction of a fixed line in the plane. Let a fixed plane coinciding with the moving plane be taken for the plane of yz, the axis of y being parallel to the direction of motion, and consider only the portion of fluid which lies on the positive side of the plane of yz. In the present case, we must evidently have $u = 0$, $w = 0$; and p, v will be functions of x and t, which have to be determined. The equation (3) is satisfied identically, and we get from (2), putting $\mu = \mu'\rho$,

$$\frac{dp}{dx} = 0, \quad \frac{dv}{dt} = \mu' \frac{d^2 v}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

The first of these equations gives $p =$ a constant, for it evidently cannot be a function of t, since the effect of the motion vanishes at an infinite distance from the plane; and if we include this constant in Π, we shall have $p = 0$. Let V be the velocity of the plane itself, and suppose

$$V = c \sin nt. \quad \text{. . . . .} \quad (9)$$

Putting in the second of equations (8)

$$v = X_1 \sin nt + X_2 \cos nt, \quad \text{. . . . .} \quad (10)$$

we get

$$nX_1 = \mu' \frac{d^2 X_2}{dx^2}, \quad nX_2 = -\mu' \frac{d^2 X_1}{dx^2} = -\frac{\mu'^2}{n} \frac{d^4 X_2}{dx^4}. \quad \text{. . . .} \quad (11)$$

The last of these equations gives

$$X_2 = \epsilon^{-\sqrt{\frac{n}{2\mu'}}x} \left(A \sin \sqrt{\frac{n}{2\mu'}}x + B \cos \sqrt{\frac{n}{2\mu'}}x\right) + \epsilon^{\sqrt{\frac{n}{2\mu'}}x} \left(C \sin \sqrt{\frac{n}{2\mu'}}x + D \cos \sqrt{\frac{n}{2\mu'}}x\right).$$

Since X_2 must not become infinite when $x = \infty$, we must have $C = 0$, $D = 0$. Obtaining X_1 from the first of equations (11), and substituting in (10), we get

$$v = \epsilon^{-\sqrt{\frac{n}{2\mu'}}x} \left\{-A \sin \left(nt - \sqrt{\frac{n}{2\mu'}}x\right) + B \cos \left(nt - \sqrt{\frac{n}{2\mu'}}x\right)\right\}.$$

Now by the equations of conditions assumed in Art. 3, we must have $v = V$ when $x = 0$, whence

$$v = c\epsilon^{-\sqrt{\frac{n}{2\mu'}}x} \sin \left(nt - \sqrt{\frac{n}{2\mu'}}x\right). \quad \text{. . . . . .} \quad (12)$$

To find the normal and tangential components of the pressure of the fluid on the plane, we must substitute the above value of v in the formulæ (4), (5), and after differentiation put $x = 0$. P_1, T_3 will then be the components of the pressure of the solid on the fluid, and therefore $-P_1$, $-T_3$, those of the pressure of the fluid on the solid. We get

$$P_1 = 0, \quad T_3 = c\rho \sqrt{\frac{n\mu'}{2}} (\sin nt + \cos nt) = \rho \sqrt{\frac{n\mu'}{2}} \left(V + \frac{1}{n} \frac{dV}{dt}\right). \quad \text{. .} \quad (13)$$

The force expressed by the first of these terms tends to diminish the amplitude of the oscillations of the plane. The force expressed by the second has the same effect as increasing the inertia of the plane.

8. The equation (12) shews that a given phase of vibration is propagated from the plane into the fluid with a velocity $\sqrt{(2\mu' n)}$, while the amplitude of oscillation decreases in geometric progression as the distance from the plane increases in arithmetic. If we suppose the time of oscillation from rest to rest to be one second, $n = \pi$; and if we suppose $\sqrt{\mu'} = .116$ inch, which, as will presently be seen, is about its value in the case of air, we get for the velocity of propagation .2908 inch per second nearly. If we enquire the distance from the plane at which the amplitude of oscillation is reduced to one half, we have only to put $\sqrt{\frac{n}{2\mu'}}x = \log_\epsilon 2$, which gives, on the same suppositions as before respecting numerical values, $x = .06415$ inch nearly.

For water the value of μ' is a good deal smaller than for air, and the corresponding value of x smaller likewise, since it varies *cæteris paribus* as $\sqrt{\mu'}$. Hence if a solid of revolution of large, or even moderately large, dimensions be suspended by a fine wire coinciding with the axis of revolution, and made to oscillate by the torsion of the wire, the effect of the fluid may be calculated with a very close degree of approximation by regarding each element of the surface of the solid as an element of an infinite plane oscillating with the same linear velocity.

For example, let a circular disk of radius a be suspended horizontally by a fine wire attached to the centre, and made to oscillate. Let r be the radius vector of any element of the disk, measured from its centre, θ the angle through which the disk has turned from its mean position. Then in equation (13), we must put $V = r\dfrac{d\theta}{dt}$, whence

$$T_3 = \rho\sqrt{\frac{n\mu'}{2}}\,r\left(\frac{d\theta}{dt} + \frac{1}{n}\frac{d^2\theta}{dt^2}\right).$$

The area of the annulus of the disk comprised between the radii r and $r + dr$ is $4\pi r dr$, both faces being taken, and if G be the whole moment of the force of the fluid on the disk,

$G = -4\pi\int_0^a r^2 T_3\,dr$, whence

$$G = -\pi\rho a^4\sqrt{\frac{n\mu'}{2}}\left(\frac{d\theta}{dt} + \frac{1}{n}\frac{d^2\theta}{dt^2}\right).$$

Let $M\gamma^2$ be the moment of inertia of the disk, and let n_1 be what n would become if the fluid were removed, so that $-n_1^2 M\gamma^2\theta$ is the moment of the force of torsion. Then when the fluid is present the equation of motion of the disk becomes

$$\left(M\gamma^2 + \pi\rho a^4\sqrt{\frac{\mu'}{2n}}\right)\frac{d^2\theta}{dt^2} + \pi\rho a^4\sqrt{\frac{n\mu'}{2}}\frac{d\theta}{dt} + n_1^2 M\gamma^2\theta = 0, \quad . \quad . \quad (14)$$

or, putting for shortness

$$\pi\rho a^4\sqrt{\frac{\mu'}{2n}} = 2\beta M\gamma^2,$$

$$(1 + 2\beta)\frac{d^2\theta}{dt^2} + 2n\beta\frac{d\theta}{dt} + n_1^2\theta = 0,$$

which gives, neglecting β^2,

$$\theta = \theta_0\epsilon^{-n\beta t}\sin(nt + \alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where

$$n = n_1(1 - \beta).$$

The observation of n and n_1, or else the observation of n and of the decrement of the arc of oscillation, would enable us to determine β, and thence μ'. The values of β determined in these two different ways ought to agree.

There would be no difficulty in obtaining a more exact solution, in which the decrement of the arc of oscillation should be taken into account in calculating the motion of the fluid, but I pass on to the problems, the solution of which forms the main object of this paper.

SECTION II.

Solution of the equations in the case of a sphere oscillating in a mass of fluid either unlimited, or confined by a spherical envelope concentric with the sphere in its position of equilibrium.

9. Suppose the sphere suspended by a fine wire, the length of which is much greater than the radius of the sphere. Neglect for the present the action of the wire on the fluid, and consider only that of the sphere. The motion of the sphere and wire being supposed to take place parallel to a fixed vertical plane, there are two different modes of oscillation possible. We have here nothing to do with the rapid oscillations which depend mainly on the rotatory inertia of the sphere, but only with the principal oscillations, which are those which are observed in pendulum experiments. In these principal oscillations the centre of the sphere describes a small arc of a curve which is very nearly a circle, and which would be rigorously such, if the line joining the centre of gravity of the sphere and the point of attachment of the wire were rigorously in the direction of the wire. In calculating the motion of the fluid, we may regard this arc as a right line. In fact, the error thus introduced would only be a small quantity of the second order, and such quantities are supposed to be neglected in the investigation. Besides its motion of translation, the sphere will have a motion of rotation about a horizontal axis, the angular motion of the sphere being very nearly the same as that of the suspending wire. This motion, which would produce absolutely no effect on the fluid according to the common theory of hydrodynamics, will not be without its influence when friction is taken into account; but the effect is so very small in practical cases that it is not worth while to take it into account. For if a be the radius of the sphere, and l the length of the suspending wire, the velocity of a point in the surface of the sphere due to the motion of rotation will be a small quantity of the order $a l^{-1}$ compared with the velocity due to the motion of translation. In finding the moment of the pressures of the fluid on the pendulum, forces arising from these velocities, and comparable with them, have to be multiplied by lines which are comparable with a, l, respectively. Hence the moment of the pressures due to the motion of rotation of the sphere will be a small quantity of the order $a^2 l^{-2}$, compared with the moment due to the motion of translation. Now in practice l is usually at least 20 or 30 times greater than a, and the whole effect to be investigated is very small, so that it would be quite useless to take account of the motion of rotation of the sphere.

The problem, then, reduces itself to this. The centre of a sphere performs small periodic oscillations along a right line, the sphere itself having a motion of translation simply: it is required to determine the motion of the surrounding fluid.

10. Let the mean position of the centre of the sphere be taken for origin, and the direction of its motion for the axis of x, so that the motion of the fluid is symmetrical with respect to this axis. Let ϖ be the perpendicular let fall from any point on the axis of x, q

the velocity in the direction of ϖ, ω the angle between the line ϖ and the plane of xy. Then p, u, and q will be functions of x, ϖ, and t, and we shall have

$$v = q \cos\omega, \quad w = q \sin\omega, \quad y = \varpi \cos\omega, \quad z = \varpi \sin\omega,$$

whence

$$\varpi^2 = y^2 + z^2, \quad \omega = \tan^{-1}\frac{z}{y}.$$

We have now to substitute in equations (2) and (3), and we are at liberty to put $\omega = 0$ after differentiation. We get

$$\frac{d}{dy} = \cos\omega\frac{d}{d\varpi} - \frac{\sin\omega}{\varpi}\frac{d}{d\omega}, \quad = \frac{d}{d\varpi} \text{ when } \omega = 0,$$

$$\frac{d^2}{dy^2} = \frac{d^2}{d\varpi^2} \text{ when } \omega = 0,$$

$$\frac{d}{dz} = \sin\omega\frac{d}{d\varpi} + \frac{\cos\omega}{\varpi}\frac{d}{d\omega}, \quad = \frac{1}{\varpi}\frac{d}{d\omega} \text{ when } \omega = 0,$$

$$\frac{d^2}{dz^2} = \frac{1}{\varpi}\frac{d}{d\varpi} + \frac{1}{\varpi^2}\frac{d^2}{d\omega^2} \text{ when } \omega = 0,$$

whence we obtain

$$\frac{dp}{dx} = \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{d\varpi^2} + \frac{1}{\varpi}\frac{du}{d\varpi}\right) - \rho\frac{du}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\frac{dp}{d\varpi} = \mu\left(\frac{d^2q}{dx^2} + \frac{d^2q}{d\varpi^2} + \frac{1}{\varpi}\frac{dq}{d\varpi} - \frac{q}{\varpi^2}\right) - \rho\frac{dq}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\frac{du}{dx} + \frac{dq}{d\varpi} + \frac{q}{\varpi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Eliminating p from (16) and (17), and putting for μ its equivalent $\mu'\rho$, we get

$$\mu'\frac{d}{d\varpi}\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi}\right)u - \mu'\frac{d}{dx}\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\varpi^2}\right)q - \frac{d}{dt}\left(\frac{du}{d\varpi} - \frac{dq}{dx}\right) = 0,$$

$$\text{or} \quad \left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\varpi^2} - \frac{1}{\mu'}\frac{d}{dt}\right)\left(\frac{du}{d\varpi} - \frac{dq}{dx}\right) = 0. \quad . \quad . \quad (19)$$

By virtue of (18), $\varpi(u\,d\varpi - q\,dx)$ is an exact differential. Let then

$$\varpi(u\,d\varpi - q\,dx) = d\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Expressing u and q in terms of ψ, we get

$$\frac{du}{d\varpi} - \frac{dq}{dx} = \frac{1}{\varpi}\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}\right)\psi.$$

Substituting in (19), and operating separately on the factor ϖ^{-1}, we obtain

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{d^2\varpi} - \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\mu'}\frac{d}{dt}\right)\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}\right)\psi = 0. \quad . \quad . \quad (20')$$

Since the operations represented by the two expressions within parentheses are evidently convertible, the integral of this equation is

$$\psi = \psi_1 + \psi_2,\text{[*]} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (21)$$

where ψ_1, ψ_2 are the integrals of the equations

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}\right)\psi_1 = 0, \quad . \ . \ . \ . \ . \ . \quad (22)$$

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\mu'}\frac{d}{dt}\right)\psi_2 = 0. \quad . \ . \ . \ . \ . \ . \quad (23)$$

11. By means of the last three equations, the expression for dp obtained from (16) and (17) is greatly simplified. We get, in the first place,

$$\frac{1}{\rho}\frac{dp}{dx} = \left\{\mu'\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi}\right) - \frac{d}{dt}\right\}\frac{1}{\varpi}\frac{d\psi}{d\varpi}: \quad . \ . \ . \ . \quad (24)$$

but by adding together equations (22) and (23), and taking account of (21), we get

$$\frac{d^2\psi}{dx^2} = -\frac{d^2\psi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\psi}{d\varpi} + \frac{1}{\mu'}\frac{d\psi_2}{dt}.$$

On substituting in (24), it will be found that all the terms in the right-hand member of the equation destroy one another, except those which contain $\dfrac{d\psi}{dt}$ and $\dfrac{d\psi_2}{dt}$, and the equation is reduced to

$$\frac{dp}{dx} = -\frac{\rho}{\varpi}\frac{d^2\psi_1}{dt\,d\varpi}.$$

The equation (17) may be reduced in a similar manner, and we get finally

$$dp = \frac{\rho}{\varpi}\left(\frac{d^2\psi_1}{dt\,dx}d\varpi - \frac{d^2\psi_1}{dt\,d\varpi}dx\right), \quad . \ . \ . \ . \ . \ . \quad (25)$$

which is an exact differential by virtue of (22).

---

[*] If we denote for shortness the operation

$$\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}$$

by D, our equation becomes

$$D\left(D - \frac{1}{\mu'}\frac{d}{dt}\right)\psi = 0,$$

which gives by the separation of symbols

$$\psi = \left\{D\left(D - \frac{1}{\mu'}\frac{d}{dt}\right)\right\}^{-1} 0$$

$$= \left(\frac{1}{\mu'}\frac{d}{dt}\right)^{-1} \left\{\left(D - \frac{1}{\mu'}\frac{d}{dt}\right)^{-1} - D^{-1}\right\} 0, \quad . \ . \ . \quad (a)$$

so that $\dfrac{d\psi}{dt}$ is composed of two parts, which are separately the integrals of (22), (23). Hence we have for the integral of (20′) $\psi = \psi_1 + \psi_2 + \Psi$, Ψ being a function of x and ϖ without t which satisfies the equation $D^2\Psi = 0$. For the equations (22), (23) will not be altered if we put $\int\psi_1 dt$, $\int\psi_2 dt$ for ψ_1, ψ_2, the arbitrary functions which would arise from the integration with respect to t being supposed to be included in Ψ. The function Ψ, which taken by itself can only correspond to steady motion, is excluded from the problem under consideration by the condition of periodicity. But we may even, independently of this condition, regard (21) as the complete integral of (20′), provided we suppose included in (21) terms which would be obtained by supposing ψ at first to vary slowly with the time, employing the integrals of (22) and (23), and then making the rate of variation diminish indefinitely. By treating the symbolical expression in the right-hand member of equation (a) as a vanishing fraction, $\dfrac{d}{dt}$ being supposed to vanish, we obtain in fact $D^{-2}0$; so that under the convention just mentioned the function Ψ may be supposed to be included in $\psi_1 + \psi_2$. The same remarks will apply to the equation in Section III. which answers to (20′).

12. Passing to polar co-ordinates, let r be the radius vector drawn from the origin, θ the angle which r makes with the axis of x, and let R be the velocity along the radius vector, Θ the velocity perpendicular to the radius vector: then

$$x = r\cos\theta,\quad \varpi = r\sin\theta,\quad u = R\cos\theta - \Theta\sin\theta,\quad q = R\sin\theta + \Theta\cos\theta.$$

Making these substitutions in (20), (22), (23), and (25), we obtain

$$r\sin\theta\,(Rr\,d\theta - \Theta\,dr) = d\psi, \quad \ldots\ldots \quad (26)$$

$$\frac{d^2\psi_1}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi_1}{d\theta}\right) = 0, \quad \ldots\ldots \quad (27)$$

$$\frac{d^2\psi_2}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi_2}{d\theta}\right) - \frac{1}{\mu'}\frac{d\psi_2}{dt} = 0, \quad \ldots \quad (28)$$

$$dp = \frac{\rho}{r\sin\theta}\left(\frac{d^2\psi_1}{dt\,dr}\,r\,d\theta - \frac{1}{r}\frac{d^2\psi_1}{dt\,d\theta}\,dr\right). \quad \ldots\ldots \quad (29)$$

We must now determine ψ_1 and ψ_2 by means of (27) and (28), combined with the equations of condition. When these functions are known, p will be obtained by integrating the exact differential which forms the right-hand member of (29), and the velocities R, Θ, if required, will be got by differentiation, as indicated by equation (26). Formulæ deduced from (4) and (5) will then make known the pressure of the fluid on the sphere.

13. Let ξ be the abcissa of the centre of the sphere at any instant. The conditions to be satisfied at the surface of the sphere are that when $r = r_1$, the radius vector of the surface, we have

$$R = \cos\theta\frac{d\xi}{dt},\qquad \Theta = -\sin\theta\frac{d\xi}{dt}.$$

Now r_1 differs from a by a small quantity of the first order, and since this value of r has to be substituted in functions which are already small quantities of that order, it will be sufficient to put $r = a$. Hence, expressing R and Θ in terms of ψ, we get

$$\frac{d\psi}{dr} = a\sin\theta^2\frac{d\xi}{dt},\quad \frac{d\psi}{d\theta} = a^2\sin\theta\cos\theta\frac{d\xi}{dt},\quad \text{when } r = a. \quad \ldots \quad (30)$$

When the fluid is unlimited, it will be found that certain arbitrary constants will vanish by the condition that the motion shall not become infinite at an infinite distance in the fluid. When the fluid is confined by an envelope having a radius b, we have the equations of condition

$$\frac{d\psi}{dr} = 0,\quad \frac{d\psi}{d\theta} = 0,\quad \text{when } r = b. \quad \ldots\ldots \quad (31)$$

14. We must now, in accordance with the plan proposed in Section I., introduce the condition that the function ψ shall be composed, so far as the time is concerned, of the circular functions $\sin nt$ and $\cos nt$, that is, that it shall be of the form $P\sin nt + Q\cos nt$, where P

and Q are functions of r and θ only. An artifice, however, which has been extensively employed by M. Cauchy will here be found of great use. Instead of introducing the circular functions $\sin nt$ and $\cos nt$, we may employ the exponentials $\epsilon^{\sqrt{-1}nt}$, and $\epsilon^{-\sqrt{-1}nt}$. Since our equations are linear, and since each of these exponential functions reproduces itself at each differentiation, it follows that if all the terms in any one of our equations be arranged in two groups, containing as a factor $\epsilon^{\sqrt{-1}nt}$ in one case, and $\epsilon^{-\sqrt{-1}nt}$ in the other, the two groups will be quite independent, and the equations will be satisfied by either group separately. Hence it will be sufficient to introduce one of the exponential functions. We shall thus have only half the number of terms to write down, and half the number of arbitrary constants to determine that would have been necessary had we employed circular functions. When we have arrived at our result, it will be sufficient to put each equation under the form $U + \sqrt{-1}\,V = 0$, and throw away the imaginary part, or else throw away the real part and omit $\sqrt{-1}$, since the system of quantities U, and the system of quantities V must separately satisfy the equations of the problem. Assuming then

$$\frac{d\xi}{dt} = c\epsilon^{\sqrt{-1}nt}, \qquad \psi = \epsilon^{\sqrt{-1}nt}P,$$

we have to determine P as a function of r and θ.

15. The form of the equations of condition (30) points out $\sin^2\theta$ as a factor of P, and since the operation $\sin\theta \dfrac{d}{d\theta}\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}$ performed on the function $\sin^2\theta$ reproduces the same function with a coefficient -2, it will be possible to satisfy equations (27) and (28) on the supposition that $\sin^2\theta$ is a factor of ψ_1 and ψ_2[*]. Assume then

$$\psi_1 = \epsilon^{\sqrt{-1}nt}\sin^2\theta\, f_1(r), \qquad \psi_2 = \epsilon^{\sqrt{-1}nt}\sin^2\theta\, f_2(r).$$

Putting for convenience

$$n\sqrt{-1} = \mu' m^2, \quad \dots\dots\dots\dots \quad (32)$$

and substituting in (27) and (28), we get

$$f_1''(r) - \frac{2}{r^2}f_1(r) = 0, \quad \dots\dots\dots\dots \quad (33)$$

$$f_2''(r) - \frac{2}{r^2}f_2(r) - m^2 f_2(r) = 0. \quad \dots\dots\dots \quad (34)$$

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[*] When this operation is performed on the function $\sin\theta\dfrac{dY_i}{d\theta}$, the function is reproduced with a coefficient $-i(i+1)$. Y_i here denotes a Laplace's coefficient of the i^{th} order, which contains only one variable angle, namely θ. Now ψ may be expanded in a series of quantities of the general form $\sin\theta\dfrac{dY_i}{d\theta}$. For, since we are only concerned with the differential coefficients of ψ with respect to r and θ, we have a right to suppose ψ to vanish at whatever point of space we please. Let then $\psi = 0$ when $r = a$ and $\theta = 0$. To find the value of ψ at a distance r from the origin, along the axis of x positive, it will be sufficient to put $\theta = 0$, $d\theta = 0$ in (26), and integrate from $r = a$ to r, whence $\psi = 0$. To find the value of ψ at the same distance r along the axis of x negative, it will be sufficient to leave r constant, and integrate $d\psi$ from $\theta = 0$ to $\theta = \pi$. Referring to (26), we see that the integral vanishes, since the total flux across the surface of the sphere whose radius is r must be equal to zero. Hence ψ vanishes when $\theta = 0$ or $= \pi$, and it appears from (26) that when θ is very small or very nearly equal to π, ψ varies ultimately as $\sin^2\theta$ for given values of r and t. Hence $\psi \operatorname{cosec}\theta$, and therefore $\int\psi \operatorname{cosec}\theta\, d\theta$, is finite even when $\sin\theta$ vanishes, and therefore $\int\psi \operatorname{cosec}\theta\, d\theta$ may be expanded in a series of Laplace's coefficients, and therefore ψ itself in a series of quantities of the form $\sin\theta\dfrac{dY_i}{d\theta}$. It was somewhat in this way that I first obtained the form of the function ψ.

The equations of condition (30), (31) become, on putting $f(r)$ for $f_1(r) + f_2(r)$,

$$f'(a) = ac, \qquad f(a) = \tfrac{1}{2}a^2c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

$$f'(b) = 0, \qquad f(b) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

We may obtain p from (29) by putting for ψ_1 its value $\epsilon^{\mu' m^2 t} \sin^2\theta f_1(r)$, replacing after differentiation $2f_1(r)$ by its equivalent $r^2 f_1''(r)$, and then integrating. It is unnecessary to add an arbitrary function of the time, since any such function may be supposed to be included in Π. We get

$$p = -\rho\mu' m^2 \epsilon^{\mu' m^2 t} \cos\theta\, f_1'(r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

16. The integration of the differential equation (33) does not present the least difficulty, and (34) comes under a well known integrable form. The integrals of these equations are

$$\left.\begin{aligned} f_1(r) &= \frac{A}{r} + Br^2, \\ f_2(r) &= C\epsilon^{-mr}\left(1 + \frac{1}{mr}\right) + D\epsilon^{mr}\left(1 - \frac{1}{mr}\right), \end{aligned}\right\} \quad . \quad . \quad . \qquad (38)$$

and we have to determine A, B, C, D by the equations of condition.

The solution of the problem, in the case in which the fluid is confined by a spherical envelope, will of course contain as a particular case that in which the fluid is unlimited, to obtain the results belonging to which it will be sufficient to put $b = \infty$. As, however, the case of an unlimited fluid is at the same time simpler and more interesting than the general case, it will be proper to consider it separately.

Let $+m$ denote that square root of $\mu'^{-1} n \sqrt{-1}$ which has its real part positive; then in equations (38) we must have $D = 0$, since otherwise the velocity would be infinite at an infinite distance. We must also have $B = 0$, since otherwise the velocity would be finite when $r = \infty$, as appears from (26). We get then from the equations of condition (35)

$$A = \tfrac{1}{2}a^3c + \frac{3a^2c}{2m}\left(1 + \frac{1}{ma}\right), \qquad C = -\frac{3ac}{2m}\epsilon^{ma},$$

whence

$$\xi = \frac{c}{\mu' m^2}\epsilon^{\mu' m^2 t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

$$\psi = \tfrac{1}{2}a^2c\,\epsilon^{\mu' m^2 t}\sin^2\theta\left\{\left(1 + \frac{3}{ma} + \frac{3}{m^2a^2}\right)\frac{a}{r} - \frac{3}{ma}\left(1 + \frac{1}{mr}\right)\epsilon^{-m(r-a)}\right\}, \quad . \qquad (40)$$

$$p = \tfrac{1}{2}\rho a c \mu' m^2\left(1 + \frac{3}{ma} + \frac{3}{m^2a^2}\right)\epsilon^{\mu' m^2 t}\cos\theta\,\frac{a^2}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

17. The symbolical equations (40), (41) contain the solution of the problem, the motion of the sphere being defined by the symbolical equation (39). If we wish to exhibit the actual results by means of real quantities alone, we have only to put the right-hand members

of equations (39), (40), (41) under the form $U + \sqrt{-1}\,V$, and reject the imaginary part. Putting for shortness

$$\sqrt{\frac{n}{2\mu'}} = \nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

we have $m = \nu(1 + \sqrt{-1})$, and we obtain

$$\xi = \frac{c}{n} \sin nt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$\psi = \tfrac{1}{2} a^2 c \sin^2\theta \left\{ \left[\left(1 + \frac{3}{2\nu a}\right) \cos nt + \frac{3}{2\nu a} \left(1 + \frac{1}{\nu a}\right) \sin nt \right] \frac{a}{r} \right.$$

$$\left. - \frac{3}{2\nu a} \epsilon^{-\nu(r-a)} \left[\cos(nt - \nu r + \nu a) + \left(1 + \frac{1}{\nu r}\right) \sin(nt - \nu r + \nu a) \right] \right\}, \quad . \quad (44)$$

$$p = -\tfrac{1}{2} \rho a c n \left\{ \left(1 + \frac{3}{2\nu a}\right) \sin nt - \frac{3}{2\nu a} \left(1 + \frac{1}{\nu a}\right) \cos nt \right\} \cos\theta \,.\, \frac{a^2}{r^2}. \quad . \quad . \quad . \quad . \quad (45)$$

The reader will remark that the ξ, ψ, p of the present article are not the same as the ξ, ψ, p of the preceding. The latter are the imaginary expressions, of which the real parts constitute the former. It did not appear necessary to change the notation.

When $\mu' = 0$, $\nu = \infty$, and ψ reduces itself to

$$\frac{a^3 c}{2r} \sin^2\theta \cos nt, \quad \text{or} \quad \frac{a^3}{2r} \sin^2\theta \frac{d\xi}{dt}.$$

In this case we get from (26)

$$R = a^3 \frac{d\xi}{dt} \frac{\cos\theta}{r^3}, \quad \Theta = \tfrac{1}{2} a^3 \frac{d\xi}{dt} \frac{\sin\theta}{r^3},$$

and $R\,dr + \Theta\,r\,d\theta$ is an exact differential $d\phi$ where

$$\phi = -\tfrac{1}{2} a^3 \frac{d\xi}{dt} \frac{\cos\theta}{r^2},$$

which agrees with the result deduced directly from the ordinary equations of hydrodynamics*.

18. Let us now form the expression for the resultant of the pressures of the fluid on the several elements of the surface of the sphere. Let P_r be the normal, and T_θ the tangential, component of the pressure at any point in the direction of a plane drawn perpendicular to the radius vector. The formulæ (4), (5) are general, and therefore we may replace x, y in these formulæ by x', y', where x', y' are measured in any two rectangular directions we please. Let the plane of $x'\,y'$ pass through the axis of x and the radius vector, and let the axis of x' be inclined to that of x at an angle ϑ, which after differentiation is made equal to θ. Then P_1, T_3 will become P_r, T_θ, respectively. We have

$$u' = R\cos(\theta - \vartheta) - \Theta \sin(\theta - \vartheta), \quad v' = R \sin(\theta - \vartheta) + \Theta\cos(\theta - \vartheta),$$

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[*] See Camb. Phil. Trans. Vol. VIII. p. 119.

and when $\theta = \vartheta$

$$\frac{d}{dx'} = \frac{d}{dr}, \quad \frac{d}{dy'} = \frac{d}{rd\theta},$$

$$\frac{du'}{dx'} = \frac{dR}{dr}, \quad \frac{du'}{dy'} = \frac{dR}{rd\theta} - \frac{\Theta}{r}, \quad \frac{dv'}{dx'} = \frac{d\Theta}{dr};$$

whence

$$P_r = p - 2\mu\frac{dR}{dr}, \quad T_\theta = -\mu\left(\frac{dR}{rd\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad (46)$$

In these formulæ, suppose r put equal to a after differentiation. Then P_r, T_θ will be the components in the direction of r, θ of the pressure of the sphere on the fluid. The resolved part of these in the direction of x is

$$P_r \cos\theta - T_\theta \sin\theta,$$

which is equal and opposite to the component, in the direction of x, of the pressure of the fluid on the sphere. Let F be the whole force of the fluid on the sphere, which will evidently act along the axis of x. Then, observing that $2\pi a^2 \sin\theta d\theta$ is the area of an elementary annulus of the surface of the sphere, we get

$$F = 2\pi a^2 \int_0^\pi (-P_r \cos\theta + T_\theta \sin\theta)_a \sin\theta d\theta, \quad . \quad . \quad . \quad . \quad (47)$$

the suffix a denoting that r is supposed to have the value a in the general expressions for P_r and T_θ.

The expression for F may be greatly simplified, without employing the solution of equations (27), (28), by combining these equations in their original state with the equations of condition (30). We have, in the first place, from (26)

$$R = \frac{1}{r^2 \sin\theta}\frac{d\psi}{d\theta}, \quad \Theta = -\frac{1}{r \sin\theta}\frac{d\psi}{dr}. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Now the equations (30) make known the values of ψ and $\frac{d\psi}{dr}$, and of their differential coefficients of all orders with respect to θ, when $r = a$. When the expressions for R and Θ are substituted in (46), the result will contain only one term in which the differentiation with respect to r rises to the second order. But we get from (21), (27), (28)

$$\frac{d^2\psi}{dr^2} = -\frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi}{d\theta}\right) + \frac{1}{\mu'}\frac{d\psi_2}{dt},$$

and the second of equations (30) gives the value for $r = a$ of the first term in the right-hand member of the equation just written. We obtain from (48) and (30)

$$\left(\frac{dR}{dr}\right)_a = 0,$$

$$\left(\frac{dR}{rd\theta}\right)_a = -\frac{\sin\theta}{a}\frac{d\xi}{dt} = \left(\frac{\Theta}{r}\right)_a,$$

$$\left(\frac{d\Theta}{dr}\right)_a = -\frac{1}{\mu' a \sin\theta}\left(\frac{d\psi_2}{dt}\right)_a.$$

Substituting in (47), and writing $\mu'\rho$ for μ, we get

$$F = 2\pi a \int_0^\pi \left\{ - a p_a \cos\theta + \rho \left(\frac{d\psi_2}{dt}\right)_a \right\} \sin\theta d\theta.$$

With respect to the first term in this expression, we get by integration by parts

$$\int p \cos\theta \sin\theta \, d\theta = \tfrac{1}{2} \sin^2\theta \, . \, p - \tfrac{1}{2} \int \sin^2\theta \frac{dp}{d\theta} d\theta.$$

The first term vanishes at the limits. Substituting in the second term for $\frac{dp}{d\theta}$ the expression got from (29), and putting $r = a$, we get

$$\int_0^\pi p_a \cos\theta \sin\theta \, d\theta = -\tfrac{1}{2}\rho \frac{d}{dt} \int_0^\pi \left(\frac{d\psi_1}{dr}\right)_a \sin\theta \, d\theta.$$

Substituting in the expression for F, we get

$$F = \pi\rho a \frac{d}{dt} \int_0^\pi \left\{ a \left(\frac{d\psi_1}{dr}\right)_a + 2(\psi_2)_a \right\} \sin\theta \, d\theta. \quad . \quad . \quad . \quad . \quad (49)$$

19. The above expression for F, being derived from the general equations (27), (28), combined with the equations of condition (30), holds good, not merely when the fluid is confined by a spherical envelope, but whenever the motion is symmetrical about an axis, and that, whether the motion of the sphere be or be not expressed by a single circular function of the time. It might be employed, for instance, in the case of a sphere oscillating in a direction perpendicular to a fixed rigid plane.

When the fluid is either unconfined, or confined by a spherical envelope concentric with the sphere in its position of equilibrium, the functions ψ_1, ψ_2 consist, as we have seen, of $\sin^2\theta$ multiplied by two factors independent of θ. If we continue to employ the symbolical expressions, which will be more convenient to work with than the real expressions which might be derived from them, we shall have

$$\epsilon^{\sqrt{-1}\,nt} f_1(r), \qquad \epsilon^{\sqrt{-1}\,nt} f_2(r),$$

for these factors respectively. Substituting in (49), and performing the integration with respect to θ, we get

$$F = \tfrac{4}{3} \pi\rho a n \sqrt{-1} \, \{a f_1'(a) + 2 f_2(a)\} \, \epsilon^{\sqrt{-1}\,nt}. \quad . \quad . \quad . \quad . \quad (50)$$

20. Consider for the present only the case in which the fluid is unlimited. The arbitrary constants which appear in equations (38) were determined for this case in Art. 16. Substituting in (50) we get

$$F = -\tfrac{2}{3} \pi\rho a^3 c n \sqrt{-1} \left(1 + \frac{9}{ma} + \frac{9}{m^2 a^2}\right) \epsilon^{\sqrt{-1}\,nt}.$$

Putting for m its value $\nu(1+\sqrt{-1})$, and denoting by M' the mass of the fluid displaced by the sphere, which is equal to $\frac{4}{3}\pi\rho a^3$, we get

$$F = -M'cn\left\{\left(\tfrac{1}{2}+\frac{9}{4\nu a}\right)\sqrt{-1}+\frac{9}{4\nu a}\left(1+\frac{1}{\nu a}\right)\right\}\epsilon^{\sqrt{-1}\,nt};$$

whence

$$F = -\left(\tfrac{1}{2}+\frac{9}{4\nu a}\right)M'\frac{d^2\xi}{dt^2}-\frac{9}{4\nu a}\left(1+\frac{1}{\nu a}\right)M'n\frac{d\xi}{dt}. \quad \ldots\ldots \quad (51)$$

Since $\sqrt{-1}$ has been eliminated, this equation will remain unchanged when we pass from the symbolical to the real values of F and ξ.

Let τ be the time of oscillation from rest to rest, so that $n\tau=\pi$, and put for shortness k, k' for the coefficients of M' in (51); then

$$\nu=\sqrt{\frac{\pi}{2\mu'\tau}},\quad k=\tfrac{1}{2}+\frac{9}{4\nu a},\quad k'=\frac{9}{4\nu a}\left(1+\frac{1}{\nu a}\right). \quad \ldots\ldots \quad (52)$$

The first term in the expression for the force F has the same effect as increasing the inertia of the sphere. To take account of this term, it will be sufficient to conceive a mass kM' collected at the centre of the sphere, adding to its inertia without adding to its weight. The main effect of the second term is to produce a diminution in the arc of oscillation: its effect on the time of oscillation would usually be quite insensible, and must in fact be neglected for consistency's sake, because the motion of the fluid was determined by supposing the motion of the sphere permanent, which is only allowable when we neglect the square of the rate of decrease of the arc of oscillation.

If we form the equation of motion of the sphere, introducing the force F, and then proceed to integrate the equation, we shall obtain in the integral an exponential $\epsilon^{-\delta t}$ multiplying the circular function, δ being half the coefficient of $\frac{d\xi}{dt}$ divided by that of $\frac{d^2\xi}{dt^2}$. Let M be the mass of the sphere, $M\gamma^2$ its moment of inertia about the axis of suspension, then

$$nk'M'(l+a)^2=2\delta\left\{M\gamma^2+kM'(l+a)^2\right\}.$$

In considering the diminution of the arc of oscillation, we may put $l+a$ for γ. During i oscillations, let the arc of oscillation be diminished in the ratio of A_0 to A_i, then

$$\log_e\frac{A_0}{A_i}=i\tau\delta=\frac{\pi i}{2}\,\frac{k'M'}{M+kM'}. \quad \ldots\ldots\ldots \quad (53)$$

For a given fluid and a given time of oscillation, both k and k' increase as a decreases. Hence it follows from theory, that the smaller be the sphere, its density being supposed given, the more the time of oscillation is affected, and the more rapidly the arc of oscillation diminishes, the alteration in the rate of diminution of the arc due to an alteration in the radius of the sphere being more conspicuous than the alteration in the time of oscillation.

21. Let us now suppose the fluid confined in a spherical envelope. In this case, we have

to determine the four arbitrary constants which appear in (38) by the four equations (35) and (36). We get, in the first place,

$$\frac{A}{a} + Ba^2 + C\epsilon^{-ma}\left(1 + \frac{1}{ma}\right) + D\epsilon^{ma}\left(1 - \frac{1}{ma}\right) = \tfrac{1}{2}a^2c, \quad . \quad . \quad . \quad . \quad (54)$$

$$-\frac{A}{a} + 2Ba^3 - C\epsilon^{-ma}\left(ma + 1 + \frac{1}{ma}\right) + D\epsilon^{ma}\left(ma - 1 + \frac{1}{ma}\right) = a^2c, \quad . \quad . \quad (55)$$

$$\frac{A}{b} + Bb^2 + C\epsilon^{-mb}\left(1 + \frac{1}{mb}\right) + D\epsilon^{mb}\left(1 - \frac{1}{mb}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$-\frac{A}{b} + 2Bb^3 - C\epsilon^{-mb}\left(mb + 1 + \frac{1}{mb}\right) + D\epsilon^{mb}\left(mb - 1 + \frac{1}{mb}\right) = 0. \quad . \quad . \quad (57)$$

Putting a^2cK for $af_1'(a) + 2f_2(a)$, which is the quantity that we want to find, we get from (38) and (54)

$$K = 1 - \frac{3A}{a^3c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Eliminating in succession B from (54) and (55), from (56) and (57), and from (54) and (56), we shall obtain for the determination of A, C, D three equations which remain unchanged when a and b are interchanged, and the signs of A, C, and D changed. Hence $-A$, $-C$, $-D$ are the same functions of b and a that A, C, D are of a and b. It will also assist in the further elimination to observe that C and D are interchanged when the sign of m is changed. The result of the elimination is

$$K = 1 - \frac{3b}{2m^2a^2} \cdot \frac{\eta(a, b) - \eta(b, a)}{12mab + \zeta(a, b) + \zeta(b, a)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

the functions ζ, η being defined by the equations

$$\left.\begin{aligned} \eta(a, b) &= (m^2a^2 + 3ma + 3)\,(m^2b^2 - 3mb + 3)\,\epsilon^{m(b-a)}, \\ \zeta(a, b) &= \{b\,(m^2b^2 - 3mb + 3) - a\,(m^2a^2 + 3ma + 3)\}\,\epsilon^{m(b-a)}. \end{aligned}\right\} \quad . \quad . \quad (60)$$

It turns out that K is a complicated function of m and ab^{-1}, and the algebraical expressions for the quantities which answer to k and k' in Art. 20 would be more complicated still, because $\nu(1 + \sqrt{-1})$ would have to be substituted for m in (60) and (59), and then K reduced to the form $-k + \sqrt{-1}\,k'$. To obtain numerical results from these formulæ, it would be best to substitute the numerical values of a, b, and ν in (60) and (59), and perform the reduction of K in figures.

22. If the distance of the envelope from the surface of the sphere be at all considerable, the exponential $\epsilon^{\nu(b-a)}$, which arises from $\epsilon^{m(b-a)}$, will have so large a numerical value that we may neglect the terms in the numerator and denominator of the fraction in the expression for K which contain $\epsilon^{-\nu(b-a)}$, as well as the term in the denominator which is free from exponentials, in comparison with the terms which contain $\epsilon^{\nu(b-a)}$. Thus, if $b - a$ be two inches, τ one second, and $\sqrt{\mu'} = .116$, we have $\epsilon^{\nu(b-a)} = 2424000000$, nearly; and if $b - a$ be only an

inch or half an inch, we have still the square or fourth root of the above quantity, that is, about 49234 or 222, for the value of that exponential. Hence, in practical cases, the above simplification may be made, which will cause the exponentials to disappear from the expression for K. We thus get

$$K = 1 - \frac{3b}{2m^2a^2}\,\frac{(m^2a^2 + 3ma + 3)\,(m^2b^2 - 3mb + 3)}{b\,(m^2b^2 - 3mb + 3) - a\,(m^2a^2 + 3ma + 3)}\,. \quad . \quad . \tag{61}$$

If we assume

$$3\nu a + 3 + (2\nu^2a^2 + 3\nu a)\sqrt{-1} = A'(\cos\alpha + \sqrt{-1}\,\sin\alpha),$$
$$-3\nu b + 3 + (2\nu^2b^2 - 3\nu b)\sqrt{-1} = B'(\cos\beta + \sqrt{-1}\,\sin\beta),$$
$$bB'\cos\beta - aA'\cos\alpha = C'\cos\gamma,$$
$$bB'\sin\beta - aA'\sin\alpha = C'\sin\gamma,$$

we get from (61)

$$K = 1 + \frac{3b\sqrt{-1}}{4\nu^2a^2}\cdot\frac{A'B'}{C'}\left\{\cos(\alpha + \beta - \gamma) + \sqrt{-1}\,\sin(\alpha + \beta - \gamma)\right\},$$

whence

$$\left.\begin{aligned} k &= \frac{3bA'B'}{4\nu^2a^2C'}\sin(\alpha + \beta - \gamma) - 1, \\ k' &= \frac{3bA'B'}{4\nu^2a^2C'}\cos(\alpha + \beta - \gamma); \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{62}$$

and, as before, kM' is the imaginary mass which we must conceive to be collected at the centre of the sphere, in order to allow for the inertia of the fluid, and $-k'M'n\dfrac{d\xi}{dt}$ the term in F on which depends the diminution in the arc of oscillation.

23. If we suppose $\mu' = 0$, and therefore $m = \infty$, we get from (61)

$$K = -\frac{b^3 + 2a^3}{2(b^3 - a^3)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{63}$$

and, in this case, k is the same as K with sign changed, and $k' = 0$, which agrees with the result obtained directly from the ordinary equations of hydrodynamics[*]. If, on the other hand, we make $b = \infty$, we arrive at the results already obtained in Art. 20. In both these cases it becomes rigorously exact to neglect in the expression for $K - 1$ given by (59) all the terms which are not multiplied by $\epsilon^{\nu(b-a)}$.

If the effect of the envelope be but small, which will generally be the case, it will be convenient to calculate k and k' from the formulæ (52), which apply to the case in which $b = \infty$, and then add corrections Δk, $\Delta k'$ due to the envelope. We get from (61)

$$\Delta k - \sqrt{-1}\,\Delta k' = \frac{3}{2m^2a}\,\frac{(m^2a^2 + 3ma + 3)^2}{b\,(m^2b^2 - 3mb + 3) - a\,(m^2a^2 + 3ma + 3)}\,, \quad . \tag{64}$$

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[*] See Camb. Phil. Trans. Vol. VIII. p. 120.

which may be treated, if required, as the equation (61) was treated in the preceding article. If, however, we suppose m large, and are content to retain only the most important term in (64), we get simply

$$\Delta k = \frac{3a^3}{2(b^3 - a^3)}, \quad \Delta k' = 0, \quad \ldots \ldots \quad (65)$$

so that the correction for the envelope may be calculated as if the fluid were destitute of friction.

SECTION III.

Solution of the equations in the case of an infinite cylinder oscillating in an unlimited mass of fluid, in a direction perpendicular to its axis.

24. Suppose a long cylindrical rod suspended at a point in its axis, and made to oscillate as a pendulum in an unlimited mass of fluid. The resistance experienced by any element of the cylinder comprised between two parallel planes drawn perpendicular to the axis will manifestly be very nearly the same as if the element belonged to an infinite cylinder oscillating with the same linear velocity. For an element situated very near either extremity of the rod, the resistance thus determined would, no doubt, be sensibly erroneous; but as the diameter of the rod is supposed to be but small in comparison with its length, it will be easily seen that the error thus introduced must be extremely small.

Imagine then an infinite cylinder to oscillate in a fluid, in a direction perpendicular to its axis, so that the motion takes place in two dimensions, and let it be required to determine the motion of the fluid. The mode of solution of this problem will require no explanation, being identical in principle with that which has been already adopted in the case of a sphere. In the present instance the problem will be found somewhat easier, up to the formation of the equations analogous to (33) and (34), after which it will become much more difficult.

25. Let a plane drawn perpendicular to the axis of the cylinder be taken for the plane of xy, the origin being situated in the mean position of the axis of the cylinder, and the axis of x being measured in the direction of the cylinder's motion. The general equations (2), (3) become in this case

$$\left.\begin{aligned} \frac{dp}{dx} &= \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right) - \rho\frac{du}{dt}, \\ \frac{dp}{dy} &= \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\right) - \rho\frac{dv}{dt}, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (66)$$

$$\frac{du}{dx} + \frac{dv}{dy} = 0. \quad \ldots \ldots \ldots \ldots \quad (67)$$

By virtue of (67), $u\,dy - v\,dx$ is an exact differential. Let then

$$u\,dy - v\,dx = d\chi. \quad \ldots \ldots \ldots \ldots \quad (68)$$

Eliminating p by differentiation from the two equations (66), and expressing u and v in terms of χ in the resulting equation, we get

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} - \frac{1}{\mu'}\frac{d}{dt}\right)\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\chi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

and, as before

$$\chi = \chi_1 + \chi_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\chi_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} - \frac{1}{\mu'}\frac{d}{dt}\right)\chi_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

We get from (66) and (68)

$$dp = \mu'\rho\, dx\,.\,\frac{d}{dy}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} - \frac{1}{\mu'}\frac{d}{dt}\right)\chi - \mu'\rho\, dy\,.\,\frac{d}{dx}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} - \frac{1}{\mu'}\frac{d}{dt}\right)\chi,$$

which becomes by means of (70), (71), and (72)

$$dp = \rho\left(\frac{d^2\chi_1}{dt\,dx}\,dy - \frac{d^2\chi_1}{dt\,dy}\,dx\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

26. Passing to polar co-ordinates r, θ, where θ is supposed to be measured from the axis of x, we get from (68), (71), (72), and (73)

$$R\,r\,d\theta - \Theta\,dr = d\chi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\chi_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2} - \frac{1}{\mu'}\frac{d}{dt}\right)\chi_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

$$dp = \rho\,\frac{d}{dt}\left(\frac{d\chi_1}{dr}\,r\,d\theta - \frac{d\chi_1}{r\,d\theta}\,dr\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

R, Θ in (74) being the velocities along and perpendicular to the radius vector.

27. Let a be the radius of the cylinder; and as before let the cylinder's motion be defined by the equation

$$\frac{d\xi}{dt} = c\,\epsilon^{\sqrt{-1}\,nt} = c\,\epsilon^{\mu' m^2 t}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

then we have for the equations of condition which relate to the surface of the cylinder

$$\left.\begin{aligned} R &= \frac{d\chi}{r\,d\theta} = \cos\theta\,\frac{d\xi}{dt} = c\cos\theta\,\epsilon^{\mu' m^2 t}, \\ \Theta &= -\frac{d\chi}{dr} = -\sin\theta\,\frac{d\xi}{dt} = -c\sin\theta\,\epsilon^{\mu' m^2 t}. \end{aligned}\right\}\ \text{when } r = a. \quad . \quad . \quad . \quad . \quad (79)$$

The general equations (75), (76), as well as the equations of condition (79), may be satisfied by taking

$$\chi_1 = e^{\mu' m^2 t} \sin\theta\, F_1(r), \qquad \chi_2 = e^{\mu' m^2 t} \sin\theta\, F_2(r). \quad . \quad . \quad . \quad . \quad (80)$$

Substituting in (75), (76), and (79), we get

$$F_1''(r) + \frac{1}{r} F_1'(r) - \frac{1}{r^2} F_1(r) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

$$F_2''(r) + \frac{1}{r} F_2'(r) - \frac{1}{r^2} F_2(r) - m^2 F_2(r) = 0, \quad . \quad . \quad . \quad . \quad (82)$$

$$F_1(a) + F_2(a) = a c, \qquad F_1'(a) + F_2'(a) = c, \quad . \quad . \quad . \quad . \quad (83)$$

besides which we have the condition that the velocity shall vanish at an infinite distance.

28. The integral of (81) is

$$F_1(r) = \frac{A}{r} + B r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

The integral of (82) cannot be obtained in finite terms.

To simplify the latter equation, assume $F_2(r) = F_3'(r)$. Substituting in (82), and integrating once, we get

$$F_3''(r) + \frac{1}{r} F_3'(r) - m^2 F_3(r) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

It is unnecessary to add an arbitrary constant, because such a constant, if introduced, might be got rid of by writing $F_3(r) + C$ for $F_3(r)$.

To integrate (85) by series according to ascending powers of r, let us first, instead of (85), take the equation formed from it by multiplying the second term by $1 - \delta$. Assuming in this new equation $F_3(r) = A_{,} x^{\alpha} + B_{,} x^{\beta} + \ldots$, and determining the arbitrary indices α, β...and the arbitrary constants $A_{,}$, $B_{,}$...so as to satisfy the equation, we get

$$\begin{aligned} F_3(r) &= A_{,} \left\{1 + \frac{m^2 r^2}{2(2-\delta)} + \frac{m^4 r^4}{2\,.\,4\,(2-\delta)(4-\delta)} + \ldots\right\} \\ &\quad + A_{,,} r^{\delta} \left\{1 + \frac{m^2 r^2}{2(2+\delta)} + \frac{m^4 r^4}{2\,.\,4\,(2+\delta)(4+\delta)} + \ldots\right\} \\ &= (A_{,} + A_{,,} + A_{,,}\delta \log r) \left\{1 + \frac{m^2 r^2}{2^2} + \frac{m^4 r^4}{2^2\,.\,4^2} + \ldots\right\} \\ &\quad + \tfrac{1}{2}(A_{,} - A_{,,})\,\delta \left\{\frac{m^2 r^2}{2^2} S_1 + \frac{m^4 r^4}{2^2\,.\,4^2} S_2 + \frac{m^6 r^6}{2^2\,.\,4^2\,.\,6^2} S_3 + \ldots\right\} \\ &\quad + \text{terms involving } \delta^2,\ \delta^3 \ldots \end{aligned}$$

In this expression

$$S_i = 1^{-1} + 2^{-1} + 3^{-1} \ldots + i^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Putting now

$$A_{,} = C - A_{,,}, \qquad A_{,,} = D\delta^{-1},$$

substituting in the above equation, and then making δ vanish, we get

$$F_3(r) = (C + D\log r)\left(1 + \frac{m^2r^2}{2^2} + \frac{m^4r^4}{2^2.4^2} + \ldots\right)$$

$$- D\left(\frac{m^2r^2}{2^2}S_1 + \frac{m^4r^4}{2^2.4^2}S_2 + \frac{m^6r^6}{2^2.4^2.6^2}S_3 + \ldots\right). \quad . \quad . \quad . \quad (87)$$

The series in this equation are evidently convergent for all values of r, however great; but, nevertheless, they give us no information as to what becomes of $F_3(r)$ when r becomes infinite, and yet one relation between C and D has to be determined by the condition that $F_3(r)$ shall not become infinite with r.

The equation (85) may be integrated by means of descending series combined with exponentials, by assuming $F_3(r) = \epsilon^{\pm mr}(A_{,}r^{\alpha} + B_{,}r^{\beta}\ldots)$. I have already given the integral in this form in a paper, *On the numerical calculation of a class of definite integrals and infinite series**. The result is

$$F_3(r) = C'\epsilon^{-mr}r^{-\frac{1}{2}}\left\{1 - \frac{1^2}{2.4mr} + \frac{1^2.3^2}{2.4(4mr)^2} - \frac{1^2.3^2.5^2}{2.4.6(4mr)^3} + \ldots\right\}$$

$$+ D'\epsilon^{mr}r^{-\frac{1}{2}}\left\{1 + \frac{1^2}{2.4mr} + \frac{1^2.3^2}{2.4(4mr)^2} + \frac{1^2.3^2.5^2}{2.4.6(4mr)^3} + \ldots\right\}. \quad . \quad . \quad (88)$$

These series, although ultimately divergent in all cases, are very convenient for numerical calculation when the modulus of mr is large. Moreover they give at once $D' = 0$ for the condition that $F_3(r)$ shall not become infinite with r, and therefore we shall be able to obtain the required relation between C and D, provided we can express D' as a function of C and D.

29. This may be effected by means of the integral of (85) expressed by definite integrals. This form of the integral is already known. It becomes, by a slight transformation,

$$F_3(r) = \int_0^{\frac{\pi}{2}}\left\{C'' + D''\log(r\sin^2\omega)\right\}\left(\epsilon^{mr\cos\omega} + \epsilon^{-mr\cos\omega}\right)d\omega, \quad . \quad . \quad (89)$$

C'', D'' being the two arbitrary constants. If we expand the exponentials in (89), and integrate the terms separately, we obtain, in fact, an expression of the same form as (87). This transformation requires the reduction of the definite integral

$$P_i = \int_0^{\frac{\pi}{2}}\cos^{2i}\omega\log\sin\omega\,d\omega.$$

If we integrate by parts, integrating $\cos\omega\log\sin\omega\,d\omega$, and differentiating $\cos^{2i-1}\omega$, we shall make P_i depend on P_{i-1}. Assuming $P_0 = Q_0$, $P_1 = \frac{1}{2}Q_1\ldots$, and generally

$$P_i = \frac{1.3\ldots(2i-1)}{2.4\ldots2i}Q_i,$$

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* Camb. Phil. Trans. Vol. IX. p. 182.

we get

$$Q_i = Q_0 - \{2^{-1} + 4^{-1} \ldots + (2i)^{-1}\} \frac{\pi}{2} = \frac{\pi}{2} \log (\tfrac{1}{2}) - \frac{\pi}{4} S_i.$$[*]

The equivalence of the expressions (87) and (89) having been ascertained, in order to find the relations between C, D and C'', D'', it will be sufficient to write down the two leading terms in (87) and (89), and equate the results. We thus get

$$C + D \log r = \pi C'' + \pi D'' \log r + 2\pi D'' \log (\tfrac{1}{2}),$$

whence

$$C = \pi C'' + 2\pi \log (\tfrac{1}{2}) . D'', \quad D = \pi D''. \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

There remains the more difficult step of finding the relation between D' and C'', D''. For this purpose let us seek the ultimate value of the second member of equation (89) when r increases indefinitely. In the first place we may observe that if Ω, Ω' be two imaginary quantities having their real parts positive, if the real part of Ω be greater than that of Ω', and if r be supposed to increase indefinitely, $\epsilon^{\Omega r}$ will ultimately be incomparably greater than $\epsilon^{\Omega' r}$, or even than $\log r . \epsilon^{\Omega' r}$, or, to speak more precisely, the modulus of the former expression will ultimately be incomparably greater than the modulus of either of the latter. Hence, in finding the ultimate value of the expression for $F_3(r)$ in (89), we may replace the limits 0 and $\frac{1}{2}\pi$ of ω by 0 and ω_1, where ω_1 is a positive quantity as small as we please, which we may suppose to vanish after r has become infinite. We may also, for the same reason, omit the second of the exponentials. Let $\cos \omega = 1 - \lambda$, so that

$$\sin^2 \omega = 2\lambda \left(1 - \frac{\lambda}{2}\right), \quad d\omega = \frac{d\lambda}{\sqrt{2\lambda - \lambda^2}} = \left(1 + \frac{\lambda}{4} + \ldots\right) \frac{d\lambda}{\sqrt{(2\lambda)}};$$

then the limits of λ will be 0 and λ_1, where $\lambda_1 = 1 - \cos \omega_1$. Since $\log \left(1 - \frac{\lambda}{2}\right)$ ultimately vanishes, and $1 + \frac{\lambda}{4} + \ldots$ becomes ultimately 1, we get from (89)

$$\text{limit of } F_3(r) = \epsilon^{mr} \times \text{limit of} \int_0^{\lambda_1} (C'' + D'' \log 2\lambda r)\, \epsilon^{-m\lambda r} \frac{d\lambda}{\sqrt{(2\lambda)}}.$$[†]

If now we put $\lambda = \lambda' r^{-1}$, we shall have 0 and $\lambda_1 r$ for the limits of λ', and the second of these becomes infinite with r. Hence

$$\text{limit of } F_3(r) = (2r)^{-\frac{1}{2}} \epsilon^{mr} \int_0^{\infty} (C'' + D'' \log 2\lambda')\, \epsilon^{-m\lambda'} \lambda'^{-\frac{1}{2}} d\lambda'. \quad . \quad . \quad (91)$$

Now $\int_0^{\infty} \epsilon^{-x} x^{-\frac{1}{2}} dx = \pi^{\frac{1}{2}}$, and if we differentiate both sides of the equation

$$\int_0^{\infty} \epsilon^{-x} x^{s-1} dx = \Gamma(s)$$

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[*] A demonstration by Mr Ellis of the theorem

$$\int_0^{\frac{\pi}{2}} \log \sin \theta d\theta = \frac{\pi}{2} \log (\tfrac{1}{2})$$

due to Euler will be found in the 2nd volume of the Cambridge Mathematical Journal, p. 282, or in Gregory's Examples, p. 484.

[†] The word *limit* is here used in the sense in which $f(r)$ may be called the limit of $\phi(r)$ when the ratio of $\phi(r)$ to $f(r)$ is ultimately a ratio of equality, though $f(r)$ and $\phi(r)$ may vanish or become infinite together, in which case the limit of $\phi(r)$, according to the usual sense of the word *limit*, would be said to be zero or infinity.

with respect to s, and after differentiation put $s = \frac{1}{2}$, we get

$$\int_0^\infty \epsilon^{-x} x^{-\frac{1}{2}} \log x \, dx = \Gamma'(\tfrac{1}{2}).$$

Putting $x = m\lambda'$ in these equations we get

$$\int_0^\infty \epsilon^{-m\lambda'} \lambda'^{-\frac{1}{2}} d\lambda' = \pi^{\frac{1}{2}} m^{-\frac{1}{2}}, \quad \int_0^\infty \epsilon^{-m\lambda'} \lambda'^{-\frac{1}{2}} \log \lambda' \, d\lambda' = m^{-\frac{1}{2}} \{\Gamma'(\tfrac{1}{2}) - \pi^{\frac{1}{2}} \log m)\},$$

where that value of $m^{-\frac{1}{2}}$ is to be taken which has its real part positive. Substituting in (91) we get

$$\text{limit of } F_3(r) = \left(\frac{\pi}{2mr}\right)^{\frac{1}{2}} \epsilon^{mr} \left\{C'' + \left(\pi^{-\frac{1}{2}} \Gamma' \tfrac{1}{2} - \log \frac{m}{2}\right) D''\right\}.$$

Comparing with (88) we get

$$D' = \left(\frac{\pi}{2m}\right)^{\frac{1}{2}} \left\{C'' + \left(\pi^{-\frac{1}{2}} \Gamma' \tfrac{1}{2} - \log \frac{m}{2}\right) D''\right\}. \quad . \quad . \quad . \quad . \quad (92)$$

30. We are now enabled to find the relation between C and D arising from the condition that the motion of the fluid shall not become infinitely great at an infinite distance from the cylinder. The determination of the arbitrary constants A, B, C, D will present no further difficulty. We must have $B = 0$, since otherwise the velocity would be finite at an infinite distance, and then the two equations (83), combined with the relation above mentioned, will serve to determine A, C, D. The motion of the fluid will thus be completely determined, the functions $F_1(r)$, $F_3(r)$ being given by (84) and (87). When the modulus of mr is large, the series in (87), though ultimately hypergeometrically convergent, are at first rapidly divergent, and in calculating the numerical value of $F_3(r)$ in such a case it would be far more convenient to employ equation (88). The employment of this equation for the purpose would require the previous determination of the constant C'. It will be found however that in calculating the resultant pressure of the fluid on the cylinder, which it is the main object of the present investigation to determine, a knowledge of the value of C' will not be required, and that, even though the equation (88) be employed.

Putting $D' = 0$ in (92), and eliminating C'' and D'' between the resulting equation and the two equations (90), we get

$$C = (\log \frac{m}{8} - \pi^{-\frac{1}{2}} \Gamma' \tfrac{1}{2}) D; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

and we get from (83) and (84), observing that $F_2(r) = F_3'(r)$, and that $B = 0$,

$$\frac{A}{a} + F_3'(a) = ac, \quad -\frac{A}{a} + aF_3''(a) = ac, \quad . \quad . \quad . \quad . \quad . \quad (94)$$

whence

$$\frac{a^2 c + A}{a^2 c - A} = \frac{aF_3''(a)}{F_3'(a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

This equation will determine A, because if $F_3(a)$ be expressed by (87) the second member of

(95) will only contain the ratio of C to D, which is given by (93), and if $F_3(a)$ be expressed by (88) C' will disappear, inasmuch as $D' = 0$.

31. Let us now form the expression for the resultant of the forces which the fluid exerts on the cylinder. Let F be the resultant of the pressures acting on a length dl of the cylinder, which will evidently be a force acting in the direction of the axis of x; then we get in the same way as the expression (47) was obtained

$$F = a\,dl \int_0^{2\pi} (-P_r \cos\theta + T_\theta \sin\theta)_a \, d\theta, \quad . \quad . \quad . \quad . \quad . \quad (96)$$

and P_r, T_θ are given in terms of R and Θ by the same formulæ (46) as before. When the right-hand members of these equations are expressed in terms of χ, there will be only one term in which the differentiation with respect to r rises to the second order, and we get from (70), (75), and (76)

$$\frac{d^2\chi}{dr^2} = -\frac{1}{r}\frac{d\chi}{dr} - \frac{1}{r^2}\frac{d^2\chi}{d\theta^2} + \frac{1}{\mu'}\frac{d\chi_2}{dt}.$$

We get from this equation and the equations of condition (79)

$$\left(\frac{dR}{dr}\right)_a = \frac{1}{a}\left(\frac{d\chi}{d\theta}\right)_a - \frac{1}{a^2}\left(\frac{d^2\chi}{dr\,d\theta}\right)_a = 0,$$

$$\left(\frac{dR}{r\,d\theta}\right)_a = \frac{1}{a^2}\left(\frac{d^2\chi}{d\theta^2}\right)_a = -\frac{\sin\theta}{a}\frac{d\xi}{dt} = \frac{\Theta}{a},$$

$$\left(\frac{d\Theta}{dr}\right)_a = -\left(\frac{d^2\chi}{dr^2}\right)_a = \frac{1}{a}\left(\frac{d\chi}{dr}\right)_a + \frac{1}{a^2}\left(\frac{d^2\chi}{d\theta^2}\right)_a - \frac{1}{\mu'}\left(\frac{d\chi_2}{dt}\right)_a = -\frac{1}{\mu'}\left(\frac{d\chi_2}{dt}\right)_a.$$

Hence

$$F = a\,dl \int_0^{2\pi} \left\{-p_a \cos\theta + \rho\left(\frac{d\chi_2}{dt}\right)_a \sin\theta\right\} d\theta. \quad . \quad . \quad . \quad . \quad . \quad (97)$$

We get by integration by parts

$$\int p_a \cos\theta \, d\theta = p_a \sin\theta - \int \left(\frac{dp}{d\theta}\right)_a \sin\theta \, d\theta.$$

The first term vanishes at both limits; and putting for $\dfrac{dp}{d\theta}$ its value given by (77), and substituting in (97), we get

$$F = \rho a\,dl \frac{d}{dt}\int_0^{2\pi} \left\{a\left(\frac{d\chi_1}{dr}\right)_a + (\chi_2)_a\right\} \sin\theta \, d\theta,$$

or

$$F = \pi\rho a\,dl \,.\, n\sqrt{-1}\,\{aF_1'(a) + F_3'(a)\}\,\epsilon^{\sqrt{-1}\,nt}.$$

Observing that $F_3'(a)$ or $F_2(a) = ac - F_1(a)$ from (83), and that $F_1(a) = Aa^{-1}$, where A is given by (95), and putting M' for $\pi\rho a^2 dl$, the mass of the fluid displaced, we get

$$F = M'cn\sqrt{-1}\left\{1 - 2\frac{aF_3''(a) - F_3'(a)}{aF_3''(a) + F_3'(a)}\right\}\epsilon^{\sqrt{-1}\,nt},$$

which becomes by means of the differential equation (85) which F_3 satisfies

$$F = - M'cn\sqrt{-1}\left\{1 - \frac{4F_3'(a)}{m^2 aF_3(a)}\right\}\epsilon^{\sqrt{-1}\,nt}. \quad . \quad . \quad . \quad . \quad (98)$$

Let

$$1 - \frac{4F_3'(a)}{m^2 aF_3(a)} = k - \sqrt{-1}\,k', \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

where k and k' are real, then, as before, $kM'\dfrac{d^2\xi}{dt^2}$ will be the part of F which alters the time of oscillation, and $k'M'n\dfrac{d\xi}{dt}$ the part which produces a diminution in the arc of oscillation.

When μ' vanishes, m becomes infinite, and we get from (88) and (99), remembering that $D' = 0$; $k = 1$, $k' = 0$, a result which follows directly and very simply from the ordinary equations of hydrodynamics*.

32. Every thing is now reduced to the numerical calculation of the quantities k, k', of which the analytical expressions are given. The series (87) being always convergent might be employed in all cases, but when the modulus of ma is large, it will be far more convenient to employ a series according to descending powers of a. Let us consider the ascending series first.

Let $2\mathfrak{m}$ be the modulus of ma; then

$$ma = 2\mathfrak{m}\,\epsilon^{\frac{\pi}{4}\sqrt{-1}}, \qquad \mathfrak{m} = \frac{a}{2}\sqrt{\frac{n}{\mu'}} = \frac{a}{2}\sqrt{\frac{\pi}{\mu'\tau}}, \quad . \quad . \quad . \quad (100)$$

τ being as before the time of oscillation from rest to rest. Substituting in (99) the above expression for ma, we get

$$k - \sqrt{-1}\,k' = 1 + \frac{\sqrt{-1}\,aF_3'(a)}{\mathfrak{m}^2 F_3(a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Putting for shortness

$$\log_\epsilon 4 + \pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2}) = -\Lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

we get from (87) and (93)

$$\frac{1}{D}F_3(a) = (\log\mathfrak{m} + \Lambda + \frac{\pi}{4}\sqrt{-1})\,(1 + \frac{\mathfrak{m}^2}{1^2}\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2^2} - \frac{\mathfrak{m}^6}{1^2.2^2.3^2}\sqrt{-1} + \ldots)$$

$$- \left(\frac{\mathfrak{m}^2}{1^2}S_1\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2^2}S_2 - \frac{\mathfrak{m}^6}{1^2.2^2.3^2}S_3\sqrt{-1} + \ldots\right),$$

$$\frac{1}{D}aF_3'(a) = 1 + \frac{\mathfrak{m}^2}{1^2}\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2^2} - \ldots + 2\left(\log\mathfrak{m} + \Lambda + \frac{\pi}{4}\sqrt{-1}\right)\left(\frac{\mathfrak{m}^2}{1}\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2} - \ldots\right)$$

$$- 2\left(\frac{\mathfrak{m}^2}{1}S_1\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2}S_2 - \frac{\mathfrak{m}^6}{1^2.2^2.3}S_3\sqrt{-1} + \ldots\right).$$

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* See Camb. Phil. Trans. Vol. VIII. p. 116.

Let

$$\left.\begin{array}{ll}
\dfrac{\mathfrak{m}^2}{1} - \dfrac{\mathfrak{m}^6}{1^2.2^2.3} + \ldots = M_0, & \dfrac{\mathfrak{m}^4}{1^2.2} - \dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4} + \ldots = M_e, \\
\dfrac{\mathfrak{m}^2}{1^2} - \dfrac{\mathfrak{m}^6}{1^2.2^2.3^2} + \ldots = M_0', & \dfrac{\mathfrak{m}^4}{1^2.2^2} - \dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4^2} + \ldots = M_e', \\
\dfrac{\mathfrak{m}^2}{1} S_1 - \dfrac{\mathfrak{m}^6}{1^2.2^2.3} S_3 + \ldots = N_0, & \dfrac{\mathfrak{m}^4}{1^2.2} S_2 - \dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4} S_4 + \ldots = N_e, \\
\dfrac{\mathfrak{m}^2}{1^2} S_1 - \dfrac{\mathfrak{m}^6}{1^2.2^2.3^2} S_3 + \ldots = N_0', & \dfrac{\mathfrak{m}^4}{1^2.2^2} S_2 - \dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4^2} S_4 + \ldots = N_e',
\end{array}\right\} \quad . \quad (103)$$

$$\log_e \mathfrak{m} + \Lambda = L: \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

then substituting in (101), changing the sign of $\sqrt{-1}$, and arranging the terms, we get

$$k + \sqrt{-1}\, k' = 1 + \frac{2}{\mathfrak{m}^2} \frac{-LM_0 + \frac{\pi}{4} M_e - \frac{1}{2} M_0' + N_0 + \left\{\frac{\pi}{4} M_0 + LM_e - \frac{1}{2}(1 - M_e') - N_e\right\}\sqrt{-1}}{-\frac{\pi}{4} M_0' + L(1 - M_e') + N_e' + \left\{- LM_0' - \frac{\pi}{4}(1 - M_e') + N_0'\right\}\sqrt{-1}} \quad (105)$$

33. Before going on with the calculation, it will be requisite to know the numerical value of the transcendental quantity Λ. Now

$$\pi^{-\frac{1}{2}} \Gamma'(\tfrac{1}{2}) = (\Gamma \tfrac{1}{2})^{-1} \Gamma'(\tfrac{1}{2}) = \frac{d}{ds} \log \Gamma(s) = \frac{d}{ds} \log \Gamma(1 + s) - \frac{1}{s}, \text{ for } s = \tfrac{1}{2},$$

and the value of $\frac{d}{ds} \log \Gamma(1 + s)$ may be got at once from Legendre's table of the common logarithms of $\Gamma(1 + s)$, in which the interval of s is .001. Putting l_s for the tabular number corresponding to s, we have

$$\frac{d}{ds} \log \Gamma(1 + s) = 1000 \log_e 10 \left\{\Delta l_s - \tfrac{1}{2} \Delta^2 l_s + \tfrac{1}{3} \Delta^3 l_s - \tfrac{1}{4} \Delta^4 l_s + \ldots\right\}.$$

For $s = \frac{1}{2}$

$$\Delta l_s = + 16050324, \quad \Delta^2 l_s = + 405620, \quad \Delta^3 l_s = - 359, \quad \Delta^4 l_s = + 6,^*$$

the last figure being in each case in the 12th place of decimals. We thus get

$$\pi^{-\frac{1}{2}} \Gamma'(\tfrac{1}{2}) = - 1.9635102, \quad \Lambda = + .5772158. \quad . \quad . \quad . \quad . \quad . \quad (106)$$

34. When $\mathfrak{m}$ is large, it will be more convenient to employ series according to descending powers of a. Observing that the general term of $F_2(a)$ as given by (88), in which $D' = 0$, is

$$(-1)^i C' \varepsilon^{-ma} \frac{[1.3\ldots(2i - 1)]^2}{2.4\ldots 2i\,(4ma)^i a^{\frac{1}{2}}},$$

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[*] These numbers are copied from De Morgan's Differential and Integral Calculus, p. 588.

we get for the general term of $F_3'(a)$

$$(-1)^{i-1} C'\epsilon^{-ma} \frac{[1 \cdot 3 \ldots (2i-3)]^2}{2 \cdot 4 \ldots (2i-2)(4ma)^{i-1} a^{\frac{1}{2}}} \left\{ m \frac{(2i-1)^2}{2i \cdot 4ma} - \frac{2i-1}{2a} \right\},$$

and the expression within brackets is equivalent to

$$-\frac{(2i-1)(2i+1)}{8ia},$$

whence

$$aF_3'(a) = C'\epsilon^{-ma} ma^{\frac{1}{2}} \left\{ -1 - \frac{1 \cdot 3}{2 \cdot 4ma} + \frac{1^2 \cdot 3 \cdot 5}{2 \cdot 4 (4ma)^2} - \ldots \right\},$$

and we find by actual division

$$\frac{a F_3'(a)}{F_3(a)} = -ma - \tfrac{1}{2} + \tfrac{1}{8} (ma)^{-1} \ldots$$

35. When many terms are required, the calculation of the coefficients may be facilitated in the following manner.

Assuming $a F_3'(a) = v(a) F_3(a)$, we have

$$F_3'(a) = a^{-1} v(a) F_3(a), \quad F_3''(a) = \left\{ a^{-1} v'(a) - a^{-2} v(a) + a^{-2} (va)^2 \right\} F_3(a).$$

Substituting in the differential equation (85) which F_3 has to satisfy, we get

$$av'(a) + \{v(a)\}^2 - m^2 a^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Assuming

$$v(a) = -ma + A_0 + A_1 (ma)^{-1} + A_2 (ma)^{-2} + \ldots, \quad . \quad . \quad . \quad (108)$$

and substituting in the above equation, we get

$$-ma - 1 A_1 (ma)^{-1} - 2 A_2 (ma)^{-2} - 3 A_3 (ma)^{-3} \ldots$$

$$+ \left\{ -2ma + A_0 + A_1 (ma)^{-1} + \ldots \right\} \left\{ A_0 + A_1 (ma)^{-1} + \ldots \right\} = 0,$$

which gives on equating coefficients, $A_0 = -\frac{1}{2}$, and for $i > 0$

$$2 A_{i+1} = -i A_i + A_0 A_i + A_1 A_{i-1} \ldots + A_i A_0,$$

or, assuming to avoid fractions,

$$A_i = 2^{-2i-1} B_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

$$B_{i+1} = -2i B_i + B_0 B_i + B_1 B_{i-1} \ldots + B_i B_0, \quad . \quad . \quad . \quad . \quad (110)$$

a formula by means of which the coefficients B_1, B_2, $B_3 \ldots$ may be readily calculated one after another. We get

$$\left. \begin{array}{l} B_0 = -1, \quad B_1 = +1, \quad B_2 = -4, \quad B_3 = +25, \quad B_4 = -208, \quad B_5 = +2146, \\ B_6 = -26368, \quad B_7 = +375733, \quad B_8 = -6092032. \end{array} \right\} . \quad (111)$$

We get now from (100), (101), (108), and (109)

$$k - \sqrt{-1}\, k' = 1 + 2\epsilon^{-\frac{\pi}{4}\sqrt{-1}} \mathfrak{m}^{-1} - \tfrac{1}{2} B_0 \epsilon^{-2\frac{\pi}{4}\sqrt{-1}} \mathfrak{m}^{-2} - \frac{1}{2^4} B_1 \epsilon^{-3\frac{\pi}{4}\sqrt{-1}} \mathfrak{m}^{-3} \ldots, \quad (112)$$

whence if we calculate

$$u_1 = 2\mathfrak{m}^{-1}, \quad u_2 = -\tfrac{1}{2} B_0 \mathfrak{m}^{-2}, \quad u_3 = \tfrac{1}{16} B_1 \mathfrak{m}^{-3} \ldots, \quad u_i = (-1)^{i+1} \tfrac{1}{2} B_{i-2} 8^{-i+2} \mathfrak{m}^{-i},$$

we shall have, changing the sign of $\sqrt{-1}$ in (112), and writing ϑ for $\epsilon^{\frac{\pi}{4}\sqrt{-1}}$,

$$\left.\begin{aligned} k + \sqrt{-1}\, k' &= 1 + u_1 \vartheta + u_2 \vartheta^2 - u_3 \vartheta^3 + u_4 \vartheta^4 - u_5 \vartheta^5 + \ldots \\ k &= 1 + \sqrt{\tfrac{1}{2}}\, u_1 + \sqrt{\tfrac{1}{2}}\, u_3 - u_4 + \sqrt{\tfrac{1}{2}}\, u_5 - \sqrt{\tfrac{1}{2}}\, u_7 + u_8 - \sqrt{\tfrac{1}{2}}\, u_9 \ldots \\ k' &= \sqrt{\tfrac{1}{2}}\, u_1 + u_2 - \sqrt{\tfrac{1}{2}}\, u_3 + \sqrt{\tfrac{1}{2}}\, u_5 - u_6 + \sqrt{\tfrac{1}{2}}\, u_7 - \sqrt{\tfrac{1}{2}}\, u_9 \ldots \end{aligned}\right\} \quad . \quad (113)$$

If l_1, $l_2 \ldots$ be the common logarithms of the coefficients of $\mathfrak{m}^{-1}$, $\mathfrak{m}^{-2} \ldots$ in the last two of the formulæ (113),

$l_1 = .1505150$; $\quad l_4 = \bar{2}.4948500$; $\quad l_7 = \bar{2}.3646348$;

$l_2 = \bar{1}.6989700$; $\quad l_5 = \bar{2}.2371251$; $\quad l_8 = \bar{2}.7019316$;

$l_3 = \bar{2}.6453650$; $\quad l_6 = \bar{2}.4046734$; $\quad l_9 = \bar{2}.6017045$;

and if the logarithms of the coefficients of $\mathfrak{m}^{-1}$, $\mathfrak{m}^{-2} \ldots$ in $u_1, u_2 \ldots$ be required, it will be sufficient to add .1505150 to the 1st, 3rd, 5th, &c. of the logarithms above given.

36. It will be found that when $\mathfrak{m}$ is at all large, the series (113) are at first convergent, and afterwards divergent, and in passing from convergent to divergent the quantities u_i become nearly equal for several successive terms. If after having calculated i terms of the first of the series (113) we wish to complete the series by a formula involving the differences of u_i, we have

$$\begin{aligned} u_i \vartheta^i - u_{i+1} \vartheta^{i+1} + u_{i+2} \vartheta^{i+2} - \ldots &= \vartheta^i \{1 - \vartheta (1 + \Delta) + \vartheta^2 (1 + \Delta)^2 - \ldots\} u_i \\ &= \vartheta^i \{1 + \vartheta (1 + \Delta)\}^{-1} u_i \\ &= \frac{\vartheta^i}{1 + \vartheta} \left\{1 - \frac{\vartheta}{1 + \vartheta} \Delta + \left(\frac{\vartheta}{1 + \vartheta}\right)^2 \Delta^2 - \ldots\right\} u_i, \end{aligned}$$

and $1 + \vartheta = 1 + \cos\frac{\pi}{4} + \sqrt{-1} \sin\frac{\pi}{4} = 2 \cos\frac{\pi}{8}\, \epsilon^{\frac{\pi}{8}\sqrt{-1}}, \quad \vartheta (1 + \vartheta)^{-1} = \tfrac{1}{2} \sec\frac{\pi}{8} . \epsilon^{\frac{\pi}{8}\sqrt{-1}},$

so that the quantities to be added to k, k', are

$$\left.\begin{aligned} &\text{to } k, \ (-1)^i \tfrac{1}{2} \sec\frac{\pi}{8} \left\{\cos\frac{2i-1}{8}\pi . u_i - \tfrac{1}{2} \sec\frac{\pi}{8} \cos\frac{2i}{8}\pi . \Delta u_i \right. \\ &\qquad\qquad \left. + \left(\tfrac{1}{2} \sec\frac{\pi}{8}\right)^2 \cos\frac{2i+1}{8}\pi . \Delta^2 u_i \ldots\right\} \\ &\text{to } k', \ (-1)^i \tfrac{1}{2} \sec\frac{\pi}{8} \left\{\sin\frac{2i-1}{8}\pi . u_i - \tfrac{1}{2} \sec\frac{\pi}{8} \sin\frac{2i}{8}\pi . \Delta u_i \right. \\ &\qquad\qquad \left. + \left(\tfrac{1}{2} \sec\frac{\pi}{8}\right)^2 \sin\frac{2i+1}{8}\pi . \Delta^2 u_i \ldots\right\} \end{aligned}\right\} \quad . \quad (114)$$

37. The following table contains the values of the functions k and k' calculated for 40 different values of $\mathfrak{m}$. From $\mathfrak{m} = .1$ to $\mathfrak{m} = 1.5$ the calculation was performed by means of the formula (105); the rest of the table was calculated by means of the series (113). In the former part of the calculation, six places of decimals were employed in calculating the functions M_0, &c. given by (103). The last figure was then struck out, and five-figure logarithms were employed in multiplying the four functions M_0, M'_0, M_e, and $1 - M'_e$ by $\frac{\pi}{4}$, and by L, as well as in reducing the right-hand member of (105) to the form $k + \sqrt{-1}k'$. The terms of the series (113) were calculated to five places of decimals. That these series are sufficiently convergent to be employed when $\mathfrak{m} = 1{\cdot}5$, might be presumed from the numerical values of the terms, and is confirmed by finding that they give $k = 1{\cdot}952$, and $k' = 1{\cdot}153$. For $\mathfrak{m} = 1{\cdot}5$ and a few of the succeeding values, the second and third of the series (113) were summed directly as far as $\mathfrak{m}^{-5}$ inclusively, and the remainders were calculated from the formulæ (114). Two columns are annexed, which give the values of $\mathfrak{m}^2 k$ and $\mathfrak{m}^2 k'$, and exhibit the law of the variation of the two parts of the force F, when the radius of the cylinder varies, the nature of the fluid and time of oscillation remaining unchanged. Four significant figures are retained in all the results.

$\mathfrak{m}$	k	k'	$\mathfrak{m}^2 k$	$\mathfrak{m}^2 k'$	$\mathfrak{m}$	k	k'	$\mathfrak{m}^2 k$	$\mathfrak{m}^2 k'$
0	∞	∞	0	0	2·1	1·677	·7822	7·395	3·450
·1	19·70	48·63	·1970	·4863	2·2	1·646	·7421	7·966	3·592
·2	9·166	16·73	·3666	·6691	2·3	1·618	·7059	8·557	3·734
·3	6·166	9·258	·5549	·8332	2·4	1·592	·6730	9·168	3·877
·4	4·771	6·185	·7633	·9896	2·5	1·568	·6430	9·799	4·019
·5	3·968	4·567	·9920	1·142	2·6	1·546	·6154	10·45	4·160
·6	3·445	3·589	1·240	1·292	2·7	1·526	·5902	11·12	4·303
·7	3·082	2·936	1·510	1·439	2·8	1·507	·5669	11·81	4·444
·8	2·812	2·477	1·800	1·585	2·9	1·489	·5453	12·52	4·586
·9	2·604	2·137	2·110	1·731	3·0	1·473	·5253	13·25	4·728
1·0	2·439	1·876	2·439	1·876	3·1	1·457	·5068	14·01	4·870
1·1	2·306	1·678	2·790	2·021	3·2	1·443	·4895	14·78	5·012
1·2	2·194	1·503	3·160	2·164	3·3	1·430	·4732	15·57	5·154
1·3	2·102	1·365	3·552	2·307	3·4	1·417	·4581	16·38	5·296
1·4	2·021	1·250	3·961	2·450	3·5	1·405	·4439	17·21	5·437
1·5	1·951	1·153	4·389	2·595	3·6	1·394	·4305	18·06	5·580
1·6	1·891	1·069	4·841	2·739	3·7	1·383	·4179	18·93	5·721
1·7	1·838	·9965	5·312	2·880	3·8	1·373	·4060	19·82	5·863
1·8	1·791	·9332	5·804	3·024	3·9	1·363	·3948	20·73	6·005
1·9	1·749	·8767	6·314	3·165	4·0	1·354	·3841	21·67	6·145
2·0	1·711	·8268	6·845	3·307	∞	1	0	∞	∞

The numerical calculation by means of the formulæ (103), (104), (105) becomes very laborious when many values of the functions are required. The difficulty of the calculation increases with the value of $\mathfrak{m}$ for two reasons, first, the calculation of the functions M_0, &c. becomes longer, and secondly, the moduli of the numerator and denominator of the fraction in the right-hand member of (105) go on decreasing, so that greater and greater accuracy is required in the calculation of the functions M_0, &c., and of the products LM_0, &c., in order to ensure a given degree of accuracy in the result. The calculation by the descending series (113) is on the contrary very easy.

It will be found that the first differences of $\mathfrak{m}^2 k'$ and of $\mathfrak{m}^2(k-1)$ are nearly constant, except near the very beginning of the table. Hence in the earlier part of the table the value of k or k' for a value of $\mathfrak{m}$ not found in the table will be best got by finding $\mathfrak{m}^2 k - \mathfrak{m}^2$ or $\mathfrak{m}^2 k'$ by interpolation, and thence passing to the value of k or k'. Very near the beginning of the table, interpolation would not succeed, but in such a case recourse may be had to the formulæ (103), (104), (105), the calculation of which is comparatively easy whem $\mathfrak{m}$ is small. It did not seem worth while to extend the table beyond $\mathfrak{m} = 4$, because where $\mathfrak{m}$ is greater than 4, the series (113) are so rapidly convergent that k and k' may be calculated to a sufficient degree of accuracy with extreme facility.

38. Let us now examine the progress of the functions k and k'.

When $\mathfrak{m}$ is very small, we may neglect the powers of $\mathfrak{m}$ in the numerator and denominator of the fraction in the right-hand member of equation (105), retaining only the logarithms and the constant terms. We thus get

$$k + \sqrt{-1}\,k' = 1 - \frac{\mathfrak{m}^{-2}\sqrt{-1}}{L - \frac{\pi}{4}\sqrt{-1}},$$

whence

$$\mathfrak{m}^2(k-1) = \frac{\frac{\pi}{4}}{L^2 + \left(\frac{\pi}{4}\right)^2}, \qquad \mathfrak{m}^2 k' = \frac{-L}{L^2 + \left(\frac{\pi}{4}\right)^2}, \quad . \quad . \quad . \quad (115)$$

L being given by (102) and (104), or (104) and (106). When $\mathfrak{m}$ vanishes, L, which involves the logarithm of $\mathfrak{m}^{-1}$, becomes infinite, but ultimately increases more slowly than if it varied as $\mathfrak{m}$ affected with any negative index however small. Hence it appears from (115), that $k-1$ and k' are expressed by $\mathfrak{m}^{-2}$ multiplied by two functions of $\mathfrak{m}$ which, though they ultimately vanish with $\mathfrak{m}$, decrease very slowly, so that a considerable change in $\mathfrak{m}$ makes but a small change in these functions. Now when the radius a of the cylinder varies, everything else remaining the same, $\mathfrak{m}$ varies as a, and in general the parts of the force F on which depend the alteration in the time of vibration, and the diminution in the arc of oscillation, vary as $a^2 k$, $a^2 k'$, respectively. Hence in the case of a cylinder of small radius, such as the wire used to support a sphere in a pendulum experiment, a considerable change in the radius of the cylinder produces a comparatively small change in the part of the alteration in the time and arc of vibration which is due to the resistance experienced by the wire. The simple formulæ (115) are accurate enough for the fine wires usually employed in such experiments if the theory itself be applicable; but reasons will presently be given for regarding the application of the theory to such fine wires as extremely questionable.

From $\mathfrak{m} = \cdot 3$ or $\cdot 4$ to the end of the table, the first differences of each of the functions $\mathfrak{m}^2(k-1)$ and $\mathfrak{m}^2 k'$ remain nearly constant. Hence for a considerable range of values of $\mathfrak{m}$, each of the functions may be expressed pretty accurately by $A + B\mathfrak{m}$. When $\mathfrak{m}$ is at all large, the first two terms in the 2nd and 3rd of the formulæ (113) will give k and k' with considerable accuracy, because, independently of the decrease of the successive quantities $\mathfrak{m}^{-1}$, $\mathfrak{m}^{-2}$, $\mathfrak{m}^{-3}\ldots$, the coefficients of $\mathfrak{m}^{-1}$ and $\mathfrak{m}^{-2}$ are considerably larger than those of several of the succeeding powers. If we neglect in these formulæ the terms after u_2, we get

$$k = 1 + \sqrt{2}.\mathfrak{m}^{-1}, \quad k' = \sqrt{2}.\mathfrak{m}^{-1} + \tfrac{1}{2}\mathfrak{m}^{-2}.$$

It may be remarked that these approximate expressions, regarded as functions of the radius a, have precisely the same form as the exact expressions obtained for a sphere, the coefficients only being different.

SECTION IV.

Determination of the motion of a fluid about a sphere which moves uniformly with a small velocity. Justification of the application of the solutions obtained in Sections II. *and* III. *to cases in which the extent of oscillation is not small in comparison with the radius of the sphere or cylinder. Discussion of a difficulty which presents itself with reference to the uniform motion of a cylinder in a fluid.*

39. Let a sphere move in a fluid with a uniform velocity V, its centre moving in a right line; and let the rest of the notation be the same as in Section II. Conceive a velocity equal and opposite to that of the sphere impressed both on the sphere and on the fluid, which will not affect the relative motion of the sphere and fluid, and will reduce the determination of the motion of the fluid to a problem of steady motion. Then we have for the equations of condition

$$R = 0, \qquad \Theta = 0, \text{ when } r = a; \quad \ldots\ldots \quad (116)$$

$$R = -V\cos\theta, \quad \Theta = V\sin\theta, \text{ when } r = \infty. \quad \ldots\ldots \quad (117)$$

The equations of condition, as well as the equations of motion, may be satisfied by supposing ψ to have the form $\sin^2\theta f(r)$. We get from (20′), by the same process as that by which (33), (34) were obtained,

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f(r) = 0, \quad \ldots\ldots \quad (118)$$

the only difference being that in the present case the equation (20′) cannot be replaced by the two (22), (23), which become identical, inasmuch as the velocity of the fluid is independent of the time.

The integral of (118) is

$$f(r) = Ar^{-1} + Br + Cr^2 + Dr^4, \quad \ldots\ldots \quad (119)$$

which gives

$$R = \frac{1}{r^2 \sin\theta}\frac{d\psi}{d\theta} = 2\cos\theta\,(A r^{-3} + B r^{-1} + C + D r^2),$$

$$\Theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr} = \sin\theta\,(A r^{-3} - B r^{-1} - 2C - 4 D r^2).$$

The first of the equations of condition (117) requires that

$$D = 0, \qquad C = -\tfrac{1}{2}V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

It is particularly to be remarked that inasmuch as the two arbitrary constants C, D are determined by the first of the conditions (117), none remain whereby to satisfy the second. Nevertheless it happens that the second of these conditions leads to precisely the same equations (120) as the first. The equations of condition (116) give

$$A = -\tfrac{1}{4}Va^3, \qquad B = \tfrac{3}{4}Va;$$

whence

$$\psi = \tfrac{1}{4}Va^2\left(-\frac{2r^2}{a^2} + \frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta,^{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

$$R = -V\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

$$\Theta = V\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

If now we wish to obtain the solution of the problem in its original shape, in which the sphere is in motion and the fluid at rest, except so far as it is disturbed by the sphere, we have merely to add $V\cos\theta$, $-V\sin\theta$, $\frac{1}{2}Vr^2\sin^2\theta$ to the expressions for R, Θ, ψ. We get from (121)

$$\psi = \tfrac{1}{4}Va^2\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

40. Let us now return to the problem of Section II.; let us suppose the time of oscillation to increase indefinitely, and examine what equation (40) becomes in the limit.

When τ becomes infinite, n, and therefore m, vanishes; the expression within brackets in (40) takes the form $\infty - \infty$, and its limiting value is easily found by the ordinary methods. We must retain the m^2 in the coefficient of t, because t is susceptible of unlimited increase. We get in the limit

$$\psi = \tfrac{1}{4}a^2 c\,\epsilon^{\mu' m^2 t}\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

If now we put V for $\dfrac{d\xi}{dt}$, the velocity of the sphere, we get from (39), $c\epsilon^{\mu' m^2 t} = V$. After substituting in (125), the equation will remain unchanged when we pass from the symbolical to the real values of ψ and V, and thus (125) will be reduced to (124).

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* I have already had occasion, in treating of another subject, to publish the solution expressed by this equation, which I had obtained as a limiting case of the problem of a ball pendulum. See *Philosophical Magazine* for May 1848, p. 343.

41. It appears then that by supposing the rate of alteration of the velocity of the sphere to decrease indefinitely, we obtain from the solution of the problem of Section II. the same result as was obtained in Art. 39, by treating the motion as steady. As yet, however, the method of Art. 40 is subject to a limitation from which that of Art. 39 is free. In obtaining equation (40), it was supposed that the maximum excursion of the sphere was small in comparison with its radius. Retaining this restriction while we suppose τ to become very large, we are obliged to suppose c to become very small, so that the velocity of the sphere is not merely so small that we may neglect terms depending upon its square, a restriction to which Art. 39 is also subject, but so extremely small that the space passed over by the sphere in even a long time is small in comparison with its radius.

We have seen, however, that on supposing τ very large in (40) we obtain a result identical with (124), not merely a result with which (124) becomes identical when the restriction above mentioned is introduced. This leads to the supposition that the solution expressed by (40) is in fact more general than would appear from the way in which it was obtained. That such is really the case may be shewn by a slight modification of the analysis. Instead of referring the fluid to axes fixed in space, refer it to axes originating at the centre of the sphere, and moveable with it. In the general equations of motion, the terms which contain differential coefficients taken with respect to the co-ordinates will remain unchanged, inasmuch as they represent the very same limiting ratios as before: it is only those in which differentiation with respect to t occurs that will be altered. If $\frac{d'}{dt}$ be the symbol of differentiation with respect to t when the fluid is referred to the moveable axes, we shall have

$$\frac{d}{dt} = \frac{d'}{dt} - \frac{d\xi}{dt}\frac{d}{dx};$$

but the terms arising from $\frac{d\xi}{dt}\frac{d}{dx}$ are of the order of the square of the velocity, and are therefore to be neglected. Hence the general equations have the same form whether the fluid be referred to the fixed or moveable axes. But on the latter supposition the equations of condition (30) become rigorously exact. Hence equation (40) gives correctly the solution of the problem, independently of the restriction that the maximum excursion of the sphere be small compared with its radius, provided we suppose the polar co-ordinates r, θ measured from the centre of the sphere in its actual, not its mean position. Similar remarks apply to the problem of the cylinder. Moreover, in the case of a sphere oscillating within a concentric spherical envelope, it is not necessary, in order to employ the solution obtained in Section II., that the maximum excursion of the sphere be small compared with its radius; it is sufficient that it be small compared with the radius of the envelope.

These are points of great importance, because the excursions of an oscillating sphere in a pendulum experiment are not by any means extremely small compared with the radius of the sphere; and in the case of a narrow cylinder, such as the suspending wire, so far from the maximum excursion being small compared with the radius of the cylinder, it is, on the contrary, the radius which is small compared with the maximum excursion.

42. Let us now return to the case of the uniform motion of a sphere. In order to obtain directly the expression for the resistance of the fluid, it would be requisite first to find p, then to get P_r and T_θ from (46), or at least to get the values of these functions for $r = a$, and lastly to substitute in (47) and perform the integration. We should obtain p by integrating the expression for dp got from (16) and (17). It would be requisite first to express u and q in terms of ψ, then to transform the expression for dp so as to involve polar co-ordinates, and then substitute for ψ its value given by (121); or else to express the right-hand member of (121) by the co-ordinates x, ϖ, and substitute in the expression for dp[*]. We have seen, however, that the results applicable to uniform motion may be deduced as limiting cases of those which relate to oscillatory motion, and consequently, we may make use of the expression for F already worked out. Writing V for $c\epsilon^{\sqrt{-1}\,nt}$ in the first equation of Art. 20, expressing m in terms of n, and then making n vanish, we get

$$-F = 6\pi\mu'\rho a V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

and $-F$ is the resistance required.

This equation may be employed to determine the terminal velocity of a sphere ascending or descending in a fluid, provided the motion be so slow that the square of the velocity may be neglected. It has been shewn experimentally by Coulomb[†], that in the case of very slow motions, the resistance of a fluid depends partly on the square and partly on the first power of the velocity. The formula (126) determines, in the particular case of a sphere, that part of the whole resistance which depends on the first power of the velocity, even though the part which depends on the square of the velocity be not wholly insensible.

It is particularly to be remarked, that according to the formula (126), the resistance varies not as the surface but as the radius of the sphere, and consequently the quotient of the resistance divided by the mass increases in a higher ratio, as the radius diminishes, than if the resistance varied as the surface. Accordingly, fine powders remain nearly suspended in a fluid of widely different specific gravity.

43. When the motion is so slow that the part of the resistance which depends on the square of the velocity may be neglected, we have, supposing V to be the terminal velocity,

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[*] The equations (16), (17) give, after a troublesome transformation to polar co-ordinates,

$$\frac{dp}{dr} = \frac{\mu}{r^2 \sin\theta}\frac{d}{d\theta}\left(\frac{d^2}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{d}{d\theta} - \frac{\rho}{\mu}\frac{d}{dt}\right)\psi, \quad (a)$$

$$\frac{dp}{d\theta} = -\frac{\mu}{\sin\theta}\frac{d}{dr}\left(\frac{d^2}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{d}{d\theta} - \frac{\rho}{\mu}\frac{d}{dt}\right)\psi. \quad (b)$$

The expression for dp got from these equations is an exact differential by virtue of the equation which determines ψ; and in the problems considered in Section II. and in the present Section ψ has the form $\Psi \sin^2\theta$, where Ψ is independent of θ. Hence we get from (b), by integrating partially with respect to θ,

$$p = \mu\cos\theta\frac{d}{dr}\left(\frac{d^2}{dr^2} - \frac{2}{r^2} - \frac{\rho}{\mu}\frac{d}{dt}\right)\Psi. \quad (c)$$

It is unnecessary to add an arbitrary function of r, because if $\lambda(r)$ be such a function which we suppose added to the right-hand member of (c), we must determine λ by substituting in (a). The resulting expression for $\lambda'(r)$ cannot contain θ, inasmuch as the expression for dp is an exact differential, but it is composed of terms which all involve $\cos\theta$ as a factor, and therefore we know, without working out, that these terms must destroy one another. Hence $\lambda(r)$ must be constant, or at most be a function of t, which we may suppose included in Π. $\lambda(r)$ will in fact be equal to zero if Π be the equilibrium pressure at the depth at which $\int g\,dx'$ vanishes.

[†] *Mémoires de l'Institut*, Tom. III. p. 246.

$-F = \frac{4}{3}\pi g(\sigma - \rho)a^3$, where g is the force of gravity, and σ, which is supposed greater than ρ, the density of the sphere. Substituting in (126) we get

$$V = \frac{2g}{9\mu'}\left(\frac{\sigma}{\rho} - 1\right)a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

Let us apply this equation to determine the terminal velocity of a globule of water forming part of a cloud. Putting $g = 386$, $\mu' = (.116)^2$, an inch being the unit of length, and supposing $\sigma\rho^{-1} - 1 = 1000$, in order to allow a little for the rarity of the air at the height of the cloud, we get $V = 6372 \times 1000a^2$. Thus, for a globule the one thousandth of an inch in diameter, we have $V = 1.593$ inch per second. For a globule the one ten thousandth of an inch in diameter, the terminal velocity would be a hundred times smaller, so as not to amount to the one sixtieth part of an inch per second.

We may form a very good judgment of the magnitude of that part of the resistance which varies as the square of the velocity, and which is the only kind of resistance that could exist if the pressure were equal in all directions, by calculating the numerical value of the resistance according to the common theory, imperfect though it be. It follows from this theory that if h be the height due to the velocity V, the resistance is to the weight as $3\rho h$ to $8\sigma a$. For $V = 1.593$ inch per second, the resistance is not quite the one four hundredth part of the weight; and for a sphere only the one ten thousandth of an inch in diameter, moving with the velocity calculated from the formula (127), the ratio of the resistance to the weight would be ten times as small. The terminal velocities of the globules calculated from the common theory would be 32.07 and 10.14 inches per second, instead of only 1.593 and .01593 inch. It appears then that the apparent suspension of the clouds is mainly due to the internal friction of air.

44. The resistance to the globule has here been determined as if the globule were a solid sphere. In strictness, account ought to be taken of the relative motion of the fluid particles forming the globule itself. Although it may readily be imagined that no material change would thus be made in the numerical result, it may be worth while to point out the mode of solution of the problem. Suppose the globule preserved in a strictly spherical shape by capillary attraction, which will very nearly indeed be the case. Conceive a velocity equal and opposite to that of the globule impressed both on the globule and on the surrounding fluid, which will reduce the problem to one of steady motion. Let ψ_1, &c. refer to the fluid forming the globule, and assume $\psi_1 = f_1(r)\sin^2\theta$. Then we get on changing the constants in (119)

$$f_1(r) = A_1r^{-1} + B_1r + C_1r^2 + D_1r^4.$$

The arbitrary constants A_1, B_1 vanish by the condition that the velocity shall not become infinite at the centre. There remain the two arbitrary constants C_1, D_1 to be determined, in addition to those which appeared in the former problem. But we have now four instead of two equations of condition which have to be satisfied at the surface of the sphere, which are that

$$R = 0, \quad R_1 = 0, \quad \Theta = \Theta_1, \quad T_\theta = T_{1\theta}, \quad \text{when } r = a. \quad . \quad . \quad . \quad (128)$$

We shall thus have the same number of arbitrary constants as conditions to be satisfied. Now $T_{1\theta}$ will involve μ_1 as a coefficient, just as T_θ involves $\mu'\rho$ or μ; and μ_1, which refers to water, is much larger than μ, which refers to air, although μ' is larger than μ_1'. Hence the results will be nearly the same as if we had taken $\mu_1 = \infty$, or regarded the sphere as solid.

If, however, instead of a globule of liquid descending in a gas we have a very small bubble ascending in a liquid, we must not treat the bubble as a solid sphere. We may in this case also neglect the motion of the fluid forming the sphere, but we have now arrived at the other extreme case of the general problem, and the two equations of condition which have to be satisfied at the surface of the sphere are that $R = 0$ and $T_\theta = 0$ when $r = a$, instead of $R = 0$ and $\Theta = 0$, when $r = a$.

The equation of condition $T_\theta = 0$ which applies to a bubble, as well as the fourth of equations (128), will not be the true equations, if forces arising from internal friction exist in the superficial film of a fluid which are of a different order of magnitude from those which exist throughout the mass. At the end of the memoir already referred to, Coulomb states that in very slow motions the resistance of bodies not completely immersed in a liquid is much greater than that of bodies wholly immersed, and promises to communicate a second memoir in continuation of the first. This memoir, so far as I can find out, has never appeared. Should the existence of such forces in the superficial film of a liquid be made out, the results deduced from the theory of internal friction will be modified in a manner analogous to that in which the results deduced from the common principles of hydrostatics are modified by capillary attraction. It may be remarked that we have nothing to do with forces of this kind in considering the motion of pendulums in air, or even in considering the oscillations of a sphere in water, except as regards the very minute fraction of the whole effect which relates to the resistance experienced by the suspending wire in the immediate neighbourhood of the free surface.

It may readily be seen that the effect of a set of forces in the superficial film of a liquid offering a peculiar resistance to the relative motion of the particles would be, to make the resistance of a gas to a descending globule agree still more clearly with the result obtained by regarding the globule as solid, while the resistance experienced by an ascending bubble would be materially increased, and made to approach to that which would belong to a solid sphere of the same size without mass, or more strictly, with a mass only equal to that of the gas forming the bubble. Possibly the determination of the velocity of ascent of very small bubbles may turn out to be a good mode of measuring the amount of friction in the superficial film of a liquid, if it be true that forces of this kind have any existence. But any investigation relating to such a subject would at present be premature.

45. Let us now attempt to determine the uniform motion of a fluid about an infinite cylinder. Employing the notation of Section III, and reducing the problem to one of steady motion as in Art. 39, we obtain the same equations of condition (116), (117), as in the case of the sphere. Assuming $\chi = \sin\theta F(r)$, and substituting in the equation obtained from (69) by transforming to polar co-ordinates and leaving out the terms which involve $\frac{d}{dt}$, we get

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 F(r) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

The integral of this equation may readily be obtained by multiplying the last term of the operating factor by $(1 + \delta)^2$, integrating the transformed equation, and then making δ vanish. It is

$$F(r) = Ar^{-1} + Br + Cr \log r + Dr^3, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (130)$$

which gives

$$R = \frac{d\chi}{r\,d\theta} = (Ar^{-2} + B + C \log r + Dr^2) \cos\theta,$$

$$\Theta = -\frac{d\chi}{dr} = (Ar^{-2} - B - C - C \log r - 3Dr^2) \sin\theta.$$

The first of the equations of condition (117) requires that

$$C = 0, \qquad D = 0, \qquad B = -V,$$

which also satisfies the second. We have thus only one arbitrary constant left whereby to satisfy the two equations of condition (116), and the same value of A will not satisfy these two equations.

46. It appears then that the supposition of steady motion is inadmissible. It will be remembered that, in the case of the sphere, the solution of the problem was only possible because it so happened that the values of two arbitrary constants determined by satisfying the first of the equations of condition (117) satisfied also the second, which indicates that the solution was to a certain extent tentative. We have evidently a right to conceive a sphere or infinite cylinder to exist at rest in an infinite mass of fluid also at rest, to suppose the sphere or cylinder to be then moved with a uniform velocity V, and to propose for determination the motion of the fluid at the end of the time t. But we have no right to assume that the motion approaches a permanent state as t increases indefinitely. We may follow either of two courses. We may proceed to solve the general problem in which the sphere or cylinder is supposed to move from rest, and then examine what results we obtain by supposing t to increase indefinitely, or else we may assume *for trial* that the motion is steady, and proceed to inquire whether we can satisfy all the conditions of the problem on this supposition. The former course would have the disadvantage of requiring a complicated analysis for the sake of obtaining a comparatively simple result, and it is even possible that the solution of the problem might baffle us altogether; but if we adopt the latter course, we must not forget that the equations with which we work are only provisional.

It might be objected that the impossibility of satisfying the conditions of the problem on the hypothesis of steady motion arose from our assumption that $\sin\theta$ was a factor of χ, the other factor being independent of θ. This however is not the case. For, for given values of r and t, χ is a finite function of θ from $\theta = 0$ to $\theta = \pi$. We have a right to suppose χ to vanish at any point of the axis of x positive that we please; and if we suppose χ to vanish at one such point, it may be shewn as in the note to Art. 15, that χ will vanish at all points of the axis of x positive or x negative. Hence χ may be expanded in a convergent series of sines of θ and its multiples; and since χ and its derivatives with respect to θ alter continuously with θ, the expansions of the derivatives will be got by direct differentiation*. This being true for all other pairs of values of r and t, χ can in general be expanded in a convergent series of sines of θ and its multiples; but the coefficients, instead of being constant, will be functions of r and t, or in the particular case of steady motion, functions of r alone. Now a very slight examination of the general equations will suffice to shew that the coefficients of the sines of the different multiples of θ remain perfectly independent throughout the whole process, and consequently had we employed the general expansion, we should have been led to the very same conclusions which have been deduced from the assumed form of χ.

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* See a paper *On the Critical Values of the Sums of Periodic Series.* Camb. Phil. Trans. Vol. VIII. p. 533.

47. If we take the impossibility of the existence of a limiting state of motion, which has just been established, in connexion with the results obtained in Section III., we shall be able to understand the general nature of the motion of the fluid around an infinite cylinder which is at first at rest, and is then moved on indefinitely with a uniform velocity.

The fluid being treated as incompressible, the first motion which takes place is impulsive. Since the terms depending on the internal friction will not appear in the calculation of this motion, we may employ the ordinary equations of hydrodynamics. The result, which is easily obtained, is

$$R dr + \Theta r d\theta = d\phi, \quad \text{where } \phi = -\frac{Va^2}{r}\cos\theta^{*} \quad . \quad . \quad . \quad . \quad . \quad (131)$$

As the cylinder moves on, it carries more and more of the fluid with it, in consequence of friction. For the sake of precision, let the quantity carried by the element dl of the cylinder be defined to be that which, moving with the velocity V, would have the same momentum in the direction of the motion that is actually possessed by the elementary portion of fluid which is contained between two parallel infinite planes drawn perpendicular to the axis of the cylinder, at an interval dl, the particles composing which are moving with velocities that vary from V to zero in passing from the surface outwards. The pressure of the cylinder on the fluid continually tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and the motion becomes uniform. But in the case of a cylinder, the increase in the quantity of fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on. The rate at which the quantity carried is increased, decreases continually, because the motion of the fluid in the neighbourhood of the cylinder becomes more and more nearly a simple motion of translation equal to that of the cylinder itself, and therefore the rate at which the quantity of fluid carried is increased would become smaller and smaller, even were no resistance offered by the surrounding fluid.

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[*] According to these equations, the fluid flows past the surface of the cylinder with a finite velocity. At the end of the small time t' after the impact, the friction has reduced the velocity of the fluid in contact with the cylinder to that of the cylinder itself, and the tangential velocity alters very rapidly in passing from the surface outwards. At a small distance s from the surface of the cylinder, the relative velocity of the fluid and the cylinder, in a direction tangential to the surface, is a function of the independant variables t', s, which vanishes with s for any given value of t', however small, but which for any given value of s, however small, approaches indefinitely to the quantity determined by (131) as t vanishes. The communication of lateral motion is similar to the communication of temperature when the surface of a body has its temperature instantaneously raised or lowered by a finite quantity.

The correctness of this explanation is confirmed by the following considerations. Suppose that $F(r)$ had been given by the equation

$$F(r) = Ar^{-1} + Br + Cr^{1-\delta} + Dr^3$$

instead of (130), δ being a small positive quantity. On this supposition it would have been possible to satisfy all the equations of condition, and therefore steady motion would have been possible. By determining the arbitrary constants, and substituting in χ, we should have obtained

$$\psi = aV\left\{-\frac{\delta}{2-\delta}\frac{a}{r} - \frac{r}{a} + \frac{2}{2-\delta}\left(\frac{r}{a}\right)^{1-\delta}\right\}\sin\theta,$$

$$R = V\left\{-\frac{\delta}{2-\delta}\left(\frac{a}{r}\right)^2 - 1 + \frac{2}{2-\delta}\left(\frac{r}{a}\right)^{-\delta}\right\}\cos\theta,$$

$$\Theta = V\left\{-\frac{\delta}{2-\delta}\left(\frac{a}{r}\right)^2 + 1 - \frac{2(1-\delta)}{2-\delta}\left(\frac{r}{a}\right)^{-\delta}\right\}\sin\theta.$$

Since δ is supposed to be extremely small, it follows from these expressions that when r is not greater than a moderate multiple of a, the velocities R, Θ are extremely small; but, however small be δ, we have only to go far enough from the cylinder in order to find velocities as nearly equal to $-V\cos\theta$, $+V\sin\theta$ as we please. But the distance from the cylinder to which we must proceed in order to find velocities R, Θ which do not differ from their limiting values $-V\cos\theta$, $+V\sin\theta$ by more than certain given quantities, increases indefinitely as δ decreases. Hence, restoring to the fluid and the cylinder the velocity V, we see that in the neighbourhood of the cylinder the motion of the fluid does not sensibly differ from a motion of translation, the same as that of the cylinder itself, while the distance to which the cylinder exerts a sensible influence in disturbing the motion of the fluid increases indefinitely as δ decreases.

48. When we have formed the equations of motion of a fluid on any particular dynamical hypothesis, it becomes a perfectly definite mathematical problem to determine the motion of the fluid when a given solid, initially at rest as well as the fluid, is moved in a given manner, or to discuss the character of the analytical solution in any extreme case proposed. It is quite another thing to enquire how far the principles which furnished the mathematical data of the problem may be modified in extreme cases, or what will be the nature of the actual motion in such cases. Let us regard in this point of view the case considered in the preceding article as a mathematical problem. When the quantity of fluid carried with the cylinder becomes considerable compared with the quantity displaced, it would seem that the motion must become unstable, in the sense in which the motion of a sphere rolling down the highest generating line of an inclined cylinder may be said to be unstable. But besides the instability, it may not be safe in such an extreme case to neglect the terms depending on the square of the velocity, not that they become unusually large in themselves, but only unusually large compared with the terms retained, because when the relative motions of neighbouring portions of the fluid become very small, the tangential pressures which arise from friction become very small likewise.

Now the general character of the motion must be nearly the same whether the velocity of the cylinder be constant, or vary slowly with the time, so that it does not vary materially when the cylinder passes through a space equal to a small multiple of its radius. To return to the problem considered in Section III., it would seem that when the radius of the cylinder is very small, the motion which would be expressed by the formulæ of that Section would be unstable. This might very well be the case with the fine wires used in supporting the spheres employed in pendulum experiments. If so, the quantity of fluid carried by the wire would be diminished, portions being continually left behind and forming eddies. The resistance to the wire would on the whole be increased, and would moreover approximate to a resistance which would be a function of the velocity. Hence, so far as depends on the wire, the arc of oscillation would be more affected by the resistance of the air than would follow from the formulæ of Section III. Whether the effect on the time of oscillation would be greater or less than that expressed by the formulæ is difficult to say, because the increase of resistance would tend to increase the effect on the time of vibration, while on the other hand the approximation of the law of resistance to that of a function of the velocity would tend to diminish it.

SECTION V.

On the effect of internal friction in causing the motion of a fluid to subside. Application to the case of oscillatory waves.

49. We have already had instances of the effect of friction in causing a gradual subsidence in the motion of a solid oscillating in a fluid; but a result may easily be obtained from the equations of motion in their most general shape, which shews very clearly the effect of friction in continually consuming a portion of the work of the forces acting on the fluid.

Let P_1, P_2, P_3 be the three normal, and T_1, T_2, T_3 the three tangential pressures in the direction of three rectangular planes parallel to the co-ordinate planes, and let D be the symbol of differentiation with respect to t when the particle and not the point of space remains the same. Then the general equations applicable to a heterogeneous fluid, (the equations (10) of my former paper,) are

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = 0, \quad . \quad . \quad . \quad . \quad . \quad (132)$$

with the two other equations which may be written down from symmetry. The pressures P_1, &c. are given by the equations

$$P_1 = p - 2\mu\left(\frac{du}{dx} - \delta\right), \quad T_1 = -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right), \quad . \quad . \quad . \quad (133)$$

and four other similar equations. In these equations

$$3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

At the end of the time t let V be the *vis viva* of a limited portion of the fluid, occupying the space which lies inside the closed surface S, and let $V + DV$ be the *vis viva* of the same mass at the end of the time $t + Dt$. Then

$$V = \iiint \rho\,(u^2 + v^2 + w^2)\,dx\,dy\,dz,$$

$$DV = 2Dt \iiint \rho \left(u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}\right) dx\,dy\,dz, \quad . \quad . \quad . \quad (135)$$

the triple integrals extending throughout the space bounded by S. Substituting now for $\frac{Du}{Dt}$, &c. their values given by the equations of the system (132), we get

$$\begin{aligned} DV = {} & 2Dt \iiint \rho\,(uX + vY + wZ)\,dx\,dy\,dz \\ & - 2Dt \iiint \left\{u\left(\frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz}\right) + v\left(\frac{dP_2}{dy} + \frac{dT_1}{dz} + \frac{dT_3}{dx}\right) \right. \\ & \left. + w\left(\frac{dP_3}{dz} + \frac{dT_2}{dx} + \frac{dT_1}{dy}\right)\right\} dx\,dy\,dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (136) \end{aligned}$$

The first part of this expression is evidently twice the work, during the time Dt, of the external forces which act all over the mass. The second part becomes after integration by parts

$$\begin{aligned} & - 2Dt \iint (uP_1 + vT_3 + wT_2)\,dy\,dz - 2Dt \iint (vP_2 + wT_1 + uT_3)\,dz\,dx \\ & - 2Dt \iint (wP_3 + uT_2 + vT_1)\,dx\,dy \\ & + 2Dt \iiint \left\{\frac{du}{dx}P_1 + \frac{dv}{dy}P_2 + \frac{dw}{dz}P_3 + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)T_1 + \left(\frac{dw}{dx} + \frac{du}{dz}\right)T_2 \right. \\ & \left. + \left(\frac{du}{dy} + \frac{dv}{dx}\right)T_3\right\} dx\,dy\,dz. \end{aligned}$$

The double integrals in this expression are to be extended over the whole surface S. If dS be an element of this surface, l', m', n' the direction-cosines of the normal drawn outwards at dS, we may write $l'dS$, $m'dS$, $n'dS$ for $dydz$, $dzdx$, $dxdy$. The second part of DV thus becomes

$$- 2Dt \iint \{u(l'P_1 + m'T_3 + n'T_2) + v(m'P_2 + n'T_1 + l'T_3) + w(n'P_3 + l'T_2 + m'T_1)\}\,dS.$$

The coefficients of u, v, w in this expression are the resolved parts, in the direction of x, y, z, of the pressure on a plane in the direction of the elementary surface dS, whence it appears that the expression itself denotes twice the work of the pressures applied to the surface of the portion of fluid that we are considering.

On substituting for P_1, &c. their values given by the equations (133), (134), we get for the last part of DV

$$\begin{aligned} & + 2Dt \iiint p\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) dx\,dy\,dz \\ & - 2Dt \iiint \mu\left\{2\left(\frac{du}{dx}\right)^2 + 2\left(\frac{dv}{dy}\right)^2 + 2\left(\frac{dw}{dz}\right)^2 - \frac{2}{3}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)^2 \right. \\ & \left. + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 + \left(\frac{dw}{dx} + \frac{du}{dz}\right)^2 + \left(\frac{du}{dy} + \frac{dv}{dx}\right)^2\right\} dx\,dy\,dz. \end{aligned}$$

In this expression p denotes, in the case of an elastic fluid, the pressure statically corresponding to the density which actually exists about the point whose co-ordinates are x, y, z, and the part of the expression which contains p denotes twice the work converted into *vis viva* in consequence of internal expansions, and arising from the forces on which the elasticity depends. The last part of the expression is essentially negative, or at least cannot be positive, and can only vanish in one very particular case. It denotes the *vis viva* consumed, or twice the work lost in the system during the time dt, in consequence of internal friction. According to the very important theory of Mr Joule, which is founded on a set of most striking and satisfactory experiments, the work thus apparently lost is in fact converted into heat, at such a rate, that the work expressed by the descent of 772 lbs through one foot, supplies the quantity of heat required to raise 1 lb. of water through 1° of Fahrenheit's thermometer.

50. The triple integral containing μ can only vanish when the differential coefficients of u, v, w satisfy the five following equations,

$$\left.\begin{array}{c}\dfrac{du}{dx}=\dfrac{dv}{dy}=\dfrac{dw}{dz},\\ \dfrac{dv}{dz}+\dfrac{dw}{dy}=0,\qquad \dfrac{dw}{dx}+\dfrac{du}{dz}=0,\qquad \dfrac{du}{dy}+\dfrac{dv}{dx}=0.\end{array}\right\}\quad\ldots\quad(137)$$

These equations give immediately the following expressions for the differentials of u, v, w, in which the co-ordinates alone are supposed to vary, the time being constant:

$$\left.\begin{array}{l}du=\delta dx-\omega'''dy+\omega''dz,\\ dv=\delta dy-\omega'dz\;+\omega'''dx,\\ dw=\delta dz-\omega''dx\;+\omega'dy.\end{array}\right\}\quad\ldots\ldots\quad(138)$$

In these equations δ, ω', ω'', ω''' are certain functions of which the forms are defined by the equations (138), but need not at present be considered. It follows from equations (138) that the motion of each element of the fluid within the surface S is compounded of a motion of translation, a motion of rotation, and a motion of dilatation alike in all directions. So far as regards the first two kinds of motion, the fluid element moves like a solid, and of course there is nothing to call internal friction into play. For the reasons stated in my former paper, I was led to assume that a motion of dilatation alike in all directions, (which of course can only exist in the case of an elastic fluid,) has no effect in causing the pressure to differ from the statical pressure corresponding to the actual density, that is, in occasioning a violation of the functional relation commonly supposed to exist between the pressure, density, and temperature. The reader will observe that this is a totally different thing from assuming that a motion of dilatation has no effect on the pressure at all.

When the fluid is incompressible $\delta=0$, and it may be proved without difficulty that ω', ω'', ω''' are constant, that is to say, constant so far as the co-ordinates are concerned. In this case we get by integrating equations (137)

$$\left.\begin{array}{l}u=a-\omega'''y+\omega''z,\\ v=b-\omega'z\;+\omega'''x,\\ w=c-\omega''x\;+\omega'y.\end{array}\right\}\quad\ldots\ldots\ldots\quad(139)$$

Hence, in the case of an incompressible fluid, unless the whole mass comprised within the surface S move together like a solid, there cannot fail to be a certain portion of *vis viva* lost by internal friction. In the case of an elastic fluid, the motion which may take place without causing a loss of *vis viva* in consequence of friction is somewhat more general, and corresponds to velocities $u + \Delta u$, $v + \Delta v$, $w + \Delta w$, where u, v, w are the same as in (139), and

$$\Delta u = \delta x + 2(\alpha x + \beta y + \gamma z)\, x - \alpha (x^2 + y^2 + z^2),$$

with similar expressions for Δv and Δw. In these expressions α, β, γ are three constants symmetrically related to x, y, z, and δ is a constant which has the same relation to each of the co-ordinates*.

51. By means of the expression given in Art. 49, for the loss of *vis viva* due to internal friction, we may readily obtain a very approximate solution of the problem: To determine the rate at which the motion subsides, in consequence of internal friction, in the case of a series of oscillatory waves propagated along the surface of a liquid.

Let the vertical plane of xy be parallel to the plane of motion, and let y be measured vertically downwards from the mean surface; and for simplicity's sake suppose the depth of the fluid very great compared with the length of a wave, and the motion so small that the square of the velocity may be neglected. In the case of motion which we are considering, $u dx + v dy$ is an exact differential $d\phi$ when friction is neglected, and

$$\phi = c\epsilon^{-my} \sin (mx - nt), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

where c, m, n are three constants, of which the last two are connected by a relation which it is not necessary to write down. We may continue to employ this equation as a near approximation when friction is taken into account, provided we suppose c, instead of being constant, to be a parameter which varies slowly with the time. Let V be the *vis viva* of a given portion of the fluid at the end of the time t, then

$$V = \rho c^2 m^2 \iiint \epsilon^{-2my}\, dx\, dy\, dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

But by means of the expression given in Art. 49, we get for the loss of *vis viva* during the time dt, observing that in the present case μ is constant, $w = 0$, $\delta = 0$, and $u dx + v dy = d\phi$, where ϕ is independent of z,

$$4\mu dt \iiint \left\{ \left(\frac{d^2\phi}{dx^2}\right)^2 + \left(\frac{d^2\phi}{dy^2}\right)^2 + 2\left(\frac{d^2\phi}{dx\, dy}\right)^2 \right\} dx\, dy\, dz,$$

which becomes, on substituting for ϕ its value,

$$8\mu c^2 m^4\, dt \iiint \epsilon^{-2my}\, dx\, dy\, dz.$$

But we get from (141) for the decrement of *vis viva* of the same mass arising from the variation of the parameter c

$$-2\rho m^2 c \frac{dc}{dt}\, dt \iiint \epsilon^{-2my}\, dx\, dy\, dz.$$

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* (*See Note* C *at the end.*)

Equating the two expressions for the decrement of *vis viva*, putting for m its value $2\pi\lambda^{-1}$, where λ is the length of a wave, replacing μ by $\mu'\rho$, integrating, and supposing c_0 to be the initial value of c, we get

$$c = c_0 e^{-\frac{16\pi^2\mu' t}{\lambda^2}}.$$

It will presently appear that the value of $\sqrt{\mu'}$ for water is about 0·0564, an inch and a second being the units of space and time. Suppose first that λ is two inches, and t ten seconds. Then $16\pi^2\mu' t\lambda^{-2} = 1{\cdot}256$, and $c : c_0 :: 1 : 0{\cdot}2848$, so that the height of the waves, which varies as c, is only about a quarter of what it was. Accordingly, the ripples excited on a small pool by a puff of wind rapidly subside when the exciting cause ceases to act.

Now suppose that λ is 40 fathoms or 2880 inches, and that t is 86400 seconds or a whole day. In this case $16\pi^2\mu' t\lambda^{-2}$ is equal to only 0·005232, so that by the end of an entire day, in which time waves of this length would travel 574 English miles, the height would be diminished by little more than the one two hundredth part in consequence of friction. Accordingly, the long swells of the ocean are but little allayed by friction, and at last break on some shore situated at the distance of perhaps hundreds of miles from the region where they were first excited.

52. It is worthy of remark, that in the case of a homogeneous incompressible fluid, whenever $u\,dx + v\,dy + w\,dz$ is an exact differential, not only are the ordinary equations of fluid motion satisfied[*], but the equations obtained when friction is taken into account are satisfied likewise. It is only the equations of condition which belong to the boundaries of the fluid that are violated. Hence any kind of motion which is possible according to the ordinary equations, and which is such that $u\,dx + v\,dy + w\,dz$ is an exact differential, is possible likewise when friction is taken into account, provided we suppose a certain system of normal and tangential pressures to act at the boundaries of the fluid, so as to satisfy the equations of condition. The requisite system of pressures is given by the system of equations (133). Since μ disappears from the general equations (1), it follows that p is the same function as before. But in the first case the system of pressures at the surface was $P_1 = P_2 = P_3 = p$, $T_1 = T_2 = T_3 = 0$. Hence if ΔP_1 &c. be the additional pressures arising from friction, we get from (133), observing that $\delta = 0$, and that $u\,dx + v\,dy + w\,dz$ is an exact differential $d\phi$,

$$\Delta P_1 = -2\mu\frac{d^2\phi}{dx^2}, \qquad \Delta P_2 = -2\mu\frac{d^2\phi}{dy^2}, \qquad \Delta P_3 = -2\mu\frac{d^2\phi}{dz^2}, \quad . \quad (142)$$

$$\Delta T_1 = -2\mu\frac{d^2\phi}{dy\,dz}, \qquad \Delta T_2 = -2\mu\frac{d^2\phi}{dz\,dx}, \qquad \Delta T_3 = -2\mu\frac{d^2\phi}{dx\,dy}. \quad . \quad (143)$$

Let dS be an element of the bounding surface, l', m', n' the direction-cosines of the normal drawn outwards, ΔP, ΔQ, ΔR the components in the direction of x, y, z of the additional pressure on a plane in the direction of dS. Then by the formulæ (9) of my former paper applied to the equations (142), (143) we get

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[*] It is here supposed that the forces X, Y, Z are such that $X\,dx + Y\,dy + Z\,dz$ is an exact differential.

$$\Delta P = -2\mu\left\{l'\frac{d^2\phi}{dx^2} + m'\frac{d^2\phi}{dx\,dy} + n'\frac{d^2\phi}{dx\,dz}\right\}, \quad . \quad . \quad . \quad . \qquad (144)$$

with similar expressions for ΔQ and ΔR, and ΔP, ΔQ, ΔR are the components of the pressure which must be applied at the surface, in order to preserve the original motion unaltered by friction.

53. Let us apply this method to the case of oscillatory waves, considered in Art. 51. In this case the bounding surface is nearly horizontal, and its vertical ordinates are very small, and since the squares of small quantities are neglected, we may suppose the surface to coincide with the plane of xz in calculating the system of pressures which must be supplied, in order to keep up the motion. Moreover, since the motion is symmetrical with respect to the plane of xy, there will be no tangential pressure in the direction of z, so that the only pressures we have to calculate are ΔP_2 and ΔT_3. We get from (140), (142), and (143), putting $y = 0$ after differentiation,

$$\Delta P_2 = -2\mu m^2 c \sin(mx - nt), \qquad \Delta T_3 = 2\mu m^2 c \cos(mx - nt). \quad . \quad . \quad . \qquad (145)$$

If u_1, v_1 be the velocities at the surface, we get from (140), putting $y = 0$ after differentiation,

$$u_1 = mc \cos(mn - nt), \qquad v_1 = -mc \sin(mx - nt). \quad . \quad . \quad . \quad . \qquad (146)$$

It appears from (145) and (146) that the oblique pressure which must be supplied at the surface in order to keep up the motion is constant in magnitude, and always acts in the direction in which the particles are moving.

The work of this pressure during the time dt corresponding to the element of surface $dx\,dz$, is equal to $dx\,dz\,(\Delta T_3 . u_1 dt + \Delta P_1 . v_1 dt.)$ Hence the work exerted over a given portion of the surface is equal to

$$2\mu m^3 c^2 dt \iint dx\,dz.$$

In the absence of pressures ΔP_2, ΔT_3 at the surface, this work must be supplied at the expence of *vis viva*. Hence $4\mu m^3 c^2 dt \iint dx\,dz$ is the *vis viva* lost by friction, which agrees with the expression obtained in Art. 51, as will be seen on performing in the latter the integration with respect to y, the limits being $y = 0$ to $y = \infty$.

PART II.

COMPARISON OF THEORY AND EXPERIMENT.

SECTION I.

Discussion of the Experiments of Baily, Bessel, Coulomb, and Dubuat.

54. THE experiments discussed in this Section will be taken in the order which is most convenient for discussion, which happens to be almost exactly the reverse of the chronological order. I commence with the experiments of the late Mr Baily, which are described in the *Philosophical Transactions* for 1832, in a memoir entitled "On the Correction of a Pendulum for the Reduction to a Vacuum: together with Remarks on some anomalies observed in Pendulum experiments."

The object of these experiments was, to determine by actual observation the correction to the time of vibration due to the presence of the air in the case of a great number of pendulums of various forms. This was effected by placing each pendulum in succession in a vacuum apparatus, by which means the pendulum, without being dismounted, could be swung alternately under the full atmospheric pressure, and in air so highly rarefied as nearly to approach to a vacuum. The paper, as originally presented to the Royal Society, contained the results obtained with 41 pendulums, the same body being counted as a different pendulum when swung in a different manner. Out of these, 14 are of such forms as to admit of comparison with theory. An addition to the paper contains the results obtained with 45 pendulums more, of which 24 admit of comparison with theory. The details of these additional experiments are omitted, the results only being given.

Baily has exhibited the results obtained with the several pendulums in each of two ways, first, by the value of the factor n by which the correction for buoyancy must be multiplied in order to amount to the whole effect of the air as given by observation, and, secondly, by the weight of air which must be conceived to be attached to the centre of gyration of the pendulum, adding to its inertia without adding to its weight, in order that the increased inertia, combined with the buoyancy of the air, may account for the whole effect observed. I shall uniformly write $\mathfrak{n}$ for Baily's n, in order to distinguish it from the n of Part I. of the present paper, which has a totally different meaning. In the case of a pendulum oscillating in air, it will be sufficient, unless the pendulum be composed of extremely light materials, to add together the effects of buoyancy and inertia. Hence if the pendulum consist of a sphere attached to a fine wire of which the effect is neglected, or else of a uniform cylindrical rod, we may suppose $\mathfrak{n} = 1 + k$, where k is the factor so denoted in Part I.; so that if M' be the mass of air displaced, kM' will be the mass which we must suppose collected at the centre of the sphere, or distributed uniformly along the axis of the cylinder, in order to express the effect of the inertia of the air. The second mode of exhibiting the effect of the air was suggested by Mr Airy, and is better adapted than the former for investigating the effect of the several pieces of which a pendulum of complicated form is composed. Since the value of the factor **n** and that of the weight of air are merely two different expressions for the result of the same experiment, it would be sufficient to compare either with the result calculated from theory. In some cases, however, I have computed both. In almost all the calculations I have employed 4-figure logarithms. The experimental result is sometimes exhibited to four figures, but no reliance can be placed on the last. In fact, in the best observations, the mean error in different determinations of **n** for the same pendulum appears to have been about the one-hundredth part of the whole, and that it should be so small, is a proof of the extreme care with which the experiments must have been performed.

55. I commence with the 13th set of experiments—*Results with plain cylindrical rods*—page 441. This set contains three pendulums, each consisting of a long rod attached to a knife-edge apparatus. The result obtained with each pendulum furnishes an equation for the determination of μ', and the theory is to be tested by the accordance or discordance of the values so obtained. The principal steps of the calculation are contained in the following table.

Determination of $\sqrt{\mu'}$ *by means of* Baily's *experiments with plain cylindrical rods.*

Pendulum rod.	No.	Diameter $2a$.	Time of vibration τ.	**n** by experiment.	Correction for confined space (by theory.)	Deduced value of k by experiment.	Corresponding value of **m**.	Resulting value of $\sqrt{\mu'}$.
Copper, 58·8 inches long	21	0·410	1·0136	2·932	− 0·009	1·923	1·5445	0·1166
Brass, 56·4	43	0·185	0·9933	4·083	− 0·002	3·081	0·7000	0·1175
Steel, 56·4	44	0·072	0·9933	7·530		6·530	0·2822	0·1134

In this table the first column explains itself. The next contains the reference number. In the case of the copper rod I have replaced 42 by 21, under which number the details of the experiment will be found. The diameters of the rods are expressed in decimals of an inch. The time of vibration of the pendulum No. 21 may be got from the tables at the end of Baily's memoir, which contain the details of the experiments. Nos. 43 and 44 belong to the "additional experiments," of which all the details are suppressed. Baily has not even given the times of vibration, not having been aware of the circumstance, indicated by the theory of this paper, that the factor **n** and the weight of air which must be conceived as dragged by the pendulum are functions of the time of vibration. Accordingly, in the cases of the pendulums Nos. 43 and 44, and in all similar cases, I have calculated the time of vibration by the ordinary formulæ of dynamics. In calculating τ, I have added 1·55 inch, the length of the shank of the knife-edge apparatus, to the length of the rods. The result so obtained is abundantly accurate enough for my purpose. Had the rod, retaining its actual length, been supposed to begin directly at the knife-edge, the error thence resulting in the value of τ, or rather the correspoding error in the calculated value of **n** or k, might just have been sensible. The fifth column in the above table is copied from Baily's table. The next contains a small correction necessary to reduce the value of **n** got from observation to what would have been got from observations made in an unlimited mass of fluid. It is calculated from the formula

$2a^2(b^2-a^2)^{-1}$ or $2a^2b^{-2}$ nearly, which is obtained from the ordinary equations of hydrodynamics, and therefore it cannot be regarded as more than a rude approximation. It will be useful, however, as affording an estimate of the magnitude of the effect produced by confining the air. The diameter of the vacuum tube (whether external or internal is not specified) is stated to have been six inches and a half, whence $2b = 6{\cdot}5$. The values of k given in the next column are obtained by applying the correction for confined space to Baily's values of $\mathfrak{n}$, and subtracting unity. The value of $\mathfrak{m}$ corresponding to each value of k was got by interpolation from the table near the end of Section III. of the former part of this paper. For $k = 1{\cdot}923$ the interpolation is easy. The value $3{\cdot}081$ happens to be almost exactly found in the table. For $k = 6{\cdot}530$, a remark already made will be found to be of importance, namely, that the first differences of $\mathfrak{m}^2(k-1)$ are nearly constant. The last column contains the value of $\sqrt{\mu'}$ obtained from the equation

$$\mathfrak{m} = \frac{a}{2}\sqrt{\frac{\pi}{\mu'\tau}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (147)$$

which contains the definition of $\mathfrak{m}$.

It will be observed that the three values of $\sqrt{\mu'}$ are nearly identical. Of course any theory professing to account for a set of experiments by means of a particular value of a disposable constant, when applied to the experiments would lead to nearly the same numerical value of the constant if the experiments were made under nearly the same circumstances. But in the present case the circumstances of the experiments are widely different. The diameter of the steel rod is little more than the sixth part of that of the copper rod, and the value of k obtained by experiment for the steel rod is more than three times as great as that obtained for the copper rod. It is a simple consequence of the ordinary theory of hydrodynamics that in the case of a long rod oscillating in an unlimited fluid $k = 1$, and we see that this value of k must be multiplied, in round numbers, by 2, by 3, and by $6\frac{1}{2}$, in order to account for the observed effect. The value $1{\cdot}5445$ of $\mathfrak{m}$ is so large that the descending series comes into play in the calculation of the function k, while $0{\cdot}2822$ is so small that the ascending series are rapidly convergent. Hence the near agreement of the values of $\sqrt{\mu'}$ deduced from the three experiments is a striking confirmation of the theory. The mean of the three is $0{\cdot}1158$, but of course the last figure cannot be trusted. I shall accordingly assume as the value of the square root of the index of friction of air in its average state of pressure, temperature, and moisture

$$\sqrt{\mu'} = 0{\cdot}116.$$

It is to be remembered that $\sqrt{\mu'}$ expresses a length divided by the square root of a time, and that the numerical value above given is adapted to an English inch as the unit of length, and a second of mean solar time as the unit of time.

56. I now proceed to compare the observed values of $\mathfrak{n}$ with those calculated from theory with the assumed value of $\sqrt{\mu}$. I begin with the same cylindrical rods as before, together with the long brass tubes Nos. 35 to 38. The diameter of this tube was 1·5 inch, and its length 56 inches. The ends were open, but as the included air was treated by Mr Baily in the reduction of his experiments as if it formed part of the pendulum, we may regard the pendulum as a solid rod. The tube was furnished with six agate planes, represented in the wood-cut at page 417, which rested on fixed knife-edges. The pendulums Nos. 35, 36, 37, and 38 consisted of the same tube swung on the planes marked A, C, a, c. In air the pendulum swung at the rate of about 90080 vibrations in a day, so that $\tau = 0{\cdot}9596$ nearly. The values of $\mathfrak{n}$ obtained with the end planes A, c were slightly though sensibly greater than the values obtained with the mean planes C, a. I shall suppose the mean of the four values of $\mathfrak{n}$, namely 2·290, to be the result of the experiments. In the following table the difference between the theoretical and experimental values of $\mathfrak{n}$ is exhibited both by decimals and as a fractional part of the former of these values.

Baily's results with a long brass tube and with long cylindrical rods.

No.	$2a$	$\mathfrak{m}$	k	Add for confined space.	Total $\mathfrak{n}$ by theory.	$\mathfrak{n}$ by experiment.	Difference.
35 to 38	1·5	5·849	1·242	0·122	2·364	2·290	$-$ 0·074, or $-\frac{1}{31}$
21 or 42	0·410	1·555	1·917	0·009	2·926	2·932	$+$ 0·006, or $+\frac{1}{489}$
43	0·185	0·7089	3·055	0·002	4·057	4·083	$+$ 0·026, or $+\frac{1}{157}$
44	0·072	0·2759	6·670		7·670	7·530	$-$ 0·140, or $-\frac{1}{54}$

It will be seen at once how closely the experiments are represented by theory. The largest proportionate difference occurs in the case of the brass tube, and even that is less than one thirtieth. A glance at Baily's wood-cut at page 417 will shew that the six planes with which the tube was furnished caused the whole figure to deviate sensibly from the cylindrical form. Moreover the resistance experienced by each element of the cylinder has been calculated by supposing the element in question to belong to an infinite cylinder oscillating with the same linear velocity, and the resistance thus determined must be a little too great in the immediate neighbourhood of the ends of the cylinder, where the free motion of the air is less impeded than it would be if the cylinder were prolonged. Lastly, the correction for confined space is calculated according to the ordinary equations of hydrodynamics, and on that account, as well as on account of the abrupt termination of the cylinder, will be only approximate. The small discrepancy between theory and observation, as well as the small difference (amounting to about the 1-83rd of the whole) detected by experiment between the results obtained with the extreme planes and those obtained with the mean planes, may reasonably be attributed to some such causes as those just mentioned. In the case of the steel rod or wire, the difference between theory and observation may be altogether removed by supposing a very small error to have existed in the measurement of the diameter of the rod. Since, as we have seen, the observation is satisfied by $\mathfrak{m} = 0{\cdot}2822$, and (147) gives $a \propto \mathfrak{m}$ when μ' and τ are constant, it is sufficient, in order to satisfy the experiment, to increase the diameter of the rod in the ratio of 0·2759 to 0·2822, or to suppose an error of only 0.0017 inch in defect to have existed in the measurement of the diameter.

57. I proceed next to the experiments on spheres attached to fine wires. The pendulums of this construction comprise four $1\frac{1}{2}$-inch spheres, Nos. 1, 2, 3, and 4; three 2-inch

spheres, Nos. 5, 6, and 7; and one 3-inch sphere, No. 66. Nos. 8 and 9 are the same spheres as Nos. 5 and 7 respectively, swung by suspending the wire over a cylinder instead of attaching it to a knife-edge apparatus. As this mode of suspension was not found very satisfactory, and the results are marked by Baily as doubtful cases, I shall omit the pendulums Nos. 8 and 9, more especially as with reference to the present inquiry they are merely repetitions of Nos. 5 and 7.

In the case of a sphere attached to a fine wire of which the effect is neglected, and swung in an unconfined mass of fluid, we have by the formulæ (52)

$$k = \frac{1}{2} + \frac{9}{2a}\sqrt{\frac{\mu'\tau}{2\pi}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

$2a$ being in this case the diameter of the sphere. Before employing this formula in the comparison of theory and experiment, it will be requisite to consider two corrections, one for the effect of the wire, the other for the effect of the confinement of the air by the sides of the vacuum tube.

I have already remarked at the end of Section IV., Part I., that the application of the formulæ of Section III. to the case of such fine wires as those used in pendulum experiments is not quite safe. Be that as it may, these formulæ will at any rate afford us a good estimate of the probable magnitude of the correction.

Let l be the length, a_1 the radius, V_1 the volume of the wire, V the volume of the sphere, I the moment of inertia of the pendulum, I' that of the air which we may conceive dragged by it, H the sum of the elements of the mass of the pendulum multiplied by their respective vertical distances below the axis of suspension, H' the same for the air displaced, σ the density of the air. Then the length of the isochronous simple pendulum is IH^{-1} in vacuum, and $(I + I')(H - H')^{-1}$ in air, and the time of vibration is increased by the air in the ratio of $I^{\frac{1}{2}}H^{-\frac{1}{2}}$ to $(I + I')^{\frac{1}{2}}(H - H')^{-\frac{1}{2}}$, or, on account of the smallness of σ, in the ratio of 1 to $1 + \frac{1}{2}(I'I^{-1} + H'H^{-1})$ nearly. Now $\frac{1}{2}H'H^{-1}$ is the correction for buoyancy, and therefore

$$\mathfrak{n} - 1 = \frac{I'}{H'} \cdot \frac{H}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149)$$

We have also, if k_1 be the value of the function k of Section III., Part I.,

$$I' = k\sigma V(l + a)^2 + \tfrac{1}{3}k_1\sigma V_1 l^2, \qquad H' = \sigma V(l + a) + \tfrac{1}{2}\sigma V_1 l, \quad . \quad . \quad (150)$$

and $HI^{-1} = (l + a)^{-1}$ very nearly. Substituting in (149), expanding the denominator, and neglecting V_1^2, we get

$$\mathfrak{n} - 1 = k + \frac{1}{3}\frac{V_1}{V}k_1\left(\frac{l}{l + a}\right)^2 - \frac{1}{2}\frac{V_1}{V}k\frac{l}{l + a}.$$

Now V_1 is very small compared with V, and it is only by being multiplied by the large factor k_1 that it becomes important. We may then, without any material error, replace the last term in the above equation by $\frac{1}{3}V_1V^{-1}l^2(l + a)^{-2}$, and if λ be the length of the isochronous simple pendulum, we may suppose $l + a = \lambda$, and replace $l^2 (l + a)^{-2}$ by $1 - 2a\lambda^{-1}$, since a is small compared with λ. We thus get, putting $\Delta \mathfrak{n}$ for the correction due to the wire,

$$\Delta \mathfrak{n} = \tfrac{1}{3} \frac{V_1}{V} \left(1 - \frac{2a}{\lambda}\right)(k_1 - 1).$$

Substituting for $k_1 - 1$ from (115), and for $\mathfrak{m}$ from (147), in which equations, however, k_1, a_1 must be supposed to be written for k, a, expressing V_1, V in terms of the diameters of the wire and sphere, and neglecting as before a^2 in comparison with λ^2, we get

$$\Delta \mathfrak{n} = \frac{(2\lambda - 3 \times 2a)\, \mu' \tau}{\left\{L^2 + \left(\frac{\pi}{4}\right)^2\right\} (2a)^2}, \quad \ldots \ldots \ldots \quad (151)$$

where

$$-L = \log_e \frac{4}{2a_1} \sqrt{\frac{\mu' \tau}{\pi}} - 0{\cdot}5772. \quad \ldots \ldots \ldots \quad (152)$$

It is by these formulæ that I have computed the correction for the wire in the following table. In the experiments, the time of oscillation was so nearly one second that it is sufficient in the formulæ (148), (151), and (152) to put $\tau = 1$, and take λ for the length of the seconds pendulum, or 39·14 inches.

With respect to the correction for confined space, it seems evident that the vacuum tube must have impeded the free motion of the air, and consequently increased the resistance experienced by the pendulum when it was swung in air, and that the increase of resistance caused by the cylindrical tube must have been somewhat less than that which would have been produced by a spherical envelope of the same radius surrounding the sphere. The effect of a spherical envelope has been investigated in Section II., Part I.; but as we are obliged at last to have recourse to estimation, it is needless to be very precise in calculating the increase of resistance due to such an envelope, and we may accordingly employ the expression obtained from the ordinary theory of hydrodynamics. According to this theory, the increase of the factor k, which is due to the envelope, is equal to $\frac{3}{2} a^3 (b^3 - a^3)^{-1}$, or $\frac{3}{2} a^3 b^{-3}$ nearly, when b is large compared with a. The increase due to a cylindrical envelope whose axis is vertical, and consequently perpendicular to the direction of oscillation of the sphere, may be estimated at about two-thirds of the increase due to a spherical envelope of the same diameter. I have accordingly taken $+ a^3 b^{-3}$ for the correction for confined space, and have supposed $2b = 6{\cdot}5$ inches.

The diameter of the wire employed in the pendulums Nos. 1, 2, 3, 5, 6, and 7, is stated to have been about the $\frac{1}{70}$th of an inch, and that of the wire employed with the heavy brass sphere No. 66, about 0·023 inch. The ivory sphere No. 4 was swung with a fine wire weighing rather more than half a grain. Taking the weight at half a grain, and the specific gravity of silver at 10·5, we have for this wire $2a_1 = 0{\cdot}00251$ nearly. The diameters of the three brass spheres in the following table are taken from page 447 of Baily's memoir. The several parts of which, according to theory, $\mathfrak{n}$ is composed, are exhibited separately.

Baily's results with spheres suspended by fine wires.

No. and kind.	Diameter of sphere. $2a$	Diameter of wire. $2a_1$	n By theory.						n By experiment.	Difference.
			For buoyancy.	For inertia on common theory.	Additional for inertia on account of internal friction.	Correction for wire.	Correction for confined space (estimated.)	Total.		
$1\frac{1}{2}$-INCH SPHERES.										
No. 1, Platina.	1·44	0·01429	1	0·5	0·289	0·035	0·011	1·835	1·881	+ 0·046, or $+\frac{1}{40}$
No. 2, Lead.	1·46	0·01429	1	0·5	0·285	0·035	0·011	1·831	1·871	+ 0·040, or $+\frac{1}{46}$
No. 3, Brass.	1·465	0·01429	1	0·5	0·284	0·035	0·011	1·830	1·834	+ 0·004, or $+\frac{1}{457}$
No. 4, Ivory.	1·46	0·00251	1	0·5	0·285	0·016	0·011	1·812	1·872	+ 0·060, or $+\frac{1}{30}$
2-INCH SPHERES.										
No. 5, Lead.	2·06	0·01429	1	0·5	0·202	0·012	0·032	1·746	1·738	− 0·008, or $-\frac{1}{218}$
No. 6, Brass.	2·065	0·01429	1	0·5	0·202	0·012	0·032	1·746	1·751	+ 0·005, or $+\frac{1}{349}$
No. 7, Ivory.	2·06	0·01429	1	0·5	0·202	0·012	0·032	1·746	1·755	+ 0·009, or $+\frac{1}{194}$
3-INCH SPHERE.										
No. 66, Brass.	3·030	0·023	1	0·5	0·137	0·005	0·101	1·743	1·748	+ 0·005, or $+\frac{1}{349}$

The mean error in different determinations of n for the same sphere was about 0·01 or 0·02, and this does not include errors arising from small errors in specific gravities, &c. Hence, if we except the spheres Nos. 1, 2, and 4, the discrepancies between theory and experiment are altogether insignificant. In considering the confirmation thence arising to the theory, it must be borne in mind that the theory did not furnish a single disposable constant, inasmuch as $\sqrt{\mu'}$ was already determined from the experiments with cylindrical rods. The result obtained with the brass sphere No. 3 happens to agree almost exactly with theory. However, as the results obtained with this sphere exhibited some anomalies, it seems best to exclude it from consideration. The value of n, then, which belongs to a $1\frac{1}{2}$ inch sphere, appears to exceed by a minute quantity the value deduced from theory. The difference is indeed so small that it might well be attributed to errors of observation, were it not that all the spheres tell the same tale. Thus the error + 0·046 in the case of the platina sphere corresponds to an error of less than the fortieth part of a second in the observation of an interval of time amounting to $4\frac{1}{2}$ hours. If the apparent defect, amounting to about 0·04 or 0·05, in the theoretical result be real, it may be attributed with probability to an error in the correction for the wire. This would be no objection to the theory, for it will be remembered that the theory itself indicated the probable failure of the formulæ generally applicable to a long cylinder when the cylinder comes to be of such extreme fineness as the wires employed in pendulum experiments.

58. The preceding experiments of Baily's are the most important for the purposes of the present paper, inasmuch as they were performed on pendulums of simple and very different forms; but there still remain three sets of experiments, the fourteenth, fifteenth, and sixteenth, in which the pendulum consisted of a combination of a sphere and a rod, so that the results can be compared with theory. The details of these experiments being suppressed, I

have been obliged to calculate the time of oscillation from the ordinary formulæ of dynamics, but the results will no doubt be accurate enough for the purpose required. In all the calculations I have supposed the rod to reach up to the axis of suspension, and have consequently added 1·55 inch (the length of the shank of the knife-edge apparatus) to the length of the rod, and have added to the weight of the rod a quantity bearing to the whole weight the ratio of 1·55 inch to the whole length.

In the case of the spheres attached to the ends of the rods (sets 14 and 16) the process of calculation is as follows. Let l be the length of the rod increased by 1·55 inch, W_1 its weight, increased as above explained, a the radius and W the weight of the sphere, λ the length of the isochronous simple pendulum. Then supposing the masses of the rod and sphere to be respectively distributed along the axis, and collected at the centre, which will be quite accurate enough for the present purpose, and putting α for the ratio of a to l, we have by the ordinary formula

$$\lambda = \frac{\frac{1}{3} W_1 + (1+\alpha)^2 W}{\frac{1}{2} W_1 + (1+\alpha) W} l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

whence τ, the time of vibration, is known. The formula (148) then gives k, which applies to the sphere, and (147) gives $\mathfrak{m}$, the a in this formula being the radius of the rod, from whence k_1, which applies to the rod, may be got by interpolation from the table in Part I. Let Δk, Δk_1 be the corrections which must be applied to k, k_1 on account of the confined space of the vacuum apparatus, and let S_1, S be the specific gravities of the rod and sphere respectively; then we get by means of the formulæ (149), (150)

$$\mathfrak{n} - 1 = \frac{\frac{1}{3}(k_1 + \Delta k_1)\dfrac{W_1}{S_1} + (1+\alpha)^2 (k + \Delta k)\dfrac{W}{S}}{\frac{1}{3} W_1 + (1+\alpha)^2 W} \times \frac{\frac{1}{2} W_1 + (1+\alpha) W}{\frac{1}{2}\dfrac{W_1}{S_1} + (1+\alpha)\dfrac{W}{S}}. \quad . \quad (154)$$

The first of the two factors connected by the sign × in this equation is equal to $\sigma^{-1} I' I^{-1}$, and if we want to calculate the weight of air which we must conceive attached to the centre of gyration of the pendulum in order to allow for the inertia of the air, we have only to multiply the factor just mentioned by σ and by the weight of the whole pendulum. The following table contains the comparison of theory and experiment in the case of the 14th set. The rods here mentioned are the same as those which composed the pendulums Nos. 21, 43, and 44, and the spheres are the three brass spheres of Nos. 3, 5, and 66. It appears from p. 432 of Baily's paper that his results are all reduced to a standard pressure and temperature, on the supposition that the effect of the air on the time of vibration is proportional to its density. The theory of the present paper shews that this will only be the case if μ' be constant, which however there is reason for supposing it to be when the pressure alone varies. Be that as it may, no material error can be produced by reducing the observations in this way, because the difference of density in any pair of experiments did not much differ from the density of air at the standard pressure and temperature. The standard pressure and temperature taken were 29·9218 inches of mercury and 32° F, and the assumed specific gravity of air at this pressure and temperature was the 1-770th of that of water, so that in the calculations from theory it is to be supposed that $\sigma^{-1} = 770$.

If w be the weight of the whole pendulum, w' that of the air which we must suppose attached to the pendulum at its centre of gyration in order to express the effect of the inertia of the air, S the *vibrating* specific gravity of the pendulum, the effects of buoyancy and inertia are as σS^{-1} to $w'w^{-1}$; but they are also as 1 to $\mathfrak{n}-1$, according to the definition of the factor $\mathfrak{n}$, and therefore

$$w' = (\mathfrak{n} - 1)\frac{\sigma}{S}w, \quad \dots\dots\dots \quad (155)$$

a formula which may be employed to calculate w' when $\mathfrak{n}$ is known.

Baily's results with spheres at the ends of long rods.

No. and nature of pendulum.	Value of $\mathfrak{n}$.			Weight of adhesive air, in grains.		
	By theory.	By experiment.	Difference.	By theory.	By experiment.	Difference.
No. 45 – $1\frac{1}{2}$-inch sphere with copper rod.	2·525	2·458	– 0·067, or – $\frac{1}{38}$	4·863	4·564	– 0·299, or – $\frac{1}{16}$
No. 46 – 2-inch sphere with ditto.	2·202	2·234	+ 0·032, or + $\frac{1}{69}$	5·005	5·076	+ 0·071, or + $\frac{1}{70}$
No. 47 – 3-inch sphere with ditto.	1·957	1·873	– 0·084, or – $\frac{1}{23}$	7·071	6·425	– 0·646, or – $\frac{1}{11}$
No. 48 – $1\frac{1}{2}$-inch sphere with brass rod.	2·375	2·356	– 0·019, or – $\frac{1}{125}$	1·447	1·417	– 0·030, or – $\frac{1}{48}$
No. 49 – 2-inch sphere with ditto.	2·060	1·982	– 0·078, or – $\frac{1}{26}$	2·135	1·973	– 0·162, or – $\frac{1}{13}$
No. 50 – 3-inch sphere with ditto.	1·631	1·933?	+ 0·302 ?	4·411	4·868 ?	+ 0·457 ?
No. 51 – $1\frac{1}{2}$-inch sphere with steel rod.	2·099	2·344?	+ 0·245 ?	0·682	0·834 ?	+ 0·152 ?
No. 52 – 2-inch sphere with ditto.	1·920	1·793	– 0·127, or – $\frac{1}{15}$	1·457	1·259	– 0·198, or – $\frac{1}{7}$
No. 53 – 3-inch sphere with ditto.	1·781	1·759	– 0·022, or – $\frac{1}{81}$	3·742	3·670	– 0·072, or – $\frac{1}{52}$

With respect to the two experiments marked ? Baily remarks, "These two experiments (with the pendulums Nos. 50 and 51) are very unsatisfactory; and are marked as such in my journal. It was consequently my intention to have repeated them: but the subject was overlooked till it was too late. I should propose their being rejected altogether." If these two experiments be struck out, it will be seen that the differences between theory and experiment are very small, especially when the difficulty of this set of experiments is considered, arising from the frequency of the coincidences with the mean solar clock.

59. On account of the difficulty which Baily experienced in obtaining accurate results with the long rods and spheres attached, he divided the brass and steel rods near the centre of oscillation, and after having cut off an inch from each portion inserted the spheres where the rods had been divided. The results thus obtained constitute the 15th set of experiments. He afterwards removed the lower segments of the rods, and obtained the results contained in the 16th set. I shall give the computation of the latter set first, inasmuch as the formulæ to be employed are exactly the same as those required for the 14th set. The experiments belonging to this set in which the spheres were swung with iron wires have already been computed under the head of spheres attached to fine wires.

Baily's results with the spheres at the end of the short rods.

No. and nature of pendulum.	Value of n.			Weight of adhesive air.		
	By theory.	By experiment.	Difference.	By theory.	By experiment.	Difference.
No. 60 – 1½-inch sphere with brass rod.	2·149	2·198	+ 0·049, or + $\frac{1}{44}$	1·011	1·047	+ 0·036, or + $\frac{1}{28}$
No. 61 – 2-inch sphere with ditto. ……	1·879	1·901	+ 0·022, or + $\frac{1}{85}$	1·619	1·513	– 0·106, or – $\frac{1}{15}$
No. 62 – 3-inch sphere with ditto. ……	1·787	1·830	+ 0·043, or + $\frac{1}{42}$	3·970	4·202	+ 0·232, or + $\frac{1}{17}$
No. 63 – 1½-inch sphere with steel rod.	1·960	1·904	– 0·056, or – $\frac{1}{35}$	0·570	0·537	– 0·033, or – $\frac{1}{16}$
No. 64 – 2-inch sphere with ditto. ……	1·796	1·785	– 0·011, or – $\frac{1}{163}$	1·239	1·227	– 0·012, or – $\frac{1}{103}$
No. 65 – 3-inch sphere with ditto. ……	1·758	1·779	+ 0·021, or + $\frac{1}{84}$	3·609	3·720	+ 0·111, or + $\frac{1}{32}$

Here again the differences between theory and experiment are extremely small. In the case of the pendulum No. 61, Baily's two results 1·901 and 1·513 appear to be inconsistent, as not agreeing with the formula (155).

60. The following table contains the values of τ, k, and k_1 deduced from the given data, and employed in the calculations of which the results are contained in the two preceding tables. It is added, partly to facilitate a comparison of the circumstances of the different experiments, partly to assist in the re-computation of any of the experiments, or the detection of any numerical error which I may have committed. I may here observe that I have not, generally speaking, re-examined the calculations, except where an error was apparent, but that each step requiring addition, subtraction, multiplication, or division, was checked immediately after it was performed. I have not thought it requisite to check in this manner the taking of logarithms or antilogarithms out of a table.

Values of τ, k, and k_1 employed in the calculation of the theoretical results employed in the two preceding tables.

Sphere.	Rod.	Long rods.				Short rods.			
		No.	τ	k	k_1	No.	τ	k	k_1
1½-inch	copper	45	1·090	0·7968	1·951				
2-inch	copper	46	1·158	0·7170	1·981				
3-inch	copper	47	1·227	0·6523	2·010				
1½-inch	brass	48	1·155	0·8055	3·222	60	0·9517	0·7772	3·012
2-inch	brass	49	1·198	0·7207	3·264	61	0·9806	0·7005	3·042
3-inch	brass	50	1·222	0·6520	3·288	62	0·9982	0·6373	3·062
1½-inch	steel	51	1·190	0·8099	7·272	63	0·9868	0·7824	6·649
2-inch	steel	52	1·199	0·7208	7·299	64	0·9954	0·7021	6·679
3-inch	steel	53	1·231	0·6525	7·396	65	1·0030	0·6377	6·714

The corrections for confined space employed are, for the spheres, (Δk), 0·0115, 0·0321, 0·1013; and for the rods, (Δk_1), 0·009, 0·002, 0·000. These corrections are to be added to the values of k, k_1 given in the preceding table before going on with the calculation.

61. In the 14th set of experiments, the weight of adhesive air due to the spheres alone has been computed by Baily by subtracting from the whole weight, as given by observation, the weight due to the rods as given by the 13th set of experiments, taking account of the change of weight corresponding to the change in the position of the centre of gyration, the point at which the air is supposed to be attached. According to theory, this process is not legitimate, inasmuch as the weight dragged by a rod in a function of the time of vibration, which is altered when a sphere is attached to the end of the rod. But in the 15th set of experiments the spheres did not materially affect the time of vibration, inasmuch as they were inserted nearly at the centre of oscillation of the rods, and therefore in this case the process is legitimate. Accordingly, I think it is a sufficient comparison between theory and experiment in the case of the 15th set, to compare the weights of air due to the spheres alone, as calculated by Baily, with the weights calculated according to the theory of this paper with the assumed value of $\sqrt{\mu'}$. I have exhibited separately the weight corresponding to the correction for confined space, in order to enable the reader to form an estimate of the extent to which the results may be affected by the uncertainty relating to the amount of this correction.

Weights of air dragged by the spheres alone, as deduced from Baily's results with the spheres at the centre of oscillation of the long rods.

By Theory.				By Experiment.			
	1½-inch sphere.	2-inch sphere.	3-inch sphere.		1½-inch sphere.	2-inch sphere.	3-inch sphere.
In free air	0·431	1·060	3·002	From exper^{ts} with brass rod	0·446	1·180	3·382
Additional for confined space	0·006	0·048	0·476	From exper^{ts} with steel rod	0·405	1·039	3·371
Total	0·437	1·108	3·478	Mean	0·425	1·109	3·377
Diff^{ce}. th. & exp., as decimal	−0·012	+0·001	−0·101	Diff^{ce}, as fraction of the whole	$-\frac{1}{36}$	$+\frac{1}{1108}$	$-\frac{1}{34}$

62. I pass now to Bessel's experiments described in his memoir entitled *Untersuchungen über die Länge des einfachen Sekundenpendels*, which is printed among the memoirs of the Academy of Sciences of Berlin for the year 1826. The object of this memoir was to determine the length of the seconds' pendulum by a new method, which consisted in swinging the same sphere with wires of two different lengths, the difference of lengths being measured with extreme precision. In the calculation, the absolute length of the simple pendulum isochronous with either the long or the short compound pendulum was regarded as unknown, but the difference of the two as known, and this difference, combined with the observed times of oscillation, is sufficient for the determination of the quantity sought. Nothing more would have been required if the pendulums had been swung in a vacuum; but inasmuch as they were swung in air, a further correction was necessary to reduce the observations to a vacuum. Since it is necessary to take into account the inertia of the air, as well as its buoyancy, in reducing the observations to a vacuum, Bessel sought to determine by experiment the value of

the factor k, of which the meaning has been already explained. The value of this factor, as Bessel remarked, will depend upon the form of the body; but he does not seem, at least in his first memoir, to have contemplated the possibility of its depending on the time of oscillation, and consequently he supposed it to have the same value for the long as for the short pendulum. When the factor k is introduced, the equation obtained from the known difference of length of the two simple pendulums contains two unknown quantities, namely k, and the length of the seconds' pendulum. To obtain a second equation, Bessel made another set of experiments, in which the brass sphere was replaced by an ivory sphere, having as nearly as possible the same diameter. The results obtained with the ivory sphere furnished a second equation, in which k appeared with a much larger coefficient, on account of the lightness of ivory compared with brass. The two equations determined the two unknown quantities.

Let λ be the length of the seconds' pendulum, t_1, t_2 the times of oscillation of the brass sphere when swung with the short wire and long wire respectively, l_1, l_2 the lengths of the corresponding simple pendulums, corrected for everything except the inertia of the air, m the mass of the sphere, m_1 the mass of the fluid displaced; then

$$\lambda t_1^2 \left(1 + \frac{m_1}{m} k\right)^{-1} = l_1;$$

or, since m_1 is so small that we may neglect m_1^2,

$$\lambda t_1^2 \left(1 - \frac{m_1}{m} k\right) = l_1.$$

The long pendulum furnishes a similar equation, and the result obtained from the brass sphere is

$$\lambda (t_2^2 - t_1^2)\left(1 - \frac{m_1}{m} k\right) = l_2 - l_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

since $l_2 - l_1$ is the quantity which is regarded as accurately known. The ivory sphere in like manner furnishes the equation

$$\lambda (t'^2_2 - t'^2_1)\left(1 - \frac{m_1}{m'} k\right) = l'_2 - l'_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

where the accented letters refer to that sphere. The equation for the determination of k results from the elimination of λ between the equations (156) and (157).

Now, according to the theory of this paper, the factor k has really different values for the long and short pendulums. Let k_1 refer to the short, and k_2 to the long pendulum with the brass sphere, k_1' to the short, and k_2' to the long pendulum with the ivory sphere. Then

$$\lambda t_1^2 \left(1 - \frac{m_1}{m} k_1\right) = l_1, \quad \lambda t_2^2 \left(1 - \frac{m_1}{m} k_2\right) = l_2,$$

and therefore

$$l_2 - l_1 = \lambda t_2^2 \left(1 - \frac{m_1}{m} k_2\right) - \lambda t_1^2 \left(1 - \frac{m_1}{m} k_1\right). \quad . \quad . \quad . \quad . \quad . \quad (158)$$

In the equation resulting from the elimination of λ between (156) and (157), let the values of $l_2 - l_1$ and $l_2' - l_1'$ got from (158) and the similar equation relating to the ivory sphere be substituted. The result is

$$(t_2^2 - t_1^2)\left(1 - \frac{m_1}{m} k\right) \left\{t_2'^2 \left(1 - \frac{m_1}{m'} k'_2\right) - t_1'^2 \left(1 - \frac{m_1}{m'} k'_1\right)\right\}$$

$$= (t_2'^2 - t_1'^2)\left(1 - \frac{m_1}{m'} k\right) \left\{t_2^2 \left(1 - \frac{m_1}{m} k_2\right) - t_1^2 \left(1 - \frac{m_1}{m} k_1\right)\right\}.$$

This equation is of the form

$$P + Qm_1 + Rm_1^2 = P' + Q'm_1 + R'm_1^2,$$

and $P = P'$, and Rm_1^2, $R'm_1^2$ may be neglected, so that the equation is reduced to $Q = Q'$. It is now no longer necessary to distinguish between t_2 and t_2', and between t_1 and t_1', which may be supposed equal. Also $m : m' :: S : S'$, where S, S' are the specific gravities of the brass and ivory spheres respectively. Substituting in the equation $Q = Q'$, and solving with respect to k, we get

$$k = \frac{t_2^2 (Sk'_2 - S'k_2) - t_1^2 (Sk'_1 - S'k_1)}{(t_2^2 - t_1^2)(S - S')} \quad . \quad . \quad . \quad . \quad . \quad (159)$$

This equation contains the algebraical definition of that function k of which the numerical value is determined by combining, in Bessel's manner, the results obtained with the four pendulums. Since the equation is linear so far as regards k, k_1, &c., we may consider separately the different parts of which these quantities are composed, and add the results. For the part which relates to the spheres, regarded as suspended by infinitely fine wires, we have $k'_2 = k_2$ and $k'_1 = k_1$, since the radii of the two spheres were equal, or at least so nearly equal that the difference is insensible in the present enquiry. We get then from (159)

$$k = \frac{t_2^2 k_2 - t_1^2 k_1}{t_2^2 - t_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

which gives

$$\frac{k - k_1}{t_2^2} = \frac{k - k_2}{t_1^2} = \frac{k_2 - k_1}{t_2^2 - t_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

Since $t_2 > t_1$ and $k_2 > k_1$, the equations (161) shew that the value of k determined by Bessel's method is greater than the factor which relates to the short pendulum, which was a seconds' pendulum nearly, and even greater than that which relates to the long pendulum, as has been already remarked in Art. 6.

If k_s be the factor relating to either sphere oscillating once in a second, and if the effect of the confinement of the air be neglected, we have from the formula (148)

$$k_1 - \tfrac{1}{2} : k_2 - \tfrac{1}{2} : k_s - \tfrac{1}{2} :: t_1^{\frac{1}{2}} : t_2^{\frac{1}{2}} : 1,$$

and in Bessel's experiments $t_1 = 1{\cdot}001$, $t_2 = 1{\cdot}721$, $2a = 2{\cdot}143$ in English inches. We thus get from either of the equations (160) or (161), on substituting $0{\cdot}116$ for $\sqrt{\mu'}$, $k = 0{\cdot}786$. The value of the factor k_s, which relates to a sphere of the same size, swung as a seconds' pendulum, is only $0{\cdot}694$, and k_1 may be regarded as equal to k_s. The formula (148) gives $k_2 = 0{\cdot}755$.

63. We have next to investigate the correction for the wire. The effect of the inertia of the air set in motion by the wire was altogether neglected by Bessel, and indeed it would have been quite insensible had the parts of the correction for inertia due to the wire and to the sphere, respectively, been to each other in nearly the same ratio as the parts of the correction for buoyancy. Baily, however, was led to conclude from his experiments that the effect of the wire was probably not altogether insignificant, and the theory of this paper leads, as we have seen, to the result that the factor $\mathfrak{n}$ is very large in the case of a very fine wire.

The ivory sphere in Bessel's experiments was swung with a finer wire than the brass sphere. It was for this reason that I did not from the first suppose $k'_1 = k_1$ and $k'_2 = k_2$. Let Δk, Δk_1 &c. be the corrections due to the wire. The values of Δk_1, Δk_2, $\Delta k'_1$, $\Delta k'_2$, may be got from the formula (151), in which it is to be remembered that λ denotes the length of the isochronous simple pendulum, not, as in Bessel's notation, the length of the seconds' pendulum. It is stated by Bessel (p. 131), that the wire used with the brass sphere weighed 10·95 Prussian grains in the case of the long pendulum, and 3·58 grains in the case of the short. This gives 7·37 grains for the weight of one toise or 72 French inches. The weight of one toise of the wire employed with the ivory sphere was 6·28 − 2·04 or 4·24 grains (p. 141). The specific gravity of the wire was 7·6 (p. 40), and the weight of a cubic line (French) of water is about 0·1885 grain. From these data it results that the radii of the wires were 0·003867 and 0·002933 inch English. The formula (147) gives $\mathfrak{m}$, whence L is known from (152). The lengths of the isochronous simple pendulums were about 39·20 inches for the short pendulum, and 116·94 for the long. On substituting the numerical values we get from (151), since $k_1 = \mathfrak{n}_1 - 1$ and $k_2 = \mathfrak{n}_2 - 1$,

$$\Delta k_1 = 0{\cdot}0107, \quad \Delta k_2 = 0{\cdot}0286, \quad \Delta k'_1 = 0{\cdot}0090, \quad \Delta k'_2 = 0{\cdot}0244.$$

The specific gravities of the two spheres were about 8·190 and 1·794, whence we get from (159) $\Delta k = 0{\cdot}0308$, or 0·031 nearly.

The value of k deduced by Bessel from his experiments was 0·9459 or 0·946 nearly, which in a subsequent paper he increased to 0·956. In this paper he contemplates the possibility of its being different in the cases of the long and of the short pendulum, and remarks with justice that no sensible error would thence result in the length of the seconds' pendulum, as determined by his method, but that the factor k would belong to the system of the two pendulums.

The following is the result of the comparison of theory and experiment in the case of Bessel's experiments on the oscillations of spheres in air.

Value of k belonging to the system of a long and a short pendulum, as determined experimentally by Bessel	0·956
Value deduced from theory, including the correction for the wire, but not the correction for confined space............................	0·817
difference	+ 0·139

I cannot find that Bessel has stated exactly the distance of the centre of the sphere from the back of the frame within which it was swung, but if we may judge by the sketch of the whole apparatus which is given in Plate I., and by a comparison of figs. 2 and 3, Plate II., it must have been very small, that is to say, a small fraction of the radius of the sphere[*]. If so, although the exact calculation of the correction for confined space would form a problem of extreme difficulty, it may be shewn from theoretical considerations that the correction would be by no means insensible, so that it might wholly or in part account for the difference + 0·139 between the results of theory and observation. It is, however, not improbable, for a reason which has been already mentioned, that the theoretical correction for the wire is not quite exact.

64. The experiments performed by Bessel on a sphere vibrating in water will be more conveniently considered after the discussion of some experiments of Coulomb's, to which I now proceed. These experiments are contained in a memoir entitled *Expériences destinées à déterminer la cohérence des fluides et les lois de leur résistance dans les mouvements très-lents*, which will be found in the 3rd Volume of the *Mémoires de l'Institut*, p. 246. The experiments which I shall first consider are those which relate to the oscillations of disks suspended in water with their planes horizontal. In these experiments the disk operated upon was attached to the lower extremity of a vertical cylinder of copper, not quite half an inch in diameter, the axis of which passed through the centre of the disk. The cylinder was suspended by a fine wire attached to its upper extremity. The under portion of the cylinder, together with the attached disk, were immersed in water, the disk at the bottom of the cylinder being immersed to the depth of 4 or 5 centimetres below the surface. The upper portion carried a horizontal metallic graduated disk, by means of which the arc of oscillation could be read off, and which, on account of its size and weight, mainly determined the inertia of the system, so that the time of oscillation in the different experiments was nearly the same. The observations were taken as follows. The whole system was turned very slowly round by applying the hands of the graduated disk, taking care not to derange the vertical position of the suspending wire. The arc through which the system had been turned was read by means of the graduation, or rather the system was turned through an arc previously fixed on; the system was then left to itself, and the arc again read off to a certain number of oscillations. Thus it was the decrement of the arc of oscillation that was observed; the time of oscillation was indeed also observed, but only approximately, for the sake of determining a subsidiary quantity required in the calculation. Indeed, it will be easily seen that the experiments were not adapted to determine the effect of the fluid on the time of oscillation. The decrement of arc so determined had to be corrected for the effect of the imperfect elasticity of the wire, and of the resistance of the air against the graduated disk, and of the water against the portion of the copper cylinder immersed. The amount of the correction was determined by repeating the observation when the lower disk had been removed.

It appeared from the experiments, *first*, that with the same disk immersed, the successive amplitudes of oscillation decreased in geometric progression; *secondly*, that with different disks the moment of the resisting force was proportional to the fourth power of the radius. From these laws Coulomb concluded that each small element of any one of the disks experienced a resistance varying as the area of the element multiplied by its linear velocity. It should be observed that Coulomb was only authorized by his experiments to assert this law to be true in the case of oscillations of given period, inasmuch as the time of oscillation was nearly the same in all the experiments.

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[*] The measurement of either of Bessel's figures, figs. 5 or 6, Plate II. gives 1·53 inch for the distance of the centre of the sphere from the surface of the broad iron bar which formed the back of the frame, the surface of the bar being supposed truly vertical; and the measurement of fig. 2 giving 2·06 inches for the diameter of the sphere, it appears that the distance of the surface of the sphere from the surface of the bar was barely equal to half the radius of the sphere.

Let a be the radius of the disk in the fluid, τ the time of oscillation, θ the angular displacement of the disk, measured from its mean position, I the moment of inertia of the whole system; and let $1 : 1 - m$ be the ratio in which the arc of oscillation is diminished in one oscillation. According to the formula (15) we have

$$\epsilon^{-n\beta t}$$

for the factor which expresses the ratio of the arc of oscillation at the end of the time t to the initial arc. At the end of one oscillation $t = \tau$, and the value of the above factor is $1 - m$, which is given by observation. Putting for β its value, in which $M\gamma^2 = I$, and $n\tau = \pi$, we get

$$\log_e (1 - m) = -\frac{\rho a^4}{I}\sqrt{\frac{\pi^3 \mu' \tau}{8}}. \quad . \quad . \quad . \quad . \quad . \quad (162)$$

Let T be the time of oscillation, and I_0 the moment of inertia, when the under disk is removed: then $I = I_0 \tau^2 T^{-2}$. Also if M be the mass and R the radius of the large graduated disk, we have $I_0 = \frac{1}{2} MR^2$, neglecting, as Coulomb did, the rotatory inertia of the copper cylinder. Substituting in (162), we get

$$\log_e (1 - m)^{-1} = 2^{-\frac{1}{2}} \pi^{\frac{3}{2}} \rho \mu'^{\frac{1}{2}} \tau^{-\frac{3}{2}} T^2 a^4 R^{-2} M^{-1}. \quad . \quad . \quad . \quad . \quad (163)$$

Let W be the weight of the disk in grammes. Then the mass of the disk is equal to that of W cubic centimetres or $1000\,W$ cubic millimetres of water. Hence $M = 1000\,\rho W$, a millimetre being the unit of length. Substituting in (163), and solving with respect to $\sqrt{\mu'}$, we get

$$\sqrt{\mu'} = 1000 \times 2^{\frac{1}{2}} \log_e 10 \,.\, \pi^{-\frac{3}{2}} W R^2 T^{-2} a^{-4} \tau^{\frac{3}{2}} \log_{10} (1 - m)^{-1}, \quad . \quad . \quad . \quad (164)$$

and the same value of $\sqrt{\mu'}$ ought to result from different experiments.

The weight of the disk is stated to have been 1003 grammes, and its diameter 271 millimetres, and it made 4 oscillations in 91 seconds. Hence $W = 1003$, $R = 135{\cdot}5$, $T = 22{\cdot}75$. The last three factors in (164) vary from one experiment to another. After making experiments with three disks of different radii attached to the copper cylinder, Coulomb made another set with nothing attached, for the purpose of eliminating the effect of the imperfect elasticity of the wire. The following table contains the data furnished by experiment, together with the value of $\sqrt{\mu'}$ deduced from the several experiments. The latter is reduced to the decimal of an English inch, by including $\bar{2}{\cdot}5952$ (the logarithm of the ratio of a millimetre to an inch) in the logarithm of the constant part of the 2nd member of equation (164).

Determination of the value of $\sqrt{\mu'}$ for water from Coulomb's experiments on the decrement of the arc of oscillation of disks, oscillating in their own plane by the force of torsion.

No.	Diameter of disk $2a$ in millimetres.	Time of four oscillations 4τ.	$\log_{10}(1-m)^{-1}$	Resulting value of $\sqrt{\mu'}$ in inches.
1	195	97	0·0568	0·05519
2	140	92	0·021	0·05716
3	119	91	0·0135	0·05436
4	0	91	0·0058	

In correcting the results of the first three experiments for the imperfect elasticity of the wire, Coulomb calculated the values of m given by the four experiments, and subtracted the value given by the fourth from each of the others. But it is at the same time easier and more exact to subtract the value of $\log(1-m)^{-1}$ given by the fourth experiment from that given by each of the others. For if

$$-2c\frac{d\theta}{dt}, \qquad -2c'\frac{d\theta}{dt}$$

be the moments of two forces, each varying as the velocity, divided by the moment of inertia, the factors by which the initial arc of oscillation must be multiplied to get the arc at the end of the time t, first, when the two forces act together, secondly, when the second force acts alone, are

$$\epsilon^{-(c+c')t}, \qquad \epsilon^{-c't},$$

respectively, and that, whether the time t be great or small. Hence if we subtract the logarithm of the second factor from that of the first we shall get the logarithm of the factor due to the action of the first force alone. But if we put each factor under the form $1-m$, and subtract the m of the second factor from the m of the first, we shall not get the m due to the first force alone, unless t be small enough to allow of our neglecting the squares of ct and $c't$, or at least the product $ct \,.\, c't$. In truth, when $t=\tau$, the quantities m are sufficiently small to be treated in Coulomb's manner without any material error, since the corrected values of $\log(1-m)$, obtained in the two ways, would only differ in the 4th place of decimals.

The numbers given in the last column of the above table were calculated from the formula (164), on substituting for $\log(1-m)^{-1}$ the numbers found in the first three lines of the 4th column, corrected by subtracting 0·0058. The mean of the three results is 0·05557, but the three experiments are not equally valuable for the determination of $\sqrt{\mu'}$. For the three numbers from which $\sqrt{\mu'}$ was deduced are 0·0510, 0·0152, 0·0077, and a given error in the first of these numbers would produce a smaller error in $\sqrt{\mu'}$ than that which would be produced by the same error in the second, still more, than that which would be produced by the same error in the third. If we multiply the three values of $\sqrt{\mu'}$ by 510, 152, and 77, respectively, and divide the sum of the products by 510 + 152 + 77 or 739, we get 0.05551. We may then take 0.555 as the result of the experiments. Assuming $\sqrt{\mu'} = 0.0555$ we have

$\log(1-m)^{-1}$ from experiment	0.0568 in No. 1,	0.021 in No. 2,	0.0135 in No. 3,
.... from theory	0.0571	0.0206	0.0137
difference	− 0.0003	+ 0.0004	− 0.0002

65. So far the accordance of the theoretical and observed results is no very searching test of the truth of the theory. For, in fact, the theory is involved in the result only so far as this, that it shews that the resistance experienced by a given small element of a disk oscillating in a given period varies as the linear velocity; since the difference of periods in Coulomb's experiments was so small that the effects thence arising would be mixed up with errors of observation. This law is so simple that it might very well result from theories differing in some essential particulars from the theory of this paper. But should the numerical value of $\sqrt{\mu'}$ determined by Coulomb's experiments on disks be found to give results in accordance with theory in totally different cases, then the theory will receive a striking confirmation. Before proceeding to the discussion of other experiments, there are one or two minute corrections to be applied to the value of $\sqrt{\mu'}$ given above, which it will be convenient to consider.

In the first place, the result obtained in Art. 8 is only approximate, the approximation depending upon the circumstance that the diameter of the revolving body is large compared with a certain line determined by the values of μ' and τ. In the particular case in which the revolving solid is a circular disk, it happens that the approximate solution satisfies the general equations exactly, except so far as relates to the abrupt termination of the disk at its edge*. In consequence of this abrupt termination, the fluid annuli in the immediate neighbourhood of the edge are more retarded by the action of the surrounding fluid than they would have been were the disk continued, and consequently the resistance experienced by the disk in the immediate neighbourhood of its edge is actually a little greater than that given by the formula. I have not investigated the correction due to this cause, but it would doubtless be very small.

In the second place, the formula (15) is adapted to an indefinite succession of oscillations, whereas Coulomb did not turn the disk through an angle greater than the largest intended to be observed, and suffer one or two oscillations to pass before the observation commenced, but took for the initial arc that at which the disk had been set by the hand. Probably the disk was held in this position for a short time, so that the fluid came nearly to rest. If so, the resulting value of $\sqrt{\mu'}$, as may readily be shewn, would be a little too small. For in the course of an indefinite series of oscillations, the disk, in its forward motion, carries a certain quantity of fluid with it, and this fluid, in consequence of its inertia, tends to preserve its motion. Hence, when the disk, having attained its maximum displacement in the positive direction, begins to return, it finds the fluid moving in such a manner as to oppose its return, and therefore it experiences a greater resistance than if it had started from the same position with the fluid at rest. In fact, it appears from the expression for G in Art. 8, that the moment of the resistance vanishes, in passing from negative to positive, not when the disk has reached the end of its excursion in the positive direction, but the eighth part of a period earlier. Hence, had the observation commenced during a series of oscillations, a larger initial arc would have been necessary, to overcome the greater resistance, in order to produce, after a given number of oscillations, the same final arc as that actually observed. I have investigated the correction to be applied on account of this cause, and find it to be about $+ 0.009$, but I must refer to a note for the demonstration, in order not to interrupt the present discussion [*]. I shall assume then, in the following comparisons, that for water

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* (*See Note A at the end.*)

$$\sqrt{\mu'} = 0.0564,$$

the units being the same as before, namely, an English inch and a second. That μ' is independent of the pressure of the fluid, or at least very nearly so, appears from an experiment of Coulomb's, in which it was found that the decrement of the arc of oscillation of a disk oscillating in water was the same in an exhausted receiver as under the full atmospheric pressure.

I will here mention another experiment of Coulomb's which bears directly on one part of the theory. On covering the disk with a thin coating of tallow, the resistance was found to be the same as before; and even when the tallow was sprinkled with powdered sandstone, by means of a sieve, the increase of resistance was barely sensible. This strikingly confirms the correctness of the equations of condition assumed to hold good at the surface of a solid.

66. I will now compare the formula (148) with the results obtained by Bessel for the oscillations of the brass sphere in water, which will be found at page 65 of his memoir. This sphere was suspended so as to be immersed in the water contained in a large vessel, and was swung with two different lengths of wire, the same as those employed for the experiments in air. The times of oscillation were 1·9085 second for the long pendulum, and 1·1078 for the short. The results are

	Long pendulum.	Short pendulum.
k, by experiment	0·648	0·602
k, by theory	0·631	0·600
difference	+ 0·017	+ 0·002

The depth to which the spheres were immersed is not stated, but it was probably sufficient to render the effect of the free surface small, if not insensible. The vessel was three feet in diameter, and the water 10 inches deep, so that unless the spheres were suspended near the bottom, which is not likely to have been the case, the effect of the limitation of the fluid by the sides of the vessel must have been but trifling. The agreement of theory and observation, as will be seen, is very close.

67. In the same memoir which contains the experiments on disks, Coulomb has given the results of some experiments in which the disk immersed in the fluid was replaced by a

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[*] (*See Note* B *at the end.*)

long narrow cylinder, placed with its axis horizontal and its middle point in the prolongation of the axis of the vertical copper cylinder. In these experiments, the arcs did not decrease in geometric progression, as would have been the case if the resistance had varied as the velocity; but it was found that the results of observation could be satisfied by supposing the resistance to vary partly as the first power, and partly as the square of the velocity. In Coulomb's notation, $1 : 1 - m$ denotes the ratio in which the arc of oscillation would be altered after one oscillation, if the part of the resistance varying as the square of the velocity were destroyed. The several experiments performed with the same cylinder were found to be sufficiently satisfied by the formula deduced from the above-mentioned hypothesis respecting the resistance, when suitable numerical values were assigned to two disposable constants m and p, of which p related to the part of the resistance varying as the square of the velocity.

Conceive the cylinder divided into elementary slices by planes perpendicular to its axis. Let r be the distance of any slice from the middle point, θ the angle between the actual and the mean positions of the axis, dF that part of the resistance experienced by the slice which varies as the first power of the velocity. Then calculating the resistance as if the element in question belonged to an infinite cylinder moving with the same linear velocity, we have by the formulæ of Art. 31

$$dF = k'M'n\frac{d\xi}{dt}, \quad \text{where } M' = \pi\rho a^2 dr, \quad \frac{d\xi}{dt} = r\frac{d\theta}{dt}.$$

If G be the moment of the resistance, l the whole length of the cylinder, we have, putting $n = \pi\tau^{-1}$,

$$G = \frac{\pi^2 k'\rho a^2 l^3}{12\tau}\,\frac{d\theta}{dt};$$

whence

$$\log_\epsilon (1 - m)^{-1} = \frac{\pi^2 k'\rho a^2 l^3}{24 I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

I being the moment of inertia.

Expressing I in terms of the same quantities as in the case of the disk, we get from (147) and (165)

$$\log_{10} (1 - m)^{-1} = \log_{10}\epsilon \,.\, \frac{\pi\mu' T l^3}{3R^4} \,.\, \frac{g\rho}{W} \,.\, \frac{T}{\tau} \,.\, \mathfrak{m}^2 k', \quad . \quad . \quad . \quad (166)$$

and $g\rho$ is the weight of a cubic millimetre of water, or the 1000th part of a gramme. The numerical values of μ', T, R, W have been already given, but μ' must be reduced from square inches to square millimetres. The cylinders, of which three were tried in succession, had all the same length, namely, 249 millimetres. Their circumferences, calculated from their weights and expressed in millimetres, were 21.1, 11.2, and 0.87, and the time of four oscillations was 92^s, 91^s, 91^s. The values of $\mathfrak{m}$ calculated from these data by means of the formula (147) are 0.4332, 0.2312, and 0.01796. For the first and second of these values, $\mathfrak{m}^2 k'$ may be obtained by interpolation from the table given in Part I.; for the third it will be sufficient to employ the second of the formulæ (115).

The following are the results:

	Cylinder, No. 1.	No. 2.	No. 3.
m, by experiment	0.0400	0.0260	0.0136
m, by theory	0.0413	0.0291	0.0113
Difference	− 0.0013	− 0.0031	+ 0.0023

The differences between the results of theory and experiment are perhaps as small as could reasonably be expected, when it is considered that, notwithstanding the delicate nature of the experiments, the numerical values of two constants, m and p, had to be deduced from their results.

68. This memoir of Coulomb's contains also a notice of a set of experiments with disks and cylinders in which the water was replaced by oil. The experiments with disks shewed that with a given disk the arc of oscillation decreased in geometric progression, and that with different disks the moments of the resistances were as the fourth powers of the diameters. The absolute resistances were greater than in the case of water in the ratio of about 17·5 to 1. The details of Coulomb's experiments on cylinders oscillating in oil are entirely omitted. It is merely stated that on making the same cylinders as before, or shorter cylinders when the resistance was too great, oscillate in oil, it was found, conformably with the results obtained with planes, that the coherence of oil was to that of water as 17 to 1. The coherence is here supposed to be measured by that part of the resistance which is proportional to the first power of the velocity. On making a rough calculation of the ratio of the resistances to cylinders oscillating in oil and in water, on the supposition that $\sqrt{\mu'}$ for oil is to $\sqrt{\mu'}$ for water as 17·5 to 1, as would follow from the experiments on disks if the difference of the specific gravities of the two fluids be neglected, I found that the ratio in question ought to have been somewhere about 100 to 1, instead of only 17 to 1. It would seem from this that the theory of the present paper is not applicable to oil; but fresh experiments would be required before this point can be considered as established, on account of the theoretical doubt respecting the application of the formulæ of Section III. Part I., to extremely fine cylinders, especially in cases in which μ' is large, so that $\mathfrak{m}$ is very small. It would be interesting to make out whether what I have called internal friction is or is not of the same nature as viscosity. Coulomb and Dubuat apply the term *viscosity* to that property of water by virtue of which certain effects are produced which have been shewn in this paper to be perfectly explicable on the theory of internal friction; whereas Poisson, in one of his memoirs, expressly asserts that the terms in the equations of motion which result from what has been called in this paper internal friction belong to perfect fluids, and have nothing to do with viscosity*. Poisson does not give the slightest hint as to the grounds on which he rested his opinion.

69. I come now to the experiments of Dubuat, which are contained in an excellent work of his entitled *Principes d'Hydraulique*, of which the second edition was published in 1786.

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* *Journal de l'Ecole Polytechnique*, Tom. XIII. p. 95.

The first edition does not contain the experiments in question. Dubuat justly remarked that the time of oscillation of a pendulum oscillating in a fluid is greater than it would be in vacuum, not only on account of the buoyancy of the fluid, which diminishes the moving force, but also on account of the mass of fluid which must be regarded as accompanying the pendulum in its motion; and even determined experimentally the mass of fluid which must be regarded as carried by the oscillating body in the case of spheres and of several other solids. Thus Dubuat anticipated by about forty years the discovery of Bessel; but it was not until after the appearance of Bessel's memoir that Dubuat's labours relating to the same subject attracted attention.

Dubuat's method was as follows. Imagine a body suspended by a fine thread or wire and swung in vacuum, and let a be the length of the pendulum, reckoned from the centre of suspension to the centre of oscillation. Now imagine the same body swung in a fluid, in which its apparent weight is p, so that if P denote the weight of fluid displaced, the true weight of the body will be $p + P$. Since the moving force is diminished in the ratio of $p + P$ to p, if the inertia of the body were all that had to be overcome, it would be necessary to diminish the length of the pendulum in the same ratio, in order to preserve the same time of oscillation. But since the mass in motion consists not only of the mass of the body itself, but also of that of the fluid which it carries with it, the pendulum must be shortened still more, in order that the time of oscillation may be unaltered. Let l be the length of the pendulum so shortened, and $\mathfrak{n}$ (which for the same reason as before I write instead of Dubuat's n,) a factor greater than unity, such that $p + \mathfrak{n}P$ is the weight of the mass in motion; then

$$l = \frac{ap}{p + \mathfrak{n}P}, \qquad \text{whence } \mathfrak{n} = \frac{p}{P}\left(\frac{a}{l} - 1\right). \quad . \quad . \quad . \quad (167)$$

Dubuat's experiments on this subject consist of 44 experiments on spheres oscillating in water, (Tom. II. p. 236); 31 experiments on other solids oscillating in water, (p. 246); and 3 experiments on spheres oscillating in air, (p. 283). The following table contains a comparison of the formula (148) with Dubuat's results for spheres oscillating in water. The value of $\sqrt{\mu'}$ employed in the calculation is 0.0564 inch English, or 0.05291 inch French.

Dubuat's experiments on spheres oscillating in water.

		n		
	τ	calc.	obs.	diff.
Sphere of lead. Diameter 1·0113 inches. Weight in water 2102 grains.	$\frac{1}{2}$	1·633	1·502	−·131
	1	1·687	1·502	−·185
	2	1·766	1·522	−·244
	3	1·825	1·620	−·205
Sphere of glass. Diameter 2·645 inches. Weight in water 574 grains.	2	1·602	1·518	−·084
	4	1·644	1·569	−·075
	6	1·676	1·598	−·078
Same sphere weighing in water 2102 grains.	1	1·572	1·515	−·057
	2	1·602	1·516	−·086
	3	1·624	1·523	−·101
	4	1·644	1·546	−·098
Same sphere weighing in water 4204 grains.	1	1·572	1·537	−·035
	2	1·602	1·523	−·079
	3	1·624	1·524	−·100
	4	1·644	1·538	−·106
Same sphere weighing in water 9216 grains.	$\frac{1}{2}$	1·551	1·449	−·102
	1	1·572	1·372	−·200
	2	1·602	1·494	−·108
	3	1·624	1·494	−·130
Sphere of wood. Diameter 4·076 inches. Weight in water 2102 grains.	2	1·566	1·507	−·059
	3	1·581	1·547	−·034
	4	1·593	1·547	−·046
	6	1·614	1·567	−·057
Same sphere weighing in water 4204 grains.	1	1·547	1·375	−·172
	2	1·566	1·456	−·110
	3	1·581	1·525	−·056
	4	1·593	1·557	−·036
	6	1·614	1·549	−·065
Same sphere weighing in water 9216 grains.	1	1·547	1·57	+·023
	2	1·566	1·553	−·013
	3	1·581	1·59	+·009
	4	1·593	1·583	−·010
Another sphere of wood. Diameter $6\frac{2}{3}$ inches. Weight in water 2102 grains.	3	1·549	1·27	−·279
	4	1·557	1·394	−·163
	6	1·570	1·487	−·083
	9	1·585	1·566	−·019
	12	1·599	1·569	−·030
	18	1·621	1·565	−·056
Same sphere weighing in water 3204 grains.	10·85	1·594	1·634	+·040
Same sphere weighing in water 4204 grains.	3	1·549	1·651	+·102
	4	1·557	1·627	+·070
	6	1·570	1·654	+·084
	9	1·585	1·664	+·079
	12	1·599	1·674	+·075

70. If we strike out the experiments with the large sphere, which cannot well be compared with theory for a reason which will be explained further on, it will be observed that in seven out of the eight groups of experiments left, the signs in the last column are regularly *minus*. The preponderance of negative errors could be destroyed by using a much smaller value of $\sqrt{\mu'}$ in the reduction. We have seen, however, that the value of $\sqrt{\mu'}$ deduced from Coulomb's experiments on the decrement of the arc of oscillation of disks satisfied almost exactly Bessel's observations of the time of oscillation of a sphere about two inches in diameter oscillating in water. The very small errors which remained in this case had both the sign +, whereas in Dubuat's experiments on the 1-inch and $2\frac{1}{2}$ inch spheres, the errors, which are far larger, have all the sign −. Since the experiments of Dubuat and Bessel, though made under similar circumstances, do not lead to the same result, it is of course impossible for any theory to

satisfy them both. The numbers in the last column of the preceding table are, however, far too regular to be attributable to mere fortuitous errors of observation. If we suppose Bessel's results to have been nearly exact, there must have been something in the mode either of making or of reducing Dubuat's experiments which caused a tendency to error in one direction.

With respect to the reduction of the experiments it may be observed that the length l was measured from the centre of oscillation, whereas in the formula (148) it is supposed that the mass of which the weight is kP or $(\mathfrak{n}-1)P$ is collected at the centre of the sphere. If h be the distance of the centre of the sphere from the axis of suspension, the observed value of $\mathfrak{n}-1$ ought in strictness to be increased in the ratio of h^2 to l^2, or the calculated value diminished in the ratio of l^2 to h^2, before comparing the results of theory and experiment. In the case of the loaded spheres especially, the theoretical value of $\mathfrak{n}$ would thus be a little diminished; but except in a very few cases, in which either l or $a-l$ is small, the diminution is hardly worth considering. After having been for a good while at a loss to account for the regular occurrence of rather large negative errors, the following occurred to me as the probable solution of the difficulty.

When a pendulum oscillates in water, the arc of oscillation rapidly decreases; this rapid diminution forms in fact the grand difficulty in experiments of this kind. In Dubuat's experiments, it will be remembered, the suspending thread was lengthened or shortened till the time of oscillation was an exact number of seconds, or occasionally half a second. Now, it is probable that the observer occasionally gave the suspending thread a slight push as the pendulum was commencing its return, in order to keep the oscillations going for a sufficient time to allow of tolerable precision in rendering the time of oscillation equal to what it ought to be. If so, these pushes would slightly accelerate the oscillations, and therefore cause the length of thread fixed on by observation to be a little too great, which would make the effect of the water in retarding the oscillations appear a little too small. On inspecting the table of differences, it may be observed that sometimes when the same sphere differently loaded is swung in the same time as before, the numbers in the table of differences are altered more than appears to be attributable to merely fortuitous errors of observation. This accords very well with the conjecture just mentioned, and seems difficult to account for in any other way, inasmuch as everything relating to the fluid must have been almost exactly the same in the two cases.

The occurrences of positive differences in the case of the large wooden sphere may be accounted for by the limitation of the fluid mass by the sides and bottom of the vessel, and by the free surface, which, except in the case of very short oscillations, would have much the same effect as a rigid plane, inasmuch as it would be preserved almost exactly horizontal by the action of gravity. The vessel which contained the water was 51 inches long and 17 broad, the water was 14 inches deep, and the spheres were plunged to about 3 inches below the surface, so that the effect of the confinement of the fluid mass would have been quite sensible in the case of such large spheres. If it be objected that the same sphere gave negative differences in the case of the first group of experiments, it must be observed, that when the apparent weight of so large a sphere was only 2102 French grains, the resistance would quickly have caused the oscillations to subside if an extraneous force had not frequently been applied.

71. In Dubuat's experiments on spheres oscillating in air, the lightness of the fluid was compensated by the extreme lightness of the spheres, which were composed, the first two of paper, and the third of gold-beater's skin. In the following table the diameter $2a$ of the sphere is expressed in French inches. The value of $\sqrt{\mu'}$ employed in the reduction is the same as was before used in the reduction of observations made in air, namely 0.116 inch English, or 0.1088 inch French.

Dubuat's experiments on light spheres oscillating in air.

No.	$2a$	τ	$\mathfrak{n}$ calc.	$\mathfrak{n}$ obs.	Diff.
337	4·0416	1·51	1·61	1·51	− 0·10
338	6·625	1·84	1·57	1·63	+ 0·06
339	17·25	3·625	1·53	1·54	+ 0·01

The differences certainly appear very small when the delicacy of the experiments and the simplicity of the apparatus employed are considered.

72. The only comparison yet made in this section between theory and observation in the case of pendulum experiments, consists in comparing the observed times of vibration with the results calculated with an assumed value of $\sqrt{\mu'}$. But according to theory we ought to be able, without assigning a particular value to any new disposable constant, to calculate the rate of decrease of the arc of vibration. I have not met with any experiments made with a view of investigating the decrease in the arc of vibration in the case of extremely small vibrations, such as those employed in pendulum experiments. The experiments of Newton and others, in which the arc of vibration was so large that the resistance depended mainly on the square of the velocity, would be quite useless for my purpose. The pendulum experiments of Bessel and Baily contain however the requisite information, or at least some portion of it, for the arcs are registered for the sake of giving the data for calculating the small reduction to indefinitely small vibrations.

In Bessel's experiments the arc is registered for the end of equal intervals of time during the motion. The number of such registrations in one experiment amounts in some cases to eleven, and is never less than three. So far the observations are just what are wanted; but there are other causes which prevent an exact comparison between theory and experiment. In the first place the spheres were swung so close to the back of the frame that the increase of resistance due to the confinement of the air must have been very sensible. In the second place the effect of the wire must have been very sensible, especially in the case of the long pendulum. For the table of Section III. Part I., shews that for the wire (for which $\mathfrak{m}$ is very small) the value of k' is much larger than that of k, whereas for spheres of the size of those employed, when the time of oscillation is only one or two seconds, k' is a good deal smaller than k. Hence, if the formulæ of that section applied to such fine wires, the effect of the wire on the arc of vibration would be much greater than its effect on the time of vibration, and therefore would be quite sensible. But it has been shewn in Section IV., that the effect of the wire in diminishing the arc of vibration is probably greater than would be given by the formula, and therefore the uncertainty depending on the wire is likely to amount to a very sensible fraction of the whole amount. Again, since Bessel's experiments were all made in air, no data are afforded whereby to eliminate the portion of the observed result which was due to friction at the point of support, imperfect elasticity of the wire, or gradual dissipation of vis viva by communication of motion to the supporting frame. Moreover in the case of the long pendulum the observations were made with rather too large arcs, for the law of the decrease of the arc of vibration deviated sensibly from that of a geometric progression. In Baily's experiments, only the initial and final arcs are registered, and not even those in the case of the "additional experiments." Hence these experiments do not enable us to make out whether it would be sufficiently exact to suppose the decrease to take place in geometric progression. Moreover, the final arc was generally so small, that a small error committed in the measurement of it would cause a very sensible error in the rate of decrease concluded from the experiment. For these reasons it would be unreasonable to expect a near accordance between the formulæ and the results of the experiments of Bessel and Baily. Still, the formulæ might be expected to give a result in defect, and yet not so much in defect as not to form a large portion of the result given by observation. On this account it will not be altogether useless to compare theory and observation with reference to the decrement of the arc of vibration.

73. Let us first consider the case of a sphere suspended by a fine wire. Let the notation be the same as was used in investigating the expression for the effect of the air on the time of vibration, except that the factors k', k'_1 come in place of k, k_1. Considering only that part of the resistance which affects the arc of vibration, we have for the portions due respectively to the sphere and to the element of the wire whose length is ds, and distance from the axis of suspension s,

$$k' M' n\,(l + a)\frac{d\theta}{dt}, \qquad k'_1\,\frac{M_1'}{l}\,ds\,.\,ns\,\frac{d\theta}{dt},$$

and if we take the moment of the resistance, and divide by twice the moment of inertia, the coefficient of $\frac{d\theta}{dt}$ in the result, taken negatively, and multiplied by t, will be the index of ϵ in the expression for the arc. Hence if a_0 be the initial arc of vibration, and a_t the arc at the end of the time t

$$\log_\epsilon a_0 - \log_\epsilon a_t = \frac{k'M'\,(l + a)^2 + \frac{1}{3}\,k_1'M_1'\,l^2}{M\,(l + a)^2 + \frac{1}{3}\,M_1 l^2}\cdot\frac{\pi t}{2\tau}, \quad . \quad . \quad . \quad . \qquad (168)$$

$M'(l + a)^2$ being as before taken for the moment of inertia of the sphere, which will be abundantly accurate enough. If then we put $\mathbf{l}$ for the Napierian logarithm of the ratio of the arc at the beginning to the arc at the end of an oscillation, we must put $t = \tau$ in (168), whence, neglecting the effect of the wire, we obtain

$$\text{I} = \frac{\pi k'}{2} \cdot \frac{\sigma}{S}. \quad \ldots \ldots \ldots \quad (169)$$

If now $\Delta k'$ be the correction to be applied to k' in this formula on account of the wire, since k', k_1' are combined together in the expression for the arc just as k, k_1 in the expression for the time, we get

$$\Delta k' = \frac{k_1'}{k_1} \Delta k, \quad \ldots \ldots \ldots \quad (170)$$

and the approximate formulæ (115) give

$$\Delta k' = -\frac{4L}{\pi} \Delta k, \quad \ldots \ldots \ldots \quad (171)$$

whence the numerical value of $\Delta k'$ is easily deduced from that of Δk, which has been already calculated. We get also from (52)

$$k' = k - \tfrac{1}{2} + \tfrac{4}{9}(k - \tfrac{1}{2})^2, \quad \ldots \ldots \ldots \quad (172)$$

whence k' may be readily deduced from k, which has been already calculated.

74. Before comparing these formulæ with Bessel's experiments, it will be proper to enquire how far the latter are satisfied by supposing the arcs of oscillation to decrease in geometric progression. In Bessel's tables the arc is registered in the column headed μ. This letter denotes the number of French lines read off on a scale placed behind the wire, and a little above the sphere, and is reckoned from the position of instantaneous rest of the wire on one side of the vertical to the corresponding position on the other side. The distance of the scale from the axis of suspension being given, as well as the correction to be applied to μ on account of parallax, the arc of oscillation may be readily deduced. However, for our present purpose, any quantity to which the arc is proportional will do as well as the arc itself, and μ, though strictly proportional to the tangent of the arc, may be regarded as proportional to the arc itself, inasmuch as the initial arc usually amounted to only about 50′ on each side of the vertical.

Now we may form a very good judgment as to the degree of accuracy of the geometric formula by comparing the arc observed in the middle of an experiment with the geometric mean of the initial and final arcs. I have treated in this way Bessel's experiments, Nos. 1, 2, 3, 4, and 5. Each of these is in fact a group of six experiments, four with the long pendulum and two with the short, so that the whole consists of 20 experiments with the long pendulum, and 10 with the short. In the case of the long pendulum, the observed value of μ regularly fell short of the calculated value, and that by a tolerably constant quantity. The mean difference amounted to 0.688 line, and the mean error in this quantity to 0.109. This mean error was not due entirely to errors of observation, or variations in the state of the air, &c., but partly also to slight variations in the initial arc, larger differences usually accompanying larger initial arcs. The initial arc usually corresponded to $\mu = 39$ or 40 lines, and the final to $\mu = 15$

or 16 lines. In the case of the short pendulum, the differences in 8 cases out of 10 had the same sign as before. The mean difference was 0.025, and the mean error 0.043. The arcs of oscillation were nearly the same as before; but inasmuch as the axis of suspension was nearer to the scale than before, the initial value of μ was only about 12 or 13 lines, and the final value about 7 lines. When the results of some of the experiments were laid down on paper, by abscissæ taken proportional to the times and ordinates to the logarithms of μ, it was found that in the case of the long pendulum the line so drawn was decidedly curved, the concavity being turned toward the side of the positive ordinates. The curvature of the line belonging to the short pendulum could hardly be made out, or at least separated from the effects of errors of observation. The experiments 9, 10, 11, having been treated numerically in the same way as the experiments 1—5, led to much the same result. In the 16 experiments with the ivory sphere and short pendulum contained in the experiments Nos. 12, 13, 14, and 15, the excess of the calculated over the observed value of μ was more apparent, the mean excess amounting to 0.129. The reason of this probably was, that the observations with the ivory sphere were made through a somewhat wider range of arc than those with the brass sphere.

It appears then that at least in the case of the long pendulum a correction is necessary, in order to clear the observed decrease in the arc of oscillation from the effect of that part of the resistance which increases with the arc more rapidly than if it varied as the first power of the velocity, and so to reduce the observed rate of decrease to what would have been observed in the case of indefinitely small oscillations.

75. In Coulomb's experiments it appeared that the resistance was composed of two terms, one involving the first power, and the other the square of the velocity. If we suppose the same law to hold good in the present case, and denote the amplitude of oscillation at the end of the time t, measured as an angle, by a, we shall obtain

$$\frac{da}{dt} = -Aa - Ba^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

where A and B are certain constants. We must now endeavour to obtain A from the results of observation. Since the substitution for a of a quantity proportional to a will only change the constant B in (173), and the numerical value of this constant is not required for comparison with theory, we may substitute for a the number of lines read off on the scale as entered in Bessel's tables in the columns headed μ.

I have employed four different methods to obtain A from the observed results. The one I am about to give is the shortest of the four, and is sufficiently accurate for the purpose.

The equation (173) gives after dividing by a

$$\frac{d \log a}{dt} = -A - Ba. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

Now, as has been already observed, the arcs of vibration decrease nearly in geometric progression. If this law were strictly true, we should have

$$a = a_0 \left(\frac{a_2}{a_0}\right)^{\frac{t}{T}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

where a_0 denotes the initial and a_2 the final arc, and T denotes the whole time of observation. We may, without committing any material error, substitute this value of a in the last term of (174). The magnitude of the error we thus commit is not to be judged of merely by the smallness of B. The approximate expression (175) is rather to be regarded as a well-chosen formula of interpolation, and in fact $T^{-1} \log_\epsilon (a_0 a_2^{-1})$ differs very sensibly from A. Making now this substitution in (174), integrating, and after integration restoring a in the last term by means of (175), we get

$$\log a = -At - \frac{BTa}{\log a_2 - \log a_0} + C, \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

C being an arbitrary constant. To determine the three constants A, B, C, let a_1 be the arc observed at the middle of the experiment, apply the last equation to the arcs a_0, a_1, a_2, and take the first and second differences of each member of the equation. Let Δ_1 denote the sum of the two first differences, so that $\Delta_1 t$ is the same thing as T. Then we may take for the two equations to determine A and B

$$\Delta_1 \log a_0 = -A\Delta_1 t - \frac{B\Delta_1 t \,.\, \Delta_1 a_0}{\Delta_1 \log a_0}; \qquad \Delta^2 \log a_0 = -\frac{B\Delta_1 t \,.\, \Delta^2 a_0}{\Delta^2 \log a_0}.$$

Eliminating B, and passing from Napierian to common logarithms, which will be denoted by Log., we get

$$A = \frac{-\Delta_1 \operatorname{Log} a_0}{\operatorname{Log} \epsilon \,.\, \Delta_1 t} \left\{1 - \frac{\Delta^2 \operatorname{Log} a_0 \,.\, \Delta_1 a_0}{\Delta_1 \operatorname{Log} a_0 \,.\, \Delta^2 a_0}\right\}. \quad . \quad . \quad . \quad (177)$$

If we suppose the part of $-\dfrac{da}{dt}$ which does not vary as the first power of a to be $a^2 \phi'(a)$ instead of Ba^2, we shall get in the same way

$$A = \frac{-\Delta_1 \operatorname{Log} a_0}{\operatorname{Log} \epsilon \,.\, \Delta_1 t} \left\{1 - \frac{\Delta^2 \operatorname{Log} a_0 \,.\, \Delta_1 \phi(a_0)}{\Delta_1 \operatorname{Log} a_0 \,.\, \Delta^2 \phi(a_0)}\right\}. \quad . \quad . \quad . \quad . \quad (178)$$

76. I have not attempted to deduce evidence for or against the truth of equation (173) from Bessel's experiments. The approximate formula (175) so nearly satisfied the observations, that almost any reasonable formula of interpolation which introduced one new disposable constant would represent the experiments within the limits of errors of observation. It may be observed, that the factor outside the brackets in equations (177) and (178) is the first approximate value of A got by using only the initial and final arcs, and supposing the arcs to decrease in geometric progression. In the case of the long pendulum, the value of A, corrected in accordance with the formula (178), would be very sensibly different according as we supposed $\phi(a)$ to be equal to Ba, in which case (178)

would reduce itself to (177), or equal to Ba^2. In the case of the long pendulum with the brass sphere, the corrected value of A, deduced from the formula (177), was equal to about 0.77 of the first approximate value.

I have not considered it necessary to go through all Bessel's experiments, as it was not to be expected that the formula should account for the whole observed decrement. I have only taken four experiments for each kind of pendulum, namely, I. *a*, *b*, *e*, and *f* for the long pendulum with the brass sphere; I. *c* and *d* and II. *c* and *d* for the short pendulum with the brass sphere; XII. *a*, *b*, *c*, and *d* for the long pendulum with the ivory sphere, and XII. *a′*, *b′*, *c′*, and *d′* for the short pendulum with the ivory sphere. The formula (177) gave the following results. First case, $\text{Log}\,\epsilon\,.\,\tau A = .0000759$; mean error $= .0000020$. Second case, $\text{Log}\,\epsilon\,.\,\tau A = .0000504$; mean error $= .0000075$. Third case, $\text{Log}\,\epsilon\,.\,A = .000631$; mean error $= .000046$. Fourth case, $\text{Log}\,\epsilon\,.\,A = .000167$; mean error $= .000074$. Now $\mathfrak{l} = \tau A$, and therefore, to get the values of $\mathfrak{l}$ deduced from experiment, it will be sufficient to divide the numbers above given by the modulus of the common system of logarithms. The theoretical value of $\mathfrak{l}$ will be got from (169), if we add to k' the correction $\Delta k'$ depending upon the wire. The following are the results:

	long p. brass s.	short p. brass s.	long p. ivory s.	short p. ivory s.
1000000 $\mathfrak{l}$ for sphere alone in an unlimited mass of fluid, by theory	67	50	298	222
additional for wire	27	9	114	39
	94	59	412	261
1000000 $\mathfrak{l}$ by experiment	175	116	1453	384

It appears then that the calculated rate of decrease of the arc amounts on the average to about half the rate deduced from observation. This is about what we might have expected, considering the various circumstances, all tending materially to augment the rate of decrease, which were not taken into account in the calculation.

77. Of Baily's pendulums I have compared the following with theory in regard to the decrement of the arc of vibration. No. 1 (the 1½-inch platina sphere), experiments 1 to 8; No. 3 (the brass 1½-inch sphere), experiments 9 to 16; No. 6 (the 2-inch brass sphere), experiments 33 to 40; No. 21 (the 0.410 inch long copper cylindrical rod), experiments 109 to 112; and No. 35—38 (the 1½-inch long brass tube), experiments 167 to 174. I have not thought it worth while to compute the results obtained with the other 1½-inch and 2-inch spheres, inasmuch as they were of the same size as the brass spheres, and moreover the observation of the decrement of the arc was not the object Baily had in view in making the experiments. The 3-inch sphere, and all the other cylindrical rods and combinations of cylindrical rods and spheres, belong to the "additional experiments" for which the arcs are not given.

The mode of performing the calculation will best be explained by an example. Take, for instance, the pair of experiments Nos. 1 and 2. In No. 1 the total interval was 4.22 hours, the initial arc was 0°.77, the final arc 0°.29, the mean height of the barometer 30.24 inches, and the temperature about 38½° *F*. The difference of the common logarithms of the initial and final arcs is 0.424, and this divided by the total interval gives 0.1005 for the difference of logarithms for one hour. The second experiment, treated in a similar way, gives 0·0352, which expresses the effect of friction at the point of support, communication of motion to the support itself, &c., together with the resistance of highly rarefied air at a pressure of only 0.97 inch of mercury. Since we have reason to believe that μ' is independent of the density, we may get the effect of air at a pressure of 30.24 – 0.97 or 29.27 inches of mercury by subtracting 0.0352 from 0.1005, which gives 0.0653. Reducing to 29 inches of mercury for convenience of comparison, we get 0.0649. Each pair of experiments is to be treated in the same way. Since the temperature was nearly the same in the experiments made with the same pendulum, we may suppose it constant, and equal to the mean of the temperatures in the experiments made under the full atmospheric pressure. The experiments reduced consist of four pair for each pendulum, except No. 21, for which only two pair were performed. The following are the results. For the $1\frac{1}{2}$-inch platina sphere 0.0644, mean error 0.0044. For the $1\frac{1}{2}$-inch brass sphere 0.180, mean error 0.024. For the 2-inch brass sphere 0.094, mean error 0.013. For the copper rod 0.486, mean error 0.113. For the brass tube the results were 0.145, 0.363, 0.338, 0.305. Rejecting the first result as anomalous, and taking the mean of the others, we get 0.335, mean error 0.030. To obtain **I** from the mean results above given we have only to divide by 3600 times the modulus, and multiply by τ, and for the experiments with spheres we may suppose $\tau = 1$.

The mode of calculating **I** from theory in the case of a sphere suspended by a fine wire has already been explained. For the sake of exhibiting separately the effect of the wire, I will give one intermediate step in the calculation.

	1.44 inch sphere.	1.46 inch sphere.	2.06 inch sphere.
k', for sphere alone...............	0.326	0.320	0.220
$\Delta k'$, the correction for the wire...	0.130	0.130	0.045
Total, to be substituted in (169)..	0.456	0.450	0.265

The formula (168), which applies to a sphere suspended by a wire, will be applicable to a long cylindrical rod if we suppose $M = 0$. Hence the same formula (169) that has been used for a sphere may be applied to a cylindrical rod if we suppose k' to refer to the rod. For the copper rod $k' = 1.107$, and for the tube $k' = 0.2561$. The following are the results for the three spheres and two cylinders.

	No. 1.	No. 3.	No. 6.	No. 21.	Nos. 35—38.
1000000**I**, from experiment...	41	115	60	315	206
. . . . . from theory	39	106	60	237	156
Difference.........	+ 2	+ 9	0	+ 78	+ 50

It appears that the experiments with spheres are satisfied almost exactly. The differences between the results of theory and observation are much larger in the case of the long cylinders. Large as these differences appear, they are hardly beyond the limits of errors of observation, though they would probably be far beyond the limits of errors of observation in a set of experiments performed on purpose to investigate the decrement of the arc of vibration. It was to be expected beforehand that the results of calculation would fall short of those of observation, inasmuch as only two arcs were registered in each experiment, so that no data were afforded for eliminating the effect of that part of the resistance which did not vary as the first power of the velocity.

78. I have now finished the comparison between theory and experiment, but before concluding this Section I will make a few general remarks.

When a new theory is started, it is proper to enquire how far the theory does violence to the notions previously entertained on the subject. The present theory can hardly be called new, because the partial differential equations of motion were given nearly thirty years ago by Navier, and have since been obtained, on different principles, by other mathematicians; but the application of the theory to actual experiment, except in some doubtful cases relating to the discharge of liquids through capillary tubes, and the determination of the numerical value of the constant μ', are, I believe, altogether new. Let us then, in the first instance, examine the magnitude of the tangential pressure which we are obliged by theory to suppose capable of existing in air or water.

For the sake of clear ideas, conceive a mass of air or water to be moving in horizontal layers, in such a manner that each layer moves uniformly in a given horizontal direction, while the velocity increases, in going upwards, at the rate of one inch per second for each inch of ascent. Then the sliding in the direction of a horizontal plane is equal to unity, and therefore the tangential pressure referred to a unit of surface is equal to μ or $\mu'\rho$. The absolute magnitude of this unit sliding evidently depends only on the arbitrary unit of time, which is here supposed to be a second. In the case supposed, it will be easily seen that the particles situated at one instant in a vertical line are situated at the expiration of one second in a straight line inclined at an angle of 45° to the horizon. Equating the tangential pressure $\mu'\rho$ to the normal pressure due to a height h of the fluid, we get $h = g^{-1}\mu'$, g being the force of gravity. Putting now $g = 386$, $\mu' = (0.116)^2$ for air, $\mu' = (0.0564)^2$ for water, we get $h = 0.00003486$ inch for air, and $h = 0.000008241$ inch for water, or about the one thirty-thousandth part of an inch for air, and less than the one hundred-thousandth part of an inch for water. If we enquire what must be the side of a square in order that the total tangential pressure on a horizontal surface equal to that square may amount to one grain, supposing the density of air to be to that of water as 1 to 836, and the weight of a cubic inch of water to be 252.6 grains, we get 25 feet 8 inches for air, and 1 foot 10 inches for water. It is plain that the effect of such small forces may well be insignificant in most cases.

79. In a former paper I investigated the effect of internal friction on the propagation of sound, taking the simple case of an indefinite succession of plane waves*. It appeared that the effect consisted partly in a gradual subsidence of the motion, and partly in a diminution of the velocity of propagation, both effects being greater for short waves than for long. The second effect, as I there remarked, would be contrary to the result of an experiment of

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* Camb. Phil. Trans. Vol. VIII. p. 302.

M. Biot's, unless we supposed the term expressing this effect to be so small that it might be disregarded. I am now prepared to calculate the numerical value of the term in question, and so decide whether the theory is or is not at variance with the result of M. Biot's experiment.

According to the expression given in the paper just mentioned, we have for the proportionate diminution in the velocity of propagation

$$\frac{8\pi^2\mu'^2}{9\lambda^2 V^2},$$

λ being the length of a wave, and V the velocity of sound. To take a case as disadvantageous as possible, suppose λ only equal to one inch, which would correspond to a note too shrill to be audible to human ears. Taking the velocity of sound in air at 1000 feet per second, there results for the common logarithm of the expression above written $\overline{11}.0428$, so that a wave would have to travel near 100000000000 inches, or about 1578000 miles, before the retardation due to friction amounted to one foot. It is plain that the introduction of internal friction leaves the theory of sound just as it was, so far as the velocity of propagation is concerned, at least if the sound be propagated in free air.

The effect of friction on the intensity of sound depends on the first power of μ'. In the case of an indefinite succession of plane waves, it appears that during the time t the amplitude of vibration is diminished in the ratio of 1 to ϵ^{-ct}, and therefore the intensity in the ratio of 1 to ϵ^{-2ct}, where

$$c = \frac{8\pi^2\mu'}{3\lambda^2}.$$

Putting $\lambda = 1$ and $t = 1$ we get 1 to 0·4923, or 2 to 1 nearly, for the ratio in which the intensity is altered during one second in the case of a series of waves an inch long. The rate of diminution decreases very rapidly as the length of wave increases, so that in the case of a series of waves one foot long the intensity is altered in one second in the ratio of 1 to 0·995095, or 201 to 200 nearly. It appears then that in all ordinary cases the diminution of intensity due to friction may be neglected in comparison with the diminution due to divergence. If we had any accurate mode of measuring the intensity of sound it might perhaps be just possible, in the case of shrill sounds, to detect the effect of internal friction in causing a more rapid diminution of intensity than would correspond to the increase of distance from the centre of divergence.

SECTION II.

Suggestions with reference to future experiments.

80. I am well aware that the mere proposal of experiments does not generally form a subject fit to be brought before the notice of a scientific society. Nevertheless, as it frequently happens in the division of labour that one person attends more to the theoretical, another to the experimental investigation of some branch of science, it is not always useless for the theorist to point out the nature of the information which it would be most important to obtain from experiment. I hope, therefore, that I may be permitted to offer a few hints with reference to experiments in which the theory of the internal friction of fluids is concerned. I shall omit all details, since they would properly come in connexion with the experiments.

Experiments with which the theory of internal friction in fluids has more or less to do may be performed for either of the following objects: first, to test still further the truth of the theory; secondly, to determine the index of friction of various gases, liquids, or solutions; to investigate the dependance of the index of friction of a gas on its state of pressure, temperature, and moisture; or to endeavour to make out the law according to which the index of friction of a mixture of gases depends upon the indices of friction of the separate gases; thirdly, to measure the length of the seconds' pendulum, or its variation from one part of the earth's surface to another.

81. *First object.* The theory has been already put to a pretty severe test by means of the experiments of Baily and others. Nevertheless there are some uncertainties in the comparison of theory and experiment arising from the influence of modifying causes of which the effect could only be estimated from theory, and yet was not so small as to be merged in errors of observation. Moreover, experiments on the decrement of the arc of vibration are almost wholly wanting. The following system of pendulums, meant to be swung in air and in vacuum, would afford a very good test of the theory.

No. 1. A 2-inch or $1\frac{1}{2}$-inch sphere swung with a fine wire.

No. 2. A very small sphere swung with the same kind of wire.

No. 3. A long cylindrical rod, a few tenths of an inch in diameter.

No. 4. A cylinder only three or four inches long, of the same diameter as No. 3, swung with the same kind of wire as No. 1.

The vacuum tube ought to be of sufficient size to render the estimated correction for confined space less than, or at most comparable with, errors of observation. The vacuum apparatus used by Col. Sabine would do very well. If the vacuum tube be not of sufficient size, it ought to admit of removal, and to be removed when the pendulums are swung in air.

In all the experiments the arc of oscillation ought to be carefully observed several times during the motion, the observation of the arc being quite as important for the purposes of theory as the observation of the time. Indeed, if it should be inconvenient to observe the time, the observation merely of the arc would be very valuable as a test of theory. In that case an approximate value of the time of oscillation in air would be required.

In the system proposed, Nos. 1 and 3 are the principal pendulums, Nos. 2 and 4 are introduced for the sake of making certain small corrections to the results of Nos. 1 and 3. No. 2 is meant for the elimination from No. 1 of the effect of the wire, and No. 4 for the elimination from No. 3 of the effect of the resistance experienced by a small portion of the rod near its end. The times of vibration of the four pendulums ought to be nearly the same, although for that purpose slightly different lengths of wire would be required in Nos. 1, 2, and 4.

It follows from theory that for a given pendulum the factor n is a function of the time of vibration. This is a result which seems to have been hardly so much as suspected by those who were engaged in pendulum experiments, or at most to have been mentioned as a mere possibility[*], and therefore it might be thought advisable to verify it by direct experiment. For my own part I regard it as so intimately connected with the fundamental principles of the theory, that if the theory be confirmed in other respects I think this result may be accepted on the strength of theory alone. The direct comparison with experiment would be inconvenient, because it would require a clock which kept excellent time, and yet admitted of being adjusted so as to make widely different numbers of vibrations in a day. The result could, however, be confirmed indirectly by observing the arc of vibration, an observation which is as easy with one time of vibration as with another.

82. *Second object.* According to theory, the index of friction may be deduced from experiments either on the arc or on the time of vibration. It must be left to observation to decide which give the more consistent results. Should the results obtained from the arc appear as trustworthy as those obtained from the time, it would apparently be much the easiest way of determining μ' for an elastic fluid to observe the arc, because no particular accuracy would then be required in the observation of time. As to the form of the pendulum, a cylindrical rod would apparently be the best if only a single pendulum were employed. The observation of the arc seems the only practicable way of determining the influence of temperature on the index of friction, unless the pendulum be extremely light, or unless the observer be content with the limited range of temperature which may be procured by making observations at different times of year. For in an apparatus artificially heated or cooled, it would be difficult to prevent small unknown variations of temperature, which would cause variations in the rate of vibration, in consequence of the expansion and contraction of the pendulum; and these variations would vitiate the result of the experiment, so far as the time of vibration is concerned, because the effect of the gas on the time of vibration is deduced from the small difference between two large quantities which are directly observed. But the effect of the gas on the arc of vibration produces by far the greater part of the whole diminution observed, and therefore small fluctuations of temperature would not be of much consequence, except so far as they might

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[*] It should be observed however that in a subsequent memoir (*Astronomische Nachrichten*, No. 223, p. 106), Bessel deduced from other experiments that the value of k was larger for the long than for the short pendulum.

occasion gentle currents; and even then would not be very important, because the forces thence arising would not be periodic, and dependent upon the phase of vibration of the pendulum.

The grand difficulty which besets the observation of the time of vibration of a pendulum oscillating in a liquid consists in the rapidity with which the oscillations subside. The best form of a pendulum to oscillate in a liquid would be a sphere suspended by a fine wire. The vessel containing the liquid and the sphere immersed in it ought to be so large as to render the correction for confined space insensible. But the index of friction of a liquid would probably be better determined by experiments more of the nature of those of Coulomb, or perhaps by the slow discharge of liquids through narrow tubes.

Among the gases for which μ' ought to be determined experimentally should be mentioned coal-gas, on account of the practical application which it appears possible to make of the result to the laying down of gas-pipes. The calculation of the resistance in a circular pipe is very simple, and is given in Art. 9 of my former paper. According to the equations of condition assumed in the present paper we must put $U = 0$, U denoting in that article the velocity close to the surface. It appears that the pressure spent in overcoming friction varies as the mean velocity divided by the square of the diameter of the pipe, or as the rate of supply divided by the fourth power of the diameter. This goes on the supposition that the motion is sufficiently slow to allow of our neglecting the pressure which may be spent in producing eddies, in comparison with that spent in overcoming what really constitutes internal friction.

83. *Third object.* With respect to experiments for determining the length of the seconds' pendulum, the theory of internal friction rather enables us to calculate for certain forms of pendulum the correction due to the inertia of the air than points out any particular mode of performing the experiments. Even the ordinary theory of hydrodynamics points out the importance of removing all obstacles to the free motion of the air in the neighbourhood of the pendulum if we would calculate from theory the whole correction for reduction to a vacuum.

Since the theoretical solution has been obtained in the case of a long cylindrical rod, or of such a rod combined with a sphere, we may regard a pendulum formed in this manner, and which is convertible in air, as also convertible in vacuum, for it is of small consequence whether the pendulum be or be not really convertible in vacuum, provided that if it be not we know the correction to be applied in consequence.

Note A, *Article* 65.

Let us apply the general equations (2), (3) to the fluid surrounding a solid of revolution which turns about its axis, with either a uniform or a variable motion, supposing the fluid to have been initially either at rest, or moving in annuli about the axis of symmetry.

In the first place we may observe, that the fluid will always move in annuli about the axis of symmetry. For let P be any point of space, and L any line passing through P, and lying in a plane drawn through P and through the axis of symmetry; and at the end of the time t let u' be the velocity at P resolved along L. Now consider a second case of motion, differing from the first in having the angular velocity of the solid and the initial velocity of the fluid reversed, every thing else being the same as before. It follows from symmetry, that at the end of the time t the velocity at P resolved along L will be equal to u', since the motion of the solid and the initial motion of the fluid, which form the data of the one problem, differ from the corresponding quantities in the other problem only as regards the distinction between one way round and the other way round, which has no relation to the distinction between to and fro in the direction of a line lying in a plane passing through the axis of rotation. But since all our equations are linear as regards the velocity, it follows that in the second problem the velocity will be the same as in the first, with a contrary sign, and therefore the velocity at P in the direction of the line L will be equal to $-u'$. Hence $u' = -u'$, and therefore $u' = 0$, and therefore the whole motion takes place in annuli about the axis of rotation.

Let the axis of rotation be taken for the axis of x; let ω be the angle which a plane passing through this axis and through the point P makes with the plane of xy, and let v' be the velocity at P. Then

$$u = -v' \sin \omega, \qquad v = v' \cos \omega, \qquad w = 0,$$

and all the unknown quantities of the problem are functions of t, x, and ϖ, where $\varpi = \sqrt{(x^2 + y^2)}$. Substituting in equations (2) the above values of u, v, and w, and after differentiation putting $\omega = 0$, as we are at liberty to do, we get

$$\frac{dp}{d\varpi} = 0, \qquad \frac{dp}{dx} = 0,$$

$$\mu \left(\frac{d^2 v'}{dx^2} + \frac{d^2 v'}{d\varpi^2} + \frac{1}{\varpi} \frac{dv'}{d\varpi} - \frac{v'}{\varpi^2} \right) = \rho \frac{dv'}{dt} \quad \ldots \ldots \quad (179)$$

The first two of these equations give $p =$ a constant, or rather $p =$ a function of t, which for the same reason as in Art. 7 we have a right to suppose to be equal to zero. The third equation combined with the equations of condition serves to determine v'.

Now in the particular case of an oscillating disk, the equation (179) becomes according to the mode of approximation adopted in Art. 8

$$\mu \frac{d^2 v'}{dx^2} = \rho \frac{dv'}{dt}, \quad \ldots \ldots \ldots \ldots \quad (180)$$

which in fact is the same as the second of the equations (8). The solution thus obtained is as we have seen

$$v' = \varpi f(x, t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

f denoting a function the form of which there is no need to write down, which satisfies (180) when written for v'. Now it will be seen at once that the expression (181) satisfies the exact equation (179), and therefore the approximate solution obtained by the method of Art. 8 is in fact exact, except so far as regards the termination of the disk at its edge, which is what it was required to prove.

Passing from semi-polar to polar co-ordinates, by putting $x = r\cos\theta$, $\varpi = r\sin\theta$, we get from (179), after writing $\mu'\rho$ for μ',

$$\frac{d^2v'}{dr^2} + \frac{2}{r}\frac{dv'}{dr} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv'}{d\theta}\right) - \frac{v'}{r^2\sin^2\theta} = \frac{1}{\mu'}\frac{dv'}{dt}. \quad . \quad . \quad . \quad . \quad (182)$$

Suppose now the solid to be a sphere, having its centre at the origin. Let a be its radius, $ȣ$ its angular velocity, and suppose the fluid initially at rest. Then v' is to be determined from the general equation (182) and the equations of condition

$$v' = 0 \text{ when } t = 0, \qquad v' = aȣ\sin\theta \text{ when } r = a, \qquad v' = 0 \text{ when } r = \infty.$$

All these equations are satisfied by supposing

$$v' = v''\sin\theta,$$

v'' being a function of r and t only. We get from (182)

$$\frac{d^2v''}{dr^2} + \frac{2}{r}\frac{dv''}{dr} - \frac{2v''}{r^2} = \frac{1}{\mu'}\frac{dv''}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

If we suppose $ȣ$ constant, v'' will tend indefinitely to become constant as t increases indefinitely, and in the limit $\frac{dv''}{dt} = 0$, whence we get from (183) and the equations of condition $v'' = aȣ$ when $r = a$, $v'' = 0$ when $r = \infty$,

$$v'' = \frac{ȣa^3}{r^2}, \qquad v' = \frac{ȣa^3}{r^2}\sin\theta.$$

This is the solution alluded to in Art. 8 of my paper *On the Theories of the Internal Friction of Fluids in motion, &c.*

NOTE B, *Article* 65.

Let us resume the problem of Art. 7, but instead of the motion of the plane being periodic, let us suppose that the plane and fluid are initially at rest, and that the plane is then moved with a constant velocity V, and let the notation be the same as in Art. 7.

The general equations (8) remain the same as before, but the equations of condition become in this case

$$v = 0 \text{ when } t = 0 \text{ from } x = 0 \text{ to } x = \infty,$$

$$v = V \text{ when } x = 0 \text{ from } t = 0 \text{ to } t = \infty.$$

By Fourier's theorem and another theorem of the same kind, v may be expanded between the limits 0 and ∞ of x in the following form:

$$v = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos\alpha x \cos\alpha x' \phi(x', t)\, dx' d\alpha + \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin\alpha x \sin\alpha x' \psi(x', t)\, dx' d\alpha. \quad . \quad (184)$$

In fact, v could be expanded by means of either of these expressions separately, and of course can be expanded in an infinite number of ways by the sum of the two. If however v had been expanded by means of the first expression alone, its derivatives with respect to x could not have been obtained by differentiating under the integral signs, inasmuch as the derivatives of an odd order do not vanish when $x = 0$, but would have been given by certain formulæ which I have investigated in a former paper.* A similar remark applies to the second expansion, in consequence of the circumstance that v itself and its derivatives of an even order do not vanish with x. But by combining the two expansions we may obtain the derivatives of v, up to any order i that we please to fix on, by merely differentiating under the integral signs. For we may evidently express the finite function v, and that in an infinite number of ways, as the sum of two finite functions $\phi(x, t)$, $\psi(x, t)$ which like v vanish when $x = \infty$, and which are such that the odd derivatives of the first, and the even derivatives of the second, up to the order i, as well as $\psi(x, t)$ itself, vanish when $x = 0$. Substituting now in the second equation (8) the expression for v given by (184), we see that the equation is satisfied provided

$$\frac{d\phi}{dt} + \mu'\alpha^2\phi = 0, \qquad \frac{d\psi}{dt} + \mu'\alpha^2\psi = 0.$$

These equations give

$$\phi(x', t) = \chi(x')\,\varepsilon^{-\mu'\alpha^2 t}, \qquad \psi(x', t) = \sigma(x')\,\varepsilon^{-\mu'\alpha^2 t},$$

where χ, σ denote two new arbitrary functions. Substituting in (184), and then passing to the first of the equations of condition, we get

$$0 = \chi(x) + \sigma(x),$$

whence $\sigma(x) = -\chi(x)$ and

$$v = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos\alpha(x + x')\,\varepsilon^{-\mu'\alpha^2 t}\chi(x')\, dx' d\alpha$$

$$= \frac{1}{\sqrt{\pi\mu' t}}\int_0^\infty \varepsilon^{-\frac{(x'+x)^2}{4\mu' t}}\chi(x')\, dx'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

The second of the equations of condition requires that

$$V = \frac{1}{\sqrt{\pi\mu' t}}\int_0^\infty \varepsilon^{-\frac{x'^2}{4\mu' t}}\chi(x')\, dx' = \frac{2}{\sqrt{\pi}}\int_0^\infty \varepsilon^{-s^2}\chi(2s\sqrt{\mu' t})\, ds.$$

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* On the critical values of the sums of periodic series. Camb. Phil. Trans. Vol. VIII. p. 533.

Since the second member of this equation must be independent of t, we get $\chi(x') =$ a constant, and this constant must be equal to V, since

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \epsilon^{-s^2}\,ds = 1.$$

Substituting in (185) we get

$$v = \frac{V}{\sqrt{\pi\mu' t}}\int_0^\infty \epsilon^{-\frac{(x+x')^2}{4\mu' t}}\,dx'. \qquad (186)$$

For the object of the present investigation nothing is required but the value of $\frac{dv}{dx}$ for $x = 0$, which we may denote by $\left(\frac{dv}{dx}\right)_0$. We get from (186)

$$\left(\frac{dv}{dx}\right)_0 = -\frac{V}{\sqrt{\pi\mu' t}}. \qquad (187)$$

Now suppose the plane to be moved in any manner, so that its velocity at the end of the time t is equal to $f(t)$. We may evidently obtain the result for this case by writing $f'(t')\,dt'$ for V, and $t - t'$ for t in (187), and integrating with respect to t'. We thus get

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{\pi\mu'}}\int_{-\infty}^t f'(t')\frac{dt'}{\sqrt{t-t'}} = -\frac{1}{\sqrt{\pi\mu'}}\int_0^\infty f'(t-t_1)\frac{dt_1}{\sqrt{t_1}}. \qquad (188)$$

To apply this result to the case of an oscillating disk, let $r\frac{d\theta}{dt} = rF(t)$ be the velocity of any annulus, and G the moment of the whole force of the fluid on the disk. Then

$$G = 4\pi\mu'\rho\int_0^a r^2\left(\frac{dv_0}{dx}\right)_0 dr\,;$$

and $\left(\frac{dv}{dx}\right)_0$ will be got from (188) by substituting $rF(t)$ for $f(t)$. We find thus

$$G = -\sqrt{\pi\mu'}\,.\,\rho a^4\int_0^\infty F'(t-t_1)\frac{dt_1}{\sqrt{t_1}}. \qquad (189)$$

If we suppose the angular velocity of the disk to be expressed by $A\sin nt$, where A is constant, we must put $F(t) = A\sin nt$ in (189), and we should then get after integration the same expression for G as was obtained in Art. 8 by a much simpler process. Suppose, however, that previously to the epoch from which t is measured the disk was at rest, and that the subsequent angular velocity is expressed by $A_t\sin nt$, where A_t is a slowly varying function of t. Then

$$F(t) = 0 \text{ when } t < 0, \qquad F(t) = A_t\sin nt \text{ when } t > 0.$$

On substituting in (189) we get

$$G = -\sqrt{\pi\mu'}\,.\,\rho a^4 n\int_0^t A_{t-t_1}\cos n(t-t_1)\frac{dt_1}{\sqrt{t_1}}. \qquad (190)$$

Now treating A_t as a slowly varying parameter, we get from a formula given by Mr Airy, and obtained by the method of the variation of parameters,

$$\frac{dA_t}{dt} = \frac{G}{I}\sin nt, \quad \ldots\ldots\ldots \quad (191)$$

where I denotes the moment of inertia. In the expression for G we may replace A_{t-t_1} under the integral sign by A_t outside it, because A_t is supposed to vary so slowly that A_{t-t_1} does not much differ from A_t while t_1 is small enough to render the integral of importance. Making this simplification and substituting in (191) we get

$$\frac{dA_t}{A_t dt} = -c\sin nt\int_0^t \cos n(t-t_1)\frac{dt_1}{\sqrt{t_1}}, \quad \ldots\ldots \quad (192)$$

where $c = \sqrt{(\pi\mu')}\,.\,\rho a^4 n I^{-1}$. If then A_0 be the initial and A the final value of A_t, we get from (192)

$$\log\frac{A_0}{A} = c\int_0^t\left\{\sin nt\int_0^t \cos n(t-t_1)\frac{dt_1}{\sqrt{t_1}}\right\}dt. \quad \ldots\ldots \quad (193)$$

Let now $A_0 + \Delta A_0$ be what A_0 would become if, while the final arc A and the whole time t remained the same, the motion had been going on for an indefinite time before the epoch from which t is measured, in which case the superior limit in the integral involved in the expression for G would have been ∞ in place of t. Then

$$\log\frac{A_0 + \Delta A_0}{A} = c\int_0^t\left\{\sin nt\int_0^\infty \cos n(t-t_1)\frac{dt_1}{\sqrt{t_1}}\right\}dt, \quad \ldots \quad (194)$$

whence by subtracting, member from member, equation (193) from equation (194), we get

$$\log\frac{A_0 + \Delta A_0}{A_0} = c\int_0^t\left\{\sin nt\int_t^\infty \cos n(t-t_1)\frac{dt_1}{\sqrt{t_1}}\right\}dt,$$

which becomes after integration by parts

$$\log\frac{A_0 + \Delta A_0}{A_0} = \frac{c}{4n}\left\{\sqrt{\frac{\pi}{2n}} - 2\sqrt{t}\,.\cos nt - \cos 2nt\int_t^\infty \cos nt\frac{dt}{\sqrt{t}}\right.$$
$$\left. + (2nt - \sin 2nt)\int_t^\infty \sin nt\frac{dt}{\sqrt{t}}\right\}. \quad \ldots \quad (195)$$

Now t is supposed to be very large: in Coulomb's experiments in fact 10 oscillations were observed, so that $nt = 10\pi$. But when t is at all large the two integrals

$$\int_t^\infty \cos nt\frac{dt}{\sqrt{t}}, \qquad \int_t^\infty \sin nt\frac{dt}{\sqrt{t}}$$

can be expressed under the forms

$$-P\sin nt + Q\cos nt, \qquad P\cos nt + Q\sin nt,$$

where

$$P = n^{-1}t^{-\frac{1}{2}} - 1\,.\,3\,.\,2^{-2}n^{-3}t^{-\frac{5}{2}} + \ldots, \qquad Q = 1\,.\,2^{-1}n^{-2}t^{-\frac{3}{2}} - 1\,.\,3\,.\,5\;\, 2^{-3}n^{-4}t^{-\frac{7}{2}} + \ldots,$$

series which are at first rapidly convergent, and which enable us to calculate the numerical values of the integrals with extreme facility. These expressions were first given by M. Cauchy, in the case of Fresnel's integrals, to which the integrals just written are equivalent. They may readily be obtained by integration by parts, though it is not thus that they were demonstrated by M. Cauchy. If now the above expressions be substituted for the integrals in (195) the terms containing $t^{\frac{1}{2}}$ destroy each other, and for general values of t the most important term after the first contains $t^{-\frac{1}{2}}$. Since however t is supposed to correspond to the end of an oscillation, so that nt is a multiple of π, the coefficient of this term vanishes, and the most important term that actually remains contains only $t^{-\frac{3}{2}}$. Hence neglecting insensible quantities we get from (195)

$$\log \frac{A_0 + \Delta A_0}{A_0} = \frac{c}{4n}\sqrt{\frac{\pi}{2n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

We get from (194) by performing the integrations

$$\log \frac{A_0 + \Delta A_0}{A} = c\sqrt{\frac{\pi}{2n}} \int_0^t \sin nt\,(\cos nt + \sin nt)\,dt$$

$$= \frac{c}{4n}\sqrt{\frac{\pi}{2n}} \{2nt + 1 - \cos 2nt - \sin 2nt\},$$

which becomes since nt is a multiple of π

$$\log \frac{A_0 + \Delta A_0}{A} = \frac{c}{4n}\sqrt{\frac{\pi}{2n}} \, . \, 2nt. \quad . \quad . \quad . \quad . \quad . \quad (197)$$

We get from (196) and (197)

$$2nt \log \frac{A_0 + \Delta A_0}{A_0} = \log \frac{A_0 + \Delta A_0}{A} = \log \frac{A_0 + \Delta A_0}{A_0} + \log \frac{A_0}{A},$$

whence

$$\log \frac{A_0 + \Delta A_0}{A_0} = (2nt - 1)^{-1} \log \frac{A_0}{A}, \quad . \quad . \quad . \quad . \quad . \quad (198)$$

and the same relation exists between the common logarithms of the arcs, which are proportional to the Napierian logarithms. Now Log A_0 − Log A is the quantity immediately deduced from experiment, and Log $(A_0 + \Delta A_0)$ − Log A_0 is the correction to be applied, in consequence of the circumstance that the motion began from rest. Instead of applying the proportionate correction $+ (2nt - 1)^{-1}$ to the difference of the logarithms, we may apply it to the deduced value of $\sqrt{\mu'}$, which is proportional to the difference of the logarithms. In Coulomb's experiments 10 oscillations were observed, and therefore $2nt = 20\pi$, and $(2nt - 1)^{-1} = 0.01617$, and the uncorrected value of $\sqrt{\mu'}$ being 0.0555, we get 0.0009 for the correction, giving $\sqrt{\mu'} = 0.0564$.

NOTE C. *Article* 50.

The results mentioned in this article were originally given without demonstration; but as the mode in which they were obtained is short, and by no means obvious, I have thought it advisable to add the demonstrations.

In order that the right-hand members of equations (138) may be perfect differentials, we must have

$$\frac{d\delta}{dy} + \frac{d\omega'''}{dx} = 0, \quad \frac{d\delta}{dz} + \frac{d\omega'}{dy} = 0, \quad \frac{d\delta}{dx} + \frac{d\omega''}{dz} = 0, \quad . \; . \; . \quad (a)$$

$$\frac{d\delta}{dz} - \frac{d\omega''}{dx} = 0, \quad \frac{d\delta}{dx} - \frac{d\omega'''}{dy} = 0, \quad \frac{d\delta}{dy} - \frac{d\omega'}{dz} = 0, \quad . \; . \; . \quad (b)$$

$$\frac{d\omega''}{dy} + \frac{d\omega'''}{dz} = 0, \quad \frac{d\omega'''}{dz} + \frac{d\omega'}{dx} = 0, \quad \frac{d\omega'}{dx} + \frac{d\omega''}{dy} = 0. \quad . \; . \; . \quad (c)$$

The equations (*c*) give

$$\frac{d\omega'}{dx} = 0, \quad \frac{d\omega''}{dy} = 0, \quad \frac{d\omega'''}{dz} = 0. \quad . \; . \; . \; . \; . \; . \quad (d)$$

In the particular case in which $\delta = 0$, the equations (*a*), (*b*), and (*d*) give

$$d\omega' = 0, \qquad d\omega'' = 0, \qquad d\omega''' = 0,$$

and therefore ω', ω'', and ω''' are constant as stated in Art. 50. In the general case the equations (*a*), (*b*), and (*d*) give for the differentials of ω', ω'', and ω''' the following expressions:

$$\left.\begin{aligned} d\omega' &= -\frac{d\delta}{dz}\,dy + \frac{d\delta}{dy}\,dz, \\ d\omega'' &= -\frac{d\delta}{dx}\,dz + \frac{d\delta}{dz}\,dx, \\ d\omega''' &= -\frac{d\delta}{dy}\,dx + \frac{d\delta}{dx}\,dy. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (e)$$

In order that the right-hand members of these equations may be perfect differentials, we must have

$$\frac{d^2\delta}{dy\,dz} = 0, \quad \frac{d^2\delta}{dz\,dx} = 0, \quad \frac{d^2\delta}{dx\,dy} = 0, \quad . \; . \; . \; . \; . \; . \quad (f)$$

$$\frac{d^2\delta}{dy^2} + \frac{d^2\delta}{dz^2} = 0, \quad \frac{d^2\delta}{dz^2} + \frac{d^2\delta}{dx^2} = 0, \quad \frac{d^2\delta}{dx^2} + \frac{d^2\delta}{dy^2} = 0,$$

and therefore

$$\frac{d^2\delta}{dx^2} = 0, \quad \frac{d^2\delta}{dy^2} = 0, \quad \frac{d^2\delta}{dz^2} = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (g)$$

The equations (f), (g) give

$$d\frac{d\delta}{dx} = 0, \qquad d\frac{d\delta}{dy} = 0, \qquad d\frac{d\delta}{dz} = 0,$$

so that $\frac{d\delta}{dx}$, $\frac{d\delta}{dy}$, and $\frac{d\delta}{dz}$ are constant. Substituting in (e) and integrating, and then substituting in (138) the resulting expressions for ω', ω'', and ω''', and integrating again, we shall obtain the results given in Art. 50.

G. G. STOKES.

XI. *On some points of the Integral Calculus.* By AUGUSTUS DE MORGAN, *of Trinity College, Secretary of the Royal Astronomical Society, and Professor of Mathematics in University College, London.*

[Read Feb. 24, 1851.]

THIS Paper contains some investigations on singular solutions, the integration of differential equations by elimination, the question of the general form of solution, and the general test of integrability.

SECTION I.

On the singular solution of a differential equation of the first order.

THE equation $\phi(x, y, c) = 0$ belongs, in a plane, to a family of curves, the individuals of which are distinguished by their values of c. And x and y must have real values; but these of c may be imaginary.

Let $c = a + b\sqrt{-1}$, and let $\phi(x + y\sqrt{-1}) = A + B\sqrt{-1}$, where A and B are two functions of x, y, a, and b. When $\phi = 0$, we have $A = 0$, $B = 0$; which signify a number of isolated points determined by the intersections of the curves, $A = 0$, $B = 0$. In certain cases only does it happen that, for one value of a, and one of b, both $A = 0$ and $B = 0$ can be satisfied by any point on a certain curve. One of these cases, of course, and the most important, occurs when $b = 0$.

It is impossible that A and B should have a common factor* containing a only, or b only: for then $\phi(x, y, a + b\sqrt{-1}) = 0$ would exist under a relation between x and y and one of the two, a or b, independently of the other. Neither can A and B have a common factor containing both a and b in a real form, as two distinct constants. For if M be the most complete common factor, and if $\phi = M(P + Q\sqrt{-1})$, so that $P = 0$ and $Q = 0$ are not satisfied by any but determinate values of x and y, the only differential relation between x, y, y', y'', under which ϕ always vanishes, is the second differential equation deduced from $M = 0$. But this is contrary to what we know of $\phi(x, y, c) = 0$; which is satisfied by an infinite number of relations between x, y, y', and y''.

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* By A and B having M for a common factor, in the widest sense, it should be meant that neither $A \div M$ nor $B \div M$ becomes infinite when M vanishes. When it is not so, I generally call M, in $M \times (A \div M)$, a *fictitious* factor; and I find it useful to have language to express the difference of these cases. This definition of the word *factor* is more extensive than that of common algebra: and so is that of the word *homogeneous* when we define an homogeneous function of the nth degree as that which has the form $x^n \phi\left(\frac{y}{x}\right)$.

It is possible that either A or B may contain a factor which is a function of a or b, or both, and not of x nor of y. By making A or B, whichever it may be, vanish by this factor, and the other by a relation between x, y, a, and b, we obtain from the two equations a real family of curves, satisfying $\phi = 0$. And B always has some positive power of b for a factor, which gives the most common case of this kind. Moreover, it is possible for an isolated imaginary value of c to give a curve: as in $x^2 + (c^2 + c)\,y = c\,(c^2 + c + 1)$ which gives continuous curves for values of c determined by $c^2 + c + 1 = 0$, and for no other imaginary values.

If ever it happen that a continuous series of imaginary values of c make $\phi\,(x, y, c) = 0$ the equation of a continuous series of curves, it is a sign that this equation can be thrown into the form $\psi\,(x, y, \varpi c) = 0$, where the imaginary values of c give real values to ϖc. For it is certain that the equation of a real family of curves can always be thrown into the form $\psi\,(x, y, k) = 0$, in which k is real.

Since $\phi\,(x, y, c) = 0$ gives values of c, real or imaginary, for every value of x and y, it follows that every point in the plane of xy is either a point in a continuous curve of solution, or an isolated point, or both: and it may be either or both of these in several different ways. As the point changes place, a critical pair of values of x and y may arise, at which the circumstances alter: an infinitely small change may bring us, for instance, from a point in six ways isolated and ten ways connected, to one in four ways isolated and twelve ways connected. Let $c = a + \sqrt{b}$, and, ϕ meaning $\phi\,(x, y, a)$, and $\phi_a\ \phi_{aa}$, &c, its differential coefficients with respect to a, let Taylor's theorem apply. We have then

$$\phi\,(x, y, a + \sqrt{b}) = \phi + \phi_{aa}\frac{b}{2} + \ldots + \sqrt{b}\,.(\phi_a + \phi_{aaa}\frac{b}{2.3} + \ldots).$$

If b be very small and negative, the conditions under which the equation can be satisfied are $\phi = 0$, $\phi_a = 0$, nearly. Accordingly, if a be eliminated between $\phi = 0$, $\phi_a = 0$, we have the equation of the curve at which the changes of character take place, whenever they happen by b passing through zero, under values to which Taylor's theorem will apply as far as ϕ_{aaa} inclusive.

For example, let the given equation be $y - fc - f'c\,(x - c) = 0$, in which the second equation must be $f''\,c\,.\,(x - c) = 0$. All the straight lines in this family are tangents to $y = fx$, and we know that passage of a point over a curve generally makes the number of tangents which can be drawn through the point greater or less by an even number. But $f''c = 0$ may indicate a point of contrary flexure: and a little consideration will shew that passage over *any part* of the tangent at a point of contrary flexure alters the number of tangents which can be drawn through a point.

When the modern analysis first impressed mathematicians with a sense of its power, it would seem as if they accepted it as fit to give results of the same extent as those of geometry. Perhaps they forgot that geometry makes its apparent universality of conclusion valid by preliminary restrictions. Euclid excludes one singular case of angular magnitude, the angle of a straight line and its continuation, altogether: and the extreme case of intersection of two lines is excluded by referring it to a separate name, *parallelism*. Hence it is that two sides of a triangle are *always* together greater than the third; and that two intersecting straight lines *always* make different angles on the same side with a third. Analysis, without any such saving of extremes, launched itself into competition with the guarded generalities of geometry; of which we feel the effects to this day. We do at length admit the necessity of a special inquiry into the cases of every formula which arise out of evanescence or infinitude: but we do not yet know the full interpretation of the maxim that what is true up to the limit is true at the limit; nor do we know whether this maxim is, in any sense, universally true. Among the cases which offer difficulty, the following may be noted. When there are phenomena, if such a term may be applied to mathematical results, which are generally simultaneous, but which present occasional peculiarities, it has been usually acted upon, if not expressly affirmed, that the criteria are either conclusive as to all, or inconclusive as to all. That is, that if P and Q be two tests, of which Q is deducible from P, each denoting usually the co-existence of the phenomena A, B, C, the exceptional cases are all supposed to be equally excepted from both tests. But it may happen, that while P continues, in the extreme and exceptional case, to point out the presence or absence of A, without deciding on B and C, Q may decide on B or C without reference to A.

When a family of curves, $\phi(x, y, c) = 0$, is completely drawn, we may separate from among the infinite number of curves which are equally related to all, the three following. First, a curve of *separation*, over which when a point passes, the number of individuals of the family which pass through the point undergoes an alteration. Secondly, a curve of *contact*, every point of which is touched by one or more individuals of the family. Thirdly, a curve of *resilience*, at every point of which an individual of the family stops and recedes, without intersection. These three usually go together: as in the case of a family of equal circles with their centres upon one straight line, to all of which a certain pair of parallel lines is a curve of separation, contact, and resilience. But a curve may have one of the above names without the other two, or two without the third. Thus the circles which touch an involute have contact without resilience, and the involute may or may not be a curve of separation. And a locus of cusps may be a curve of resilience without contact, and may or may not be a curve of separation. It is also possible that one individual of the family itself may be a curve of either kind to the rest.

The *singular solution* of a differential equation has been usually defined as the solution which is not any case of the general solution. In this paper I propose to apply the term to any solution whatsoever, be it contained in the general primitive or not, which results from any process that cannot introduce an arbitrary constant: reserving the phrase *extraneous solution* to signify any solution which is not a case of the general primitive. It has long been known that cases of the complete primitive, when detected by the tests for extraneous solutions, fulfil all the conditions of the latter, in a great many cases, with the single exception of the one from which the name is derived. It is also well known that none of the ordinary modes of detecting extraneous solutions are sufficient to separate them from the *other* singular solutions.

Let the equation of the family of curves be $\phi(x, y, c) = 0$, or $c = \Phi(x, y)$; and let Φ_x be the partial[*] differential coefficient of Φ with respect to x, &c.

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[*] I never use the symbol y_x except where y is explicitly a function of x. If $y = x$, for instance $y_x = 1$: but if $y - x = 0$, $\frac{dy}{dx}$ must be employed, and it is $-(y-x)_x \div (y-x)_y$.

The differential equation is $y' = -\Phi_x : \Phi_y$, say $y' = \chi(x, y)$. Not confining ourselves to the supposition that c is constant, we have $dc = \Phi_x dx + \Phi_y dy$, whence

$$(y' - \chi)dx = \frac{dc}{\Phi_y} = -\frac{\chi \,.\, dc}{\Phi_x}.$$

Now if either of the two, Φ_x and Φ_y, be made infinite by a relation between x and y, Φ *remaining finite*, it can only be in one of three ways: (1), $\Phi_x = \infty$, $\Phi_y = \infty$; (2), $\Phi_x = \infty$ only, and $x = \text{const.}$; (3), $\Phi_y = \infty$ only, and $y = \text{const.}$ For if $\Phi_x = \infty$, $\Phi_x dx + \Phi_y dy$ cannot be an ordinary differential, except either $\Phi_y = \infty$, or $dx = 0$. On any other supposition, Φ would have a permanently infinite differential coefficient, which a finite function cannot have.

Accordingly, except by $dc = 0$, giving the ordinary primitive, $y' - \chi = 0$ is solved only as follows. Either $\Phi_y = \infty$, $\Phi_x = \infty$; or $\Phi_y = \infty$, $y = \text{const.}$, which gives $\chi = 0$, $y' = 0$; or $\Phi_x = \infty$, $x = \text{const.}$, which gives $y' = \infty$, $\chi = \infty$.

That is, every mode of satisfying either of the following systems,

$\Phi_x = \infty$, $\Phi_y = \infty$; $\Phi_x = \infty$, and not Φ_y, $x = \text{const.}$; $\Phi_y = \infty$ and not Φ_x, $y = \text{const.}$

is a singular solution: and by this mode all the singular solutions are obtained.

And since $\Phi_x = -\phi_x : \phi_c$, $\Phi_y = -\phi_y : \phi_c$, these last fractions may be substituted in the tests.

Any two solutions which have a point[*] in common have a contact of at least the first order at that point, since y' is $\chi(x, y)$ for both. But it often happens in analysis, that a result which is *à priori* evident from the conditions of the question, cannot be universally verified, by reason of the difficulty of the extreme cases. If we make c a function of x and y, no doubt

$$\phi(x, y, c) = 0 \text{ gives}$$

$$\frac{\phi_x}{\phi_y} + y' + \frac{\phi_c}{\phi_y}\left(\frac{dc}{dx} + \frac{dc}{dy} . y'\right) = 0, \text{ or } y' - \chi = -\frac{\phi_c}{\phi_y}\left(\frac{dc}{dx} + \frac{dc}{dy} . y'\right).$$

If then we would have curves contained under $y' - \chi = 0$, but not under $c = \text{const.}$, we must look for them by determining c from $\phi_c : \phi_y = 0$. But if, which may happen for aught we can see to the contrary, the same value of c which is thus obtained, substituted in $\phi(x, y, c) = 0$, leads to a relation between x and y under which $dc : dx$ and $dc : dy$ are infinite, we can no longer be sure that $y' - \chi = 0$ is satisfied.

The following geometrical interpretation will add to our power over the meaning of the singular solution. Let x, y, c, be the three co-ordinates of a point in a surface, of which the equation is $\phi(x, y, c) = 0$, or $c = \Phi(x, y)$. Let the plane of xy be horizontal. Then, c being constant, $c = \Phi$ is, in the horizontal plane, the equation of the projection of a horizontal section of the surface. The projecting cylinders of these sections may be called *vertical*. Of other vertical cylinders we have especially to consider the one we may call the *tangent* cylinder, which has the same tangent plane with the surface at every point of concourse. Its equation gives $\Phi_x = \infty$, $\Phi_y = \infty$, except when it is a plane parallel to xz or yz, in which cases its equation gives $\Phi_y = \infty$, $y = \text{const.}$ and $\Phi_x = \infty$, $x = \text{const.}$

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[*] It is here understood that if $\chi(x, y)$ be of multiple value, we speak of *one* of the values.

The base-curve of the tangent cylinder is the singular solution of $y' = \chi$, to which all the projections of the horizontal sections are ordinary solutions. When the tangent cylinder meets the surface in an oblique curve, to every point of which a distinct horizontal section of course belongs, then the singular curve is touched by a primitive curve at every point. If from any point (x, y, c) of the surface, we proceed to another $(x + dx, y + dy, c + dc)$ infinitely near to it, the equation $dc = \Phi_x dx + \Phi_y dy$ usually determines the ratios of dc to dx and to dy in terms of y'. This connexion is momentarily severed when Φ_x and Φ_y, either or both, become infinite. If we take advantage of this to assume dc in such manner that Φ_x or Φ_y or both (as before) shall remain infinite when x, y, c, become $x + dx$, $y + dy$, $c + dc$, and if we repeat this process, we draw a curve through the vertical elements of the surface, being the curve of concourse with the tangent cylinder. The projection of any point in this curve of concourse is a point at which two projections have a common tangent, the two being those of the curve of concourse, and of the horizontal section passing through the point. If the tangent cylinder touch in no oblique curve, but only in horizontal sections, the singular solution consists of primary solutions only, which have no contacts with others. But if the tangent cylinder not only touch in an oblique curve, but also touch or cut in a horizontal section, then the singular curve is itself a primary curve, touched by another primary in each point of it. And if the curve of concourse have convolutions or branches, all oblique, each point of the singular curve may be the point of contact with several different primaries. To consider only extraneous solutions, to the exclusion of other singular solutions, is not to mark out, as objects of special attention, those surfaces whose tangent cylinders have oblique curves of concourse, to the exclusion of those which have them only horizontal. There is another, and a most unsystematic exclusion: that of all cases in which the tangent cylinder touches in an oblique curve, and *also* touches or cuts in a horizontal one. When $\Phi_x = \infty$, &c., gives only $c =$ const., the curve of concourse is only a horizontal section: but when it gives $c = \psi(x, y)$, and $\phi\{x, y, \psi(x, y)\} = 0$ is equivalent to $\phi(x, y, k) = 0$, k being some definite value of c, there is an oblique curve of concourse, and also a horizontal section. For instance, the primitive being $y = x + c(x + \sqrt{c})^2$, the branches of the singular solution are obtained from $x + \sqrt{c} = 0$, and $x + 2\sqrt{c} = 0$. The first gives $y = x$, the same as arises from $c = 0$; the second gives the extraneous solution. But both give curves of separation, of contact, and of resilience, to the primary curves.

The following instance will serve to illustrate that when the singular solution is itself a primary, which is neither touched nor cut by other primaries, and is nothing but a curve of separation, it is still one of those extreme cases which analysis admits when it clearly sees the way up to the limit. Let the primitive equation be

$$x^2 + mxy + ny^2 + (ax + by)c + pc^2 = q,$$

m, n, &c. being fixed constants: and let the surface be an ellipsoid. Draw the vertical cylinder which touches this surface, and let a plane parallel to xy, beginning and ending as a tangent plane to the ellipsoid, move so as to pass through all the intermediate horizontal sections. It will then be obvious, that the connexion of the primitives and singular, in the plane of xy, is as follows. The singular curve is an ellipse, in which are two points, say A and B, similarly situated on opposite sides of its centre. The primitive is an elliptic wave which, beginning from *A* (not its centre) spreads without touching the singular ellipse, until at last it touches it in one point, with a contact of the second order. Immediately afterwards two points of contact of the first order separate, which move in opposite directions as the wave proceeds, until they meet on the opposite side, and there is again a contact of the second order. The contact then disappears, and the wave diminishes, vanishing at *B*. Now the more near the ellipsoid is to a sphere, the smaller is the duration of the period of contact: and when the ellipsoid becomes a sphere, there is but an instantaneous contact, the circular wave arriving at complete coincidence with the singular circle, as to a terminal position, and then receding.

This geometrical view may fail in one particular. There may be horizontal projections with no sections belonging to them, owing to c being imaginary. The two surfaces, $\phi(x, y, c) = 0$ and $\phi(x, y, \varpi c) = 0$, have the same tangent cylinder, but meet it in different curves: they have the same horizontal *projections*, but a given projection has not *sections* in both, unless c and ϖc be both real for the value of c.

I now proceed to the consideration of the test derived from the differential equation itself. This is usually stated as follows:—The singular (meaning extraneous) solution is to be looked for among those relations which make χ_y infinite, $y' = \chi(x, y)$ being the differential equation. Remember that $\phi(x, y, c) = 0$ and $\phi_x + \phi_y\chi = 0$ are equations of identical meaning, and treat c as a function of three variables, x, y, χ, obtained from the second. Substitution of this value of c in the first will make it identical, if for χ we write its value in terms of x and y. That is,

$$\phi_x + \phi_c\left(\frac{dc}{dx} + \frac{dc}{d\chi}\chi_x\right) = 0, \qquad \phi_y + \phi_c\left(\frac{dc}{dy} + \frac{dc}{d\chi}\chi_y\right) = 0,$$

$$\chi_x\frac{dc}{d\chi} = -\left(\frac{\phi_x}{\phi_c} + \frac{dc}{dx}\right), \qquad \chi_y\frac{dc}{d\chi} = -\left(\frac{\phi_y}{\phi_c} + \frac{dc}{dy}\right),$$

$$(\chi_x + \chi_y\chi)\frac{dc}{d\chi} = -\left(\frac{dc}{dx} + \frac{dc}{dy}\chi\right).$$

If, as will most usually happen, the differential coefficients of c be of ordinary form, then χ_x and χ_y are finite or infinite with $\phi_x : \phi_c$ and $\phi_y : \phi_c$: that is, χ_x and χ_y may be substituted for Φ_x and Φ_y in the test already examined. But the possibility that $dc : d\chi$ may vanish, or that $dc : dx$ or $dc : dy$ may become infinite, must immediately strike the mind. If indeed we can be assured that the concourses of the singular curves with primitives take place at points of finite curvature, so that y'' or $\chi_x + \chi_y\chi$ or $(c_x + c_y\chi) \div c_x$ is ordinary and finite, we much increase the presumption that c_x and c_y are ordinary, but still we have not proved it. This difficulty puts the complete meaning of the test in question beyond the reach of ordinary analysis. The following mode of examination will succeed completely.

When two solutions of $y' = \chi(x, y)$ meet (and therefore touch), it follows that χ_x or χ_y, one or both, must be infinite, except only in some of the cases in which one at least of the curves is of infinite curvature. The following sketch of a proof may easily be reduced into the language of algebra and of limits.

Let P be the point of concourse; take Q and R infinitely near to it on the two curves, and let the tangents at Q and R meet in T. If the increments of the angles of revolution of the tangents have a finite ratio, that is, if the curvatures be both finite (or both nothings or infinites of the same order) then RT and QT are of the same order, and both infinitely great compared with QR. Hence the angle RTQ is infinitely great compared with QR, subject, it may be, to exception for some isolated directions of QR: and this angle is $d\chi \div (1+\chi^2)$; $d\chi$ being formed in the passage from Q to R. Hence $\chi_x dx + \chi_y dy$ is generally infinitely great compared with $\sqrt{(dx^2+dy^2)}$, and independently of dx and dy: from which it follows that χ_x or χ_y is infinite at P.

Conversely, let there be a curve on which χ_x or χ_y is infinite. Take a point P on this curve which is not of infinite curvature, and if a primitive of $y'=\chi$ can be drawn through this point, take Q and R on the given curve and on the primitive, infinitely near to P. Then if the primitive be also of finite curvature, and if tangents be drawn as before, it follows that, as $d\chi$ is infinitely great compared with $\sqrt{(dx^2+dy^2)}$, the angle QTR is infinitely great compared with QR. From this it follows that there is a contact of the first order at least, or that y' is the same in both curves, or that $y'=\chi$ is satisfied on the first curve. All this requires more amplification, and will not be received without objection except by those who give it.

We collect then the following theorem, remembering that at a point which has not infinite curvature, y'' or $\chi_x+\chi_y\chi$ is not infinite. Any relation which satisfies $\chi_x=\infty$ or $\chi_y=\infty$ certainly satisfies $y'=\chi$ whenever $\chi_x+\chi_y\chi$ is not infinite, and *may* satisfy $y'=\chi$ when $\chi_x+\chi_y\chi$ *is* infinite. And in the first case, the curve $\chi_x=\infty$ or $\chi_y=\infty$ has contact with all the primitives which meet it, but in the second, whether with or without contact, it is the locus of points of infinite curvature in the primitives (most commonly cusps). But when $\chi_x+\chi_y\chi$ is infinite, it does not follow that $y'=\chi$ is satisfied.

The following is another proof of the novel part of this theorem. Let the point (x, y) be called the pole of the straight line

$$\eta - y = \chi(x, y) \,.\, (\xi - x).$$

This line is tangent to any solution of $y'=\chi$, which passes through (x, y). And in order that a family of straight lines may all touch such a solution, the ultimate intersection of any two which are infinitely near to each other must be the pole of either. Now the intersection of the line above given with that in which x, y, χ become $x+\Delta x$, $y+\Delta y$, $\chi+\Delta\chi$, is determined by the equations

$$\xi - (x+\Delta x) = -\frac{\Delta y - \chi\Delta x}{\Delta\chi} \qquad \eta - (y+\Delta y) = -\frac{\Delta y - \chi\Delta x}{\Delta\chi}(\chi+\Delta\chi).$$

So that, when the ultimate intersection is the pole itself, we must have

$$\frac{m-\chi}{\chi_x+\chi_y m} = 0;$$

m being the limiting ratio of Δy to Δx.

If P and Q be the two contiguous points, and T the ultimate intersection of their polar lines, we now see as follows. If P be *not* on the curve $\chi_x=\infty$ or $\chi_y=\infty$, there is only one direction of PQ for which T *does* coincide with P, namely, that determined by $m = \chi$. But if P *be* on that curve, there is only one direction (and there may not even be that one) in which T does *not* coincide with P, namely, the direction determined by $m = -\chi_x \div \chi_y$.

And this exceptional direction is precisely that of the tangent of the curve. For $\chi_x = a$ has the direction of its tangent defined by $y' = -\chi_{xx} \div \chi_{xy}$; and $\chi_y = a$ by $y' = -\chi_{xy} \div \chi_{yy}$. Both of these, when χ_x and χ_y become infinite, take the same value as $-\chi_x \div \chi_y$. But if χ_x be infinite without χ_y, it is through $x =$ const., giving y' also infinite: and if χ_y be infinite without χ_x, it is through $y =$ const., giving $y' = 0$; so that the exceptional direction is tangent to the curve in those cases also.

If then, at a point on the curve which gives χ_x or χ_y infinite, we find $\chi_x + \chi_y \chi$ finite, it follows that $y' = \chi$ on that curve, or the curve is a singular solution of $y' = \chi$: while it does not follow that such is not the case, even if $\chi_x + \chi_y \chi$ be infinite. In this last case there is a singular solution if, as in the former, $\chi = -\chi_x \div \chi_y$ be found true.

I think the preceding detail will be necessary to prevent suspicion from attaching to the following short, and I believe conclusive, account of the matter. On a curve on which χ_x and χ_y are both infinite, y' is determined by $-\chi_{xx} \div \chi_{xy}$ or $-\chi_{xy} \div \chi_{yy}$, both fractions of the form $\infty \div \infty$, and both having the same value, or extreme form of value (0 or ∞) with $-\chi_x \div \chi_y$. Now if $\chi_x + \chi_y \chi = P$, we have $\chi = -\dfrac{\chi_x}{\chi_y} + \dfrac{P}{\chi_y}$, and χ is the value of y' on the curve, whenever $P \div \chi_y = 0$; that is, certainly when P is finite, and as it may happen when P is infinite. But when $\chi_y = \infty$ and not χ_x, we have $y =$ const. and $y' = 0$ on the curve; so that $y' = \chi$ is satisfied when $P \div \chi_y = 0$; and similarly for the case in which χ_x is infinite without χ_y. The reason why this proof is never given, probably lies in the want of the habit of applying the determining process for $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ to functions of two variables so called, in the case in which one of them is constant.

Nothing which precedes necessarily applies to any case in which $x = \infty$ or $y = \infty$ is the equation to be examined. And I have not made any examination of the cases in which $\chi = \infty$, since this is only an accident of the position of the axes.

As an example, let $y' = a + \sqrt{(x-y)}$. Here $\chi_x = \infty$, $\chi_y = \infty$ are satisfied only when $y = x$. And $y'' = -\frac{1}{2} + \frac{1}{2}(1-a)(x-y)^{-\frac{1}{2}}$ which is only finite for $x = y$ in the case of $a = 1$. And in this case $y = x$ is obviously a solution. When a is not $= 1$, there is no singular solution, unless $x = \infty$ or $y = \infty$, should give one. But $y = x$ is nevertheless a straight line which passes through cusps of primitives, having infinite curvature. The primitive equation is

$$x = c + 2(1-a)\log(1 - a - \sqrt{x-y}) - 2\sqrt{(x-y)}.$$

Let $yy' + x = f(y')$. This is the equation of the family of curves which always cuts at right angles the family of straight lines $y - y'x = -y' f(-y'^{-1})$: so that the first equation is that of the involutes of the singular solution of the second. The singular solution, if any, of the first is found by eliminating y' between it and $y = f'y'$: and this singular solution is the evolute.

But y'' is made infinite by the result: and the differential equation is not satisfied. Consequently, the evolute has the general characteristic of a singular solution of the equation of the involutes, but is really a curve passing through their cusps of infinite curvature. This was noticed in my *Differential Calculus*, pp. 364—367.

Let $y' = p$, $y'_1 = q$ be the differential equations of two families of curves, and let $f(x, y, y', y'_1) = 0$ be a relation which exists at every intersection. Then $f(x, y, p, q) = 0$ is an identical equation, and

$$f_p p_x + f_q q_x + f_x = 0, \quad f_p p_y + f_q q_y + f_y = 0.$$

In all the cases in which $p_x = \infty$, $p_y = \infty$ make $q_x = \infty$, $q_y = \infty$, the two families have the same singular solution, or locus of infinite curvatures, or the singular solution of one is the locus of infinite curvatures to the other.

The manner in which the test just considered was first obtained, was, in its final step, as follows:—"...d'où l'on tire $f'(x) = \dfrac{F'(x)}{F'(\phi) \times \phi'(y')}$, $f'(y) = \dfrac{F'(y)}{F'(\phi) \times \phi'(y')}$. Or l'équation primitive singulière rend $F'(\phi) = 0$; donc elle rendra infinie les deux fonctions $f'(x)$ et $f'(y)$..." I quote Lagrange's words to recall the very decided manner in which he speaks of the effect of $F'(\phi) = 0$, without apparently casting a thought upon the possible accompaniment of $\phi'(y') = \infty$, or of $F'(x) = 0$ and $F'(y) = 0$. But we are to remember that when Lagrange wrote, all general theorems were looked upon as liable to failure in particular cases: the tact of the mathematician detected the usual way in which an extreme case arose, and the unusual ways were left for consideration. The practice of demonstrating the universal truth of Taylor's theorem in one chapter, and its occasional falsehood in another, was not an isolated case, but a formal specimen of what was done whenever it was convenient. The reasonable alteration, in the case before us, would be to extend the result, and to state that χ_x and χ_y {which answer to $f'(x)$ and $f'(y)$} are either infinite or else $\frac{0}{0}$ or one of its congeneric forms. This is the test arrived at by M. Cauchy, and is subject to the following difficulty. If we could be secure of arriving at χ_y in the *form* $P \div (QR)$ we might certainly infer as above from $R = 0$. But χy is subject to all the accidents of change of form which arise from disappearance of factors, &c. and it may well happen that $P \div (QR)$ shews the form $\frac{0}{0}$, while a transformed equivalent, being the one we actually obtain in a given example, shews the ordinary value to which that form is reducible. The test deduced is, I venture to say, imperfect: for though admitting, of course, that $\frac{0}{0}$ must be examined as well as other cases, I think I have shewn that unless $\frac{0}{0}$ be, *pro vice*, infinite, there is no singular solution. The method followed is, first, a constructive solution of $y' = \chi(x, y)$, by assuming that $x = x_0$ gives $y = y_0$, and forming the polygon in the limit of which the curve of solution is seen; all done in algebraic terms, without reference to geometry, and for values both real and imaginary. This first part (Moigno, Vol. II. pp. 385-434) is a masterly specimen of a power which distinguishes all the later writings of M. Cauchy on fundamental points, that of recurring, in the treatment of elevated subjects, to the very consideration of successive finitely differing values of a function, by which a beginner learns common differentiation and integration. The conclusion is, that $y' = \chi$ has a solution in which $x = x_0$, $y = y_0$ whenever χ_y is finite and continuous from $x = x_0$, through any finite extent. This limitation is not meant to be an exclusion of what is without: and it is followed (pp. 435, &c.) by proof that when two or more solutions agree in giving $x = x_0$, $y = y_0$, we must have either $\chi = \frac{0}{0}$, or $\chi_y = \frac{0}{0}$ or $\chi_y = \infty$, for the case in question. Hence the singular solution is detected, provided it be granted that whenever only one solution gives $x = x_0$ $y = y_0$, that one is an ordinary primitive. This point is asserted in p. 435, and I cannot find any proof of it in what goes before. Until the contrary be disproved (the geometrical illustration I have given does disprove it in *real* cases) we have no right to assume that every extraneous solution has at every point, concourse with an ordinary primitive. I may also remark that M. Cauchy does not arrive at the cases in which only χ_x is infinite. Thus $y' = x(1 - x^2)^{-\frac{1}{2}}$ has $x = \pm 1$ for an extraneous solution: but χ_y is 0, neither ∞ nor $\frac{0}{0}$. And Lagrange also omitted these cases.

The whole of this examination was suggested to me by what appeared a strong *primâ facie* case against the tests derived from χ_x and χ_y: one of the strongest, I think, which was ever finally refuted. The primitive and singular have contact, generally of the first order: and y' has the same form in both. But this contact is not generally of the second order; y'' is therefore not the same in both*. Now $y' = \chi$ must give $y'' = \chi_x + \chi_y . y'$, which can only have twice as many values as χ by an escape into $\frac{0}{0}$ or some other singular form: and $\pm \infty \mp \infty$ seems of necessity to be the most common way in which this can happen. But sometimes the contact *will* be of the second order: may not then $\chi_x + \chi_y \cdot y'$ be ordinary, and y''' the first which exhibits singularity of form? And so on: for the contact may be of any order. The following investigation shewed that in cases giving contact of the second order, the general forms of χ_x and χ_y exhibit a tendency to the form $\frac{0}{0}$, though I do not find the same in reduced examples.

Let $\phi(x, y, a, b) = 0$ be the equation of an infinite number of families of curves, each family being defined by a relation between a and b, to which relation b' is the differential coefficient of b. This relation existing, the singular solution for the separated family is determined by

$$\frac{\phi_a + \phi_b . b'}{\phi_x} = 0, \text{ and } \frac{\phi_a + \phi_b . b'}{\phi_y} = 0;$$

omitting (in this sketch of main points) the cases of $x = \text{const.}$ and $y = \text{const.}$ And the first differential equation is obtained, both for primitive and singular, by eliminating a, after substitution for b, between

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* How does it happen that, though $y' = \chi(x, y)$ may, for a given point, have two solutions, with different values of y'', yet $\chi_x + \chi_y \cdot \chi$, reduced, shews no symptom whatever of the second value? An examination of instances will shew that $y'' = \psi x$ is obtained as $y'' = \psi x + P - P$, and that $P = \infty$ in the ambiguous case. So that it seems the following reasoning is inaccurate;—"Since $P - P = 0$, however great P may be, then $P - P = 0$, when P is infinite." All we ought to say is that *one* value is 0, and that the form is an indication of the possibility of more values than one.

$$\phi = 0, \quad y' + F = 0, \quad \text{where } (F = \phi_x \div \phi_y).$$

Now let a be the function of x and y which gives the singular solution. We have then, for the singular solution,

$$y'' + F_x + F_y . y' + (F_a + F_b . b')(a_x + a_y . y') = 0,$$

whence, in order that the singular solution may agree with the primitives which touch it, as to y'', and so have contact of the second order with them, we must have

$$F_a + F_b . b' = 0.$$

If then we eliminate x and y from

$$\phi = 0, \quad (\phi_a + \phi_b b' = 0 \text{ or } \phi_x = \infty\,,\ \phi_y = \infty), \quad \text{and } F_a + F_b . b' = 0,$$

we have a differential equation of the first order between a, b, b', each solution of which, be it primitive or singular, selects a family of curves out of $\phi = 0$ for which there is contact of at least the second order with the singular curve.

Take for examination $\phi_a + \phi_b . b' = 0$. We have then $\phi_a F_b - \phi_b F_a = 0$, which, combined with $\phi = 0$, gives x and y in terms of a and b. Take the pair $\phi = 0$, $y' + F = 0$, and by help of the relation between b and a, suppose a a function of x and y from the first, which, on substitution, gives χ for y' in the second. We have then

$$\phi_y + (\phi_a + \phi_b . b')\frac{da}{dy} = 0, \quad \frac{d\chi}{dy} + F_y + (F_a + F_b . b')\frac{da}{dy} = 0, \text{ \&c.}$$

$$\frac{d\chi}{dy} = -F_y + \frac{F_a + F_b . b'}{\phi_a + \phi_b . b'} . \phi_y, \quad \frac{d\chi}{dx} = -F_x + \frac{F_a + F_b . b'}{\phi_a + \phi_b . b'} . \phi_x.$$

Whence, as a general rule, by passing over exceptional cases, and leaning to the conclusion that $\frac{0}{0}$ will appear in specific cases when it appears in general forms, we should imagine that χ_x and χ_y would be infinite when the singular solution has contact of the first order only with primitives, and $\frac{0}{0}$ when of the second and higher orders. But such instances as I happen to have chosen do not confirm this conclusion.

Let us now take a *singular* solution of the equation between a, b, b'. Taking the three equations

$$\phi = 0, \quad \frac{\phi_a}{\phi_b} - \frac{F_a}{F_b} = 0, \quad F_a + F_b . b' = 0,$$

and calling the second $\psi = 0$, and the third $f + b' = 0$, we imply in them the function which b' is of a and b in the differential equation whose singular solution we are in search of. And we have (considering x, y, b', as functions of a and b),

$$\phi_a + \phi_x \frac{dx}{da} + \phi_y \frac{dy}{da} = 0,$$

$$\psi_a + \psi_x \frac{dx}{da} + \psi_y \frac{dy}{da} = 0,$$

$$f_a + f_x \frac{dx}{da} + f_y \frac{dy}{da} + \frac{db'}{da} = 0,$$

and similar equations with respect to b. Determine λ and μ from $\lambda\phi_x + \mu\psi_x + f_x = 0$ $\lambda\phi_y + \mu\psi_y + f_y = 0$. This gives

$$\frac{db'}{da} = -\lambda\phi_a - \mu\psi_a - f_a, \qquad \frac{db'}{db} = -\lambda\phi_b - \mu\psi_b - f_b.$$

The singular solution usually takes place when λ and μ are infinite; or when

$$\phi_x\psi_y - \phi_y\psi_x = 0, \text{ or } \phi_x \left(\frac{\phi_a}{\phi_b} - \frac{F_a}{F_b}\right)_y - \phi_y \left(\frac{\phi_a}{\phi_b} - \frac{F_a}{F_b}\right)_x = 0.$$

Common differentiation shews that

$$\phi_x \left(\frac{\phi_a}{\phi_b}\right)_y - \phi_y \left(\frac{\phi_a}{\phi_b}\right)_x = \frac{\phi_y^2}{\phi_b^2} \left\{\phi_a \left(\frac{\phi_x}{\phi_y}\right)_b - \phi_b \left(\frac{\phi_x}{\phi_y}\right)_a\right\}$$

$$= \frac{\phi_y^2}{\phi_b^2} (\phi_a F_b - \phi_b F_a) = 0.$$

This reduces the preceding condition to

$$\phi_x \left(\frac{F_a}{F_b}\right)_y - \phi_y \left(\frac{F_a}{F_b}\right)_x = 0, \text{ or } \left(\frac{F_a}{F_b}\right)_x - \left(\frac{F_a}{F_b}\right)_y . F = 0.$$

Now resume the equation $y'' + F_y . y' + F_x = 0$, and let us ask what is the condition under which y''' is the same both for the primitives and the singular, so that their contacts are of the third order. This happens when

$$(F_{ay} + F_{by} . b')\, y' + (F_{ax} + F_{bx} . b') = 0,$$

where $y' + F = 0$, $F_a + F_b . b' = 0$. Substitution shews that we have in this case also

$$\left(\frac{F_a}{F_b}\right)_x - \left(\frac{F_a}{F_b}\right)_y F = 0.$$

So that if $b' = \chi\,(a, b)$ be derived by eliminating x and y between

$$\phi = 0, \qquad \phi_a + \phi_b . b' = 0, \qquad \left(\frac{\phi_x}{\phi_y}\right)_a + \left(\frac{\phi_x}{\phi_y}\right)_b . b' = 0;$$

the ordinary primitive values of b in terms of a, substituted in $\phi\ (x, y, a, b) = 0$, give families which have contacts of the second order at least with their singulars. But a singular value of b in terms of a, similarly used, gives contacts of the third order.

The connexion of the preceding with the theory of the singular solutions of equations of the second order will be noticed, and also the glimpse which it gives of the general theory of singular solutions. I will conclude this section with a remark which may tend to increase confidence in that *main-survey* system of reasoning which Lagrange and others have used, though by no means intended to imply that it is to be implicitly received.

An equation of the form $\phi\ (x, y, c) = 0$ may have many different forms given to it. If $\psi\ (x, y, c) = 0$ be another equivalent equation, we know that $\phi_c : \phi_x = \psi_c : \psi_x$. Now it may and does happen that a branch of the singular solution which the first form only yields out of $\phi_r = \infty$, may in the second form be given by $\psi_c = 0$. And I believe that in every case a form may be obtained which will allow $\psi_c = 0$ to represent the complete singular solution. If this be so, the inclusiveness of the preceding reasoning is much augmented.

SECTION II.

On certain solutions of differential equations which ultimately depend on elimination.

SOME time ago I communicated to the Society a method (printed in Vol. VIII. Part II.) of which I afterwards found, as I acknowledged in a note* at the end, that it had been given by M. Chasles, from an unpublished paper of Monge, as I supposed. On looking again at the account given by M. Chasles, (*Aperçu Historique, &c.*, p. 376.) I find that he had not seen the memoir of Monge (which was never printed, though intended to appear in the *Memoirs of the Institute* for 1808): but that he had only the hint of the *reciprocal surfaces* mentioned at the end of my paper. From this M. Chasles infers that Monge must have used these surfaces in the integration of partial differential equations. So that it by no means follows that Monge *did* invent the method: this remains with M. Chasles until a prior right is established. It seems unlikely that Monge's memoir is now in existence.

The most striking peculiarity of the method is the perfectly reciprocal character of the substitutions; in transforming the equations, x and p, y and q, &c. are *interchanged.* The following extensions will shew that this reciprocity is not an essential part of all similar methods. Their peculiar feature really is, that they are founded upon the circumstance noted by Clairaut in his celebrated solution of $y = y'x + fy'$, namely, that there are cases of differentiation in which the only new differential coefficient introduced appears as a factor of the whole.

Let $y^{(n)}$ express, as usual, the n^{th} differential coefficient of y with respect to x, and let

$$V_{m,n} = \int \phi x \,.\, dy^{(n)} = \phi x \,.\, y^{(n)} - \phi' x \,.\, y^{(n-1)} + \phi'' x \,.\, y^{(n-2)} - \dots$$

ϕx being a rational function of the degree m. When m exceeds n, integrals of y will enter: we need not exclude these cases from the reasoning, though they will be of no use. Accordingly, $y^{(n+1)}$ is a factor of $V'_{m,n}$, whence it follows that $V_{m,n}$ is constant if y be any rational function of x, not exceeding the n^{th} degree.

Let $f(V_{m,n},\ V_{m',n},\ V_{m'',n}, \dots) = 0$, a differential equation of the n^{th} order.

Assume $y = C_0 + C_1 x + \dots + C_n x^n$, having $n + 1$ constants. Substitution reduces $V_{m,n}$, &c. and therefore f, to constants, and this, even though the rational functions entering in them be wholly unconnected. If then one of the constants C_0, &c. be substituted in terms of the rest from $f=0$, we have the complete integral of the equation. By differentiating $f=0$, we obtain

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* The date of this note is misprinted; it should be 1848, not 1847.

$$(P\,\phi x + P_1\,\phi_1 x + \dots)\,y^{(n+1)} = 0,$$

P, P_1, &c. being partial differential coefficients of f with respect to $V_{m,n}$, &c. and ϕx, $\phi_1 x$, &c. being the several rational functions of these last. By making the first factor vanish, and eliminating $y^{(n)}$ between this and $f=0$, we get an equation of the $(n-1)^{\text{th}}$ order, the *ordinary primitives* of which are singular solutions of $f=0$.

For instance, let the equation be

$$\frac{x^2}{2}y'' - xy' + y = (xy'' - y')\,y''.$$

Here $y = C_0 + C_1 x + C_2 x^2$, gives $C_0 = -2C_1C_2$, and the complete primitive is

$$y = -2C_1C_2 + C_1 x + C_2 x^2.$$

Differentiating the equation, and throwing out the factor y''', we have $\frac{1}{2}x^2 = xy'' - y' + xy''$, and elimination of y'' gives

$$4y'^2 - 12x^2y' + 16xy + x^4 = 0.$$

Of this $y = \frac{1}{18}x^3$ is an ordinary solution, and a singular solution of the original: but $y = \frac{1}{2}x^3$ is a singular solution of the last, and is not a solution of the original.

Let $f(V_{m,n}, V_{m',n-1}) = 0$. The preceding method gives only $n-1$ constants, or n constants with one relation between them. But if we write z for $y^{(n)}$ the preceding takes the form

$$f(\int\phi x\,.\,z'dx,\ \int\psi x\,.\,zdx) = 0.$$

If we differentiate with respect to x, and eliminate $\int\phi x\,.\,z'dx$, we have an equation of the form

$$\psi(x,\ z,\ \int\psi x\,.\,zdx) = 0,$$

which is of the first order. When z or $y^{(n)}$ is found from this, y is found by integration, but we must take care to introduce the relation known to exist among the n constants which the integration brings in.

In the same manner $f(V_{m,n},\ V_{m',n-1},\ V_{m'',n-2}) = 0$ is reducible to an equation of the second order: and so on.

If $V_{m,n} = \int\phi x\,.\,dy^{(n)}$ and $V_{m',n-1} = \int\psi x\,.\,dy^{(n-1)}$, we have

$$V_{m,n} = \int\frac{d}{dx}\left(\frac{1}{\psi x}\ \frac{dV_{m',n-1}}{dx}\right).\,\phi x dx.$$

Whence it follows that all differential coefficients of $V_{m,n}$ can be expressed in terms of those of $V_{m',n-1}$, and thence in terms of those of $V_{m'',n-2}$, and so on. If then a differential equation contain x, $V_{m,n}$, and differential coefficients only of $V_{m',n+1}$, $V_{m'',n+2}$, &c., the substitution of z for $V_{m,n}$ will reduce the equation n units of order.

The following examples will illustrate a large class of transformations, one of which is that which corresponds, in equations of one variable, to M. Chasles's method for the case of two.

According as the rational function is ϕx or ψx, let ϕ_n or ψ_n be used to represent $V_{m,n}$. Let

$\phi_n = X$, $\psi_n = Y$, so that Y is implied in terms of X, when y is assigned in terms of x. Let $dY : dX$ be Y', &c. We have then

$$\phi x \,.\, dy^{(n)} = dX, \quad \psi x \,.\, dy^{(n)} = dY, \quad Y' = \frac{\psi x}{\phi x}, \text{ say } x = \chi Y'.$$

Consequently, $f(\phi_n, \psi_n, x) = 0$ is reducible in the first instance to $f(X, Y, \chi Y') = 0$. If from this we find $Y = \varpi(X, C)$, this, and $x = \chi Y'$ enable us to find X, and thence Y, in terms of x and C. Substitute in $\phi_n = X$ and $\psi_n = Y$, eliminate $y^{(n)}$, and we have a *linear* equation of the order $n - 1$, involving the constant already obtained, and yielding $n - 1$ others in integration.

For instance, let $\dfrac{x^2}{2} y'' - xy' + y = \dfrac{2a}{x}(xy'' - y')$.

Let $\dfrac{x^2}{2} y'' - xy' + y = Y$, $\quad xy'' - y' = X$, $\quad Y' = \tfrac{1}{2}x$,

$YY' = aX$, $\quad Y^2 = aX^2 + C$, $\quad Y' = \dfrac{aX}{\sqrt{(aX^2 + C)}} = \dfrac{x}{2}$,

$X^2 = \dfrac{Cx^2}{a(4a - x^2)}$, $\quad Y^2 = \dfrac{4aC}{4a - x^2}$; eliminate y'' and

$\dfrac{x}{2} y' - y = -\tfrac{1}{2}\sqrt{\dfrac{C}{a}} \,.\, \sqrt{(4a - x^2)}$,

whence the complete value of y can be found.

By making $\phi_n = Y$, $\psi_{n-m} = X$, whence $Y' = \phi x \,.\, y^{(n+1)} \div \psi x \,.\, y^{(n-m+1)}$, we see that

$$f\left(\phi_n, \psi_{n-m}, \frac{\phi x \,.\, y^{(n+1)}}{\psi x \,.\, y^{(n-m+1)}}\right) = 0 \text{ depends on } f(X, Y, Y') = 0.$$

If this last give $Y = \varpi(X, C)$, we have $\phi_n = \varpi(\psi_{n-m}, C)$, which is one unit lower in order than the given equation, and can be reduced in the manner already pointed out.

We can make further discovery of reductions by using more differentiations. For example, the last case but one gives

$$Y'' dX = \left(\frac{\psi x}{\phi x}\right)' dx \text{ or } Y'' = \frac{1}{\phi x \,.\, y^{(n+1)}} \left(\frac{\psi x}{\phi x}\right)'$$

Hence $f\left\{\phi_n, \psi_n, x, \dfrac{1}{\phi x \,.\, y^{(n+1)}} \,.\, \left(\dfrac{\psi x}{\phi x}\right)'\right\} = 0$, depends on $f(X, Y, \chi Y', Y'') = 0$.

This last being integrated, we can express X by help of Y', in terms of x and two constants; after which, substitution in the first equations, and elimination of $y^{(n)}$, gives a linear equation of the order $n - 1$, from which the remaining constants are found.

We may also, instead of using X and Y, use functions of them similar to those already given. For example, let

$$\frac{x^2}{2} y'' - xy' + y = XY' - Y, \quad y'' = \tfrac{1}{2}X^2.$$

Then $\tfrac{1}{2}x^2 = Y''$, $\quad \dfrac{X^2}{2} Y'' - XY' + Y = xy' - y$, $\quad Y'''y''' = Xx$; &c. whence

$$f\left(\frac{x^2}{2}y'' - xy' + y,\ \ xy' - y,\ \ y'',\ \ \frac{y'''}{x},\ \&\text{c.}\right) = 0,$$

$$f\left(XY' - Y,\ \ \frac{X^2}{2}Y'' - XY' + Y,\ \ \tfrac{1}{2}X^2,\ \ \frac{X}{Y'''},\ \&\text{c.}\right) = 0,$$

depend each on the other; and one will often be directly integrable when the other is not. This reciprocity of relations always exists when the original assumptions are as in

$$\frac{x^n}{2\,.\,3\ldots n}y^{(n)} - \frac{x^{n-1}}{2\,.\,3\ldots n-1}y^{(n-1)} + \ldots = \frac{X^{n-1}}{2\,.\,3\ldots n-1}\,Y^{(n-1)} - \ldots \quad \text{and } y^{(n)} = \frac{X^n}{2\,.\,3\ldots n}.$$

But there is no case, so far as I can see, which will secure a transformation and chance of integration to *every* differential equation, except the following, which suggested the whole method, and was derived from consideration of M. Chasles's method for equations of three variables.

Let $xy' - y = Y$, $y' = X$. Then we have

$$x = Y',\quad y = XY' - Y,\quad y''Y'' = 1,\quad y'''Y''^3 + Y''' = 0,\quad Y'''y''^3 + y''' = 0,$$

$$y^{\text{v}} = -\frac{Y''Y^{\text{v}} - 3Y'''^2}{Y''^5},\qquad Y^{\text{v}} = -\frac{y''y^{\text{v}} - 3y'''^2}{y''^5},\ \text{and so on.}$$

Hence the following equations,

$$f(x, y, y', y'', y''', y^{\text{v}}, \ldots) = 0,\quad f\left(Y', XY' - Y, X, \frac{1}{Y''}, -\frac{Y'''}{Y''^3}, -\frac{Y''Y^{\text{v}} - 3Y'''^2}{Y''^5}, \ldots\right) = 0,$$

depend each on the other. If the second can be solved, then the first is solved by eliminating X between $y = XY' - Y$, and $x = Y'$; and *vice versa*. If we apply this method to $x = fy'$, $y = fy'$, $y = x\phi y' + \psi y'$, &c. we shall see that it includes most of the common cases in which the solution is reduced to elimination after integration. Thus also $y''^2 + ay''' = bxy''^3$ is reduced to $y'' - ay''' = by'$, and $y''(x\phi y' + y\psi y' + \chi y') = 1$ is ultimately linear.

If we assume $x = \phi(Y', XY' - Y)$, $y = \psi(Y', XY' - Y)$ and use α and β for Y' and $XY' - Y$ as indices of differentiation, we easily obtain

$$y' = \frac{\psi_\alpha + \psi_\beta\,.\,X}{\phi_\alpha + \phi_\beta\,.\,X}\ \text{say} = \chi(Y', XY' - Y, X),$$

$$y'' = \frac{\chi_\alpha + \chi_\beta\,.\,X}{\phi_\alpha + \phi_\beta\,.\,X} + \frac{\chi_X}{\phi_\alpha + \phi_\beta\,.\,X}\cdot\frac{1}{Y''},\ \&\text{c.}$$

This may suggest transformations available in particular cases, and in which the orders of differentiation make their first appearances simultaneously. If we try to carry this further, as by supposing x and y to be each of the form

$$\phi\left(Y'', XY'' - Y', \frac{X^2}{2}Y'' - XY' + Y\right)$$

we can then express y' by aid of Y'', at the highest: but y'' requires Y'''.

[Since* the above was communicated, I have found a more general view of the subject, by which any number of modes of transformation may be effected, no one of which alters the order of the equation.

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* The date of this addition is March 17, 1851.

In order to produce a system such as

$$X = \phi\,(x,y,y'), \quad Y = \psi\,(x,y,y'), \quad Y' = \psi_{,}\,(x,y,y'), \quad Y'' = \psi_{,,}\,(x,y,y',y''), \text{ \&c., \&c.,}$$

in which $Y^{(n)}$ shall be expressed in terms of $x, y, \ldots y^{(n)}$, and nothing higher, it is the necessary and sufficient condition that, X and Y being constant, $X = \phi$ and $Y = \psi$ shall be the two primitives of the first order belonging to one equation of the second order. For Y', or $\psi_x + \psi_y \cdot y' + \psi_{y'} y''$ divided by $\phi_x + \phi_y \cdot y' + \phi_{y'} \cdot y''$, cannot be independent of y'', unless in the form

$$Y' = \frac{\psi_{y'}}{\phi_{y'}} \left(y'' + \frac{\psi_y \cdot y' + \psi_x}{\psi_{y'}}\right) : \left(y'' + \frac{\phi_y \cdot y' + \phi_x}{\phi_{y'}}\right)$$

the two last members be identical: in which case $X = \psi$ and $Y = \phi$ are primitives of the equation of the second order obtained by making those members vanish. Such a connexion of ϕ and ψ being chosen, we have

$$X = \phi, \quad Y = \psi, \quad Y' = \frac{\psi_{y'}}{\phi_{y'}}, \quad Y'' = \left\{\left(\frac{\psi_{y'}}{\phi_{y'}}\right)_x + \text{\&c.}\right\} : (\phi_x + \text{\&c.}), \text{ \&c.,}$$

and if we can integrate $f(X, Y, Y', \text{\&c.}) = 0$ we can then integrate $f(\phi, \psi, \psi_{y'} : \phi_{y'}, \text{\&c.}) = 0$, by eliminating X and y' between $X = \phi$, $Y = \psi$, $Y' = \psi_{y'} : \phi_{y'}$. And if we can integrate the second equation, we integrate the first by eliminating x between $X = \phi$, $Y = \psi$. For instance, if we choose $X = x + y + y'$, we find, from $1 + y' + y'' = 0$, that we may have $Y = \epsilon^x (1 + y')$, whence $Y' = \epsilon^x$, $Y'' = \epsilon^x : (1 + y' + y'')$, &c. Hence the equations

$$f\left\{x + y + y',\ \epsilon^x (1 + y'),\ \epsilon^x,\ \frac{\epsilon^x}{1 + y' + y''},\ \text{\&c.}\right\} = 0, \text{ and } f(X, Y, Y', Y'', \text{\&c.}) = 0$$

have their primitives connected by the preceding rule with

$$X = x + y + y', \quad Y = \epsilon^x (1 + y') \quad Y' = \epsilon^x.$$

The case previously given is that in which the two primitives chosen of $y'' = 0$ are $X = y'$, $Y = xy' - y$. A great many remarks will suggest themselves, in extension of those commonly made on the equation $xy' - y = fy'$. Passing over these, I observe that if $F(x, y, X, Y) = 0$ be the common primitive of $X = \phi$, $Y = \psi$, then the system may be inverted as follows. Let X and Y be the variables, and x and y the constants. Then the same three equations $X = \phi$, $Y = \psi$, $Y' = \psi_{y'} : \phi_{y'}$ contain the differential equations of the first order belonging to this inverted system. If we eliminate y' between these three equations, we have them; and if we thus get

$$x = \Phi\,(X, Y, Y'), \quad y = \Psi\,(X, Y, Y'), \quad \text{we shall have } y' = \Psi_{Y'} : \Phi_{Y'}.$$

Thus in the instance above, we have

$$x = \log Y', \quad y = X - \log Y' - \frac{Y}{Y'} + 1, \quad y' = \frac{Y}{Y'} - 1 = \Psi_{Y'} : \Phi_{Y'}$$

and, x and y being constants, the first two equations are two primitives of one equation of the second order, namely $Y'' = 0$. The original primitive is $Y = \epsilon^x X + (y + x - 1)\,\epsilon^x$.

If we want to have equations of the same invertible character as $Y = xy' - y$, $X = y'$, $y = XY' - Y$, $x = Y'$, &c., we must choose a form of $\phi(x, y, X, Y) = 0$, in which X and Y may be interchanged with x and y, without alteration of the relation. The instance just cited comes from $y + Y = xX$; if we choose $yY = x + X$, we get

$$X = \frac{y}{y'} - x, \quad Y = \frac{1}{y'}, \quad Y' = \frac{1}{y}, \quad Y'' = \frac{y'^3}{y^3} \cdot \frac{1}{y''}, \text{ \&c.,}$$

with power to interchange large and small letters. If we choose $X^2y^2 + Y^2x^2 = 1$, we have

$$X^2 = \frac{1}{y(y - y'x)}, \quad Y^2 = -\frac{y'}{x(y - y'x)}, \quad Y'^2 = -\frac{y^3}{x^3y'}, \text{ \&c.,}$$

with the same power.

The following problem thus becomes of interest. Given $\phi(x, y, y', y'', \ldots)$, required means of determining all or any of the modes by which it may be transformed into the given function $\psi(X, Y, Y', \ldots)$, where $X = \xi(x, y, y')$, $Y = \eta(x, y, y')$, these equations being two primitives of the same equation of the second order.]

SECTION III.

On the number of constants which the complete primitive of an equation can contain.

It has never, till very lately, been thought necessary to prove that the number of constants in the primitive of an equation of the nth order amounts to n: and I am not aware of any attempt to prove, generally, that the number cannot exceed n. It was, at one time, usually taken for granted that the restoration of the primitive form requires n integrations and no more. Now, first, it was never shewn that $\phi(x, y, y', \ldots)$ always admits of a factor which renders it integrable, independently of relation between x and y. Secondly, in the absence of direct methods, and of proof of their possibility, it might be suspected that in certain cases, more than n integrations would be requisite, followed by changes of form, and then by differentiations which, in consequence of the changes, would not destroy the previously introduced constants. Thus the notion that $y^{\text{V}} = \phi(x, y, y', y'', y''')$ must yield y by *four* integrations would not stand, on its own *à priori* probability, against any one who should assert that it required twenty integrations, the introduction of a factor, and sixteen differentiations.

That the solution does introduce as many as n constants is proved without difficulty. Taking a given value of x, and assuming values of $y\, y' \ldots y^{(n-1)}$, independent of each other, we may construct a solution geometrically, as the limit of a polygon of curved sides, or algebraically, by forming subsequent differential coefficients from the equation. M. Cauchy prefers to shew the approximate construction of the solution of n simultaneous differential equations of the first order, and its introduction of n constants; after which he reduces the single equation of the nth order to n such simultaneous equations.

By a differential equation of the nth order we mean an expression of $y^{(n)}$ in terms of $y^{(n-1)},\ldots y, x$: and by its complete primitive we mean that relation between y and x which admits of *no other* relation between $y^{(n)}, y^{(n-1)}$, &c. except that in the equation. If we take a primitive which contains n distinct constants (that is, constants which do not admit of reduction to a smaller number by substituting one new letter for a function of two or more), and if we differentiate n times, we know that elimination of the n constants between the primitive and its n *differentiated* equations will produce one equation, and one only. For the independence of these $n+1$ equations is established by each of them containing a letter which is not to be found in any of the preceding; the second has y', the third y'', and so on. Consequently, elimination of n constants by the equations which differentiation furnishes, independently of all connexion between the meanings of y, y', y'', &c., certainly gives only one equation of the nth order.

But there *is* a connexion between the meanings of y, y', &c.: taking this into account, is it not possible that some other relation may exist between $x, y, \ldots y^{(n)}$, not indeed algebraically deducible from the equations, but due to the additional consideration of the connexion in question?

Let V_0 be a symbol for the primitive equation, and let $V_1\ V_2 \ldots V_n$ be symbols for the differentiated equations. Let E_n be the differential equation deduced from elimination; and if it be possible, let there be another, F_n. Eliminate $y^{(n)}$ between E_n and F_n, producing say W_{n-1}. If then we eliminate the $n-1$ quantities $y', \ldots y^{(n-1)}$ between the n equations $V_1, \ldots V_{n-1}, W_{n-1}$, we obtain a relation between x, y, and the n constants, which must be V_0: for, were it any other, x and y would themselves be constants. Then, since $V_1 \ldots V_{n-1}, W_{n-1}$, combined give V_0, it follows that $V_0, V_1, \ldots V_{n-1}$ combined give W_{n-1}, or the n distinct constants can be eliminated (algebraically) by help of n distinct equations only; which cannot be.

Next, observe that if $y^{(n)}$ be expressible in terms of $y^{(n-1)}, \ldots y, x$, it follows that $y^{(n+1)}$ is in an infinite number of ways expressible in terms of $y^{(n)}, \ldots y, x$. For $y^{(n)} = \phi\,(y^{(n-1)}, \ldots y, x)$ can in an infinite number of ways be made a particular case of $y^{(n)} = \psi\,(y^{(n-1)}, \ldots y, x, C)$, and each of these ways gives its own form of $y^{(n+1)} = \chi\,(y^{(n)}, \ldots y, x)$, true for all values of C.

From the above, it follows that a differential equation of the nth order cannot have fewer than n constants in its complete primitive, nor more, unless it have an infinite number. By the meaning of the terms, the differential equation is to give the only mode of expressing $y^{(n)}$ in lower orders. But if the complete primitive have $n-1$ arbitrary constants only, then $y^{(n-1)}$ can be expressed in lower orders, and $y^{(n)}$ can therefore be expressed in lower orders in an infinite number of ways.

Again, since $y^{(n)}$ can be expressed in lower orders, by the equation itself, it follows that $y^{(n+k)}$ can be thus expressed in an infinite number of ways. Hence, there cannot then be $n+k$ constants in the original primitive: for then $y^{(n+k)}$ could be expressed only in one way. This reasoning applies to all finite values of k, but does not disprove the possibility of an infinite number.

To settle this remaining point, observe that if the original primitive have an infinite number of constants, it must be possible to assign values to them in such manner as, for a given value of x, to assign given values to $y, y', \ldots y^{(n)}$. That is to say, the differential equation can be satisfied by any values of $x, y, y', \ldots y^{(n)}$, and is an identical equation. This of itself might be held sufficient to prove that there cannot be even $n + 1$ constants.

[Writers[*] often express themselves as if a given equation of the second order had but one pair of primitives of the first order, having one arbitrary constant in each: and similarly for other orders. But this is not correct: if $a = \phi(x, y, y')$ and $b = \psi(x, y, y')$ be any one pair of primitives, then $c = f(\phi, \psi)$ is also a primitive of the first order for every form of f. Thus $y = ax + b$, and $ax + by = a^2 + b^2$ equally give $y'' = 0$, but the second gives a very different pair of equations of the first order from what the first gives. An equation of the second order represents a family *of families* of curves. Now though a ***group*** is but in one way a group, yet a group of groups is in an infinite number of ways convertible into other groups of groups, differing according to the manner in which individuals from each of the old groups are put together to form one of the new ones. Each of these methods involves a differential equation of the first order with an arbitrary constant in it. The nature of this last equation dictates the principle on which the groups are formed, the value of the constant settles which group is taken into consideration.

In the common form $y = y'x + fy'$, it is said that we have only a limited selection from the lines indicated by $y'' = 0$, namely, those contained in $y = mx + fm$. But if there be an arbitrary constant in fm, we have the *complete* solution, since fc is then independent of c in value. The fact is, that $a = f(y', y'x - y)$, which answers to the above with an arbitrary constant in m, is the general mode of representing an equation of the first order formed from $y'' = 0$.

But it is further to be noticed that we are not bound to the entrance of *one* arbitrary constant *only* in each of the primitives of the first order. The equation $f(y' - a, y'x - y - b) = 0$ is really satisfied by $y'' = 0$, and by $y = Ax + B$, provided $f(A - a, -B - b) = 0$. And this is the complete primitive of $f = 0$; and any other mode of integration will introduce the new arbitrary constant only in such combinations with a or b, or both, as will render the three equivalent only to two.

Many cases have casually occurred in which a greater number of constants enter a primitive than the theory points out; but it is always found that the constants enter in a manner which reduces their effective number. Nevertheless, it has, I think, been generally received that an integral of the order $n - m$, of an equation of the nth order, cannot have more than m constants. To shew that this is not always the case, take the equation $(A + x)y' + Bx + C - 2y = 0$. It has three independent constants; and yet it is a primitive of $y''' = 0$. If we differentiate twice, we have $(A + x)y''' = 0$, and though it cannot be said that *three* constants disappear in *two* differentiations, yet it must be admitted that a third constant is driven from the differential equation, which must be $y''' = 0$. Neither do we get four constants in the original primitive; for if we integrate $(A + x)y' + Bx + C = 2y$ in the usual way, we find

$$y + Ex^2 + (2AE - B)x + \left(A^2E - \frac{C + BA}{2}\right) = 0,$$

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[*] The date of this addition is March 17, 1851.

in which the constant of integration is combined with the old ones in such manner that only three distinct arbitrary constants appear.

The integrating factor of a common differential equation is any solution of a certain partial differential equation, and therefore contains an arbitrary function, which is the equivalent of an *infinite* number of arbitrary constants. It must therefore be shewn that the primitive of a common differential equation does not contain an arbitrary function. In the case of the first order this is easy, for if $Pdx + Qdy$ be made dV by multiplication by M, the general integrating factor is $\phi V . M$, and the primitive is of the form $\psi V = \text{const.}$, which, were ψV to contain a million of arbitrary constants, amounts to no more than $V = \text{const.}$ But no such proof applies to equations of a higher order than the first; on the contrary, I think it can be shewn that all the primitives of such an equation, except only the general primitive, contain effective arbitrary functions.

Let us take $y''' = 0$. What are called *the* primitives of the second order are $y'' + a = 0$, $xy'' - y' + b = 0$, $x^2 y'' - 2xy' + 2y + c = 0$. But it is obvious that $y''' = 0$ is the equation of the third order, and the only one, which follows from

$$f(y'' + a, \quad xy'' - y' + b, \quad x^2 y'' - 2xy' + 2y + c) = 0.$$

And the complete integral of this is $y = A\dfrac{x^2}{2} + Bx + C$, subject to the condition $f(A + a, B - b, 2C + c) = 0$.

Neither can there be any *other* equation of the second order, deducible from, or deducing, $y''' = 0$, and not contained in the above. For, let $F(x, y, y', y'') = 0$ be the most general solution of $y''' = 0$, and let $y'' + a = P$, $xy'' -$ &c. $= Q$, $x^2 y'' -$ &c. $= R$; and, substituting the values of y, y', y'', thence obtained in terms of P, Q, R, x, let $F = 0$ be thus reduced to $\phi(P, Q, R, x) = 0$. Then we have $(\phi_P + \phi_Q . x + \phi_R . x^2)\, y''' + \phi_x = 0$, which, since $y''' = 0$ is to be an equation of the third order deducible, gives $\phi_x = 0$ for all values of x, so that ϕ does not contain x explicitly.

The following extension of the theorem involved in the method of treating $y = y'x + fy'$ may now be easily proved. Let $\phi(x, y, A_1, \dots A_n) = 0$ be the complete integral of an equation of the nth order, and let $U_1 + A_1 = 0, \dots U_n + A_n = 0$, be the *fundamental* integrals of this equation of the order $n - 1$. Then the general preceding primitive of the equation of the nth order is $f(U_1 + A_1, \dots U_n + A_n) = 0$, f being any form whatsoever; and the original primitive of *this* equation, independently considered, is $\phi(x, y, B_1, \dots B_n)$ in which the n new constants $B_1, \dots B_n$ are subject to the condition $f(A_1 - B_1, \dots A_n - B_n) = 0$. This process must be repeated on the last primitive in order to produce the most general equation of the order $n - 2$; and so on.

For illustration, I return to the case of $y''' = 0$, and indicate the mode of finding the general primitive of the first order. Let $y = ax^2 + bx + c$ be first taken as a primitive: its general equation of the third order is $y''' = 0$, of which the immediate primitive is

$$x^2 y'' - 2xy' + 2y - 2c = f(xy'' - y' + b, \; y'' - 2a),$$

which however, strictly belongs to $y = ax^2 + bx + c$ only when $0 = f(0, 0)$. The complete primitive of the last is $y = Ax^2 + Bx + C$ subject to the condition $2C - 2c = f(b - B, 2A - 2a)$, or it is

$$y = Ax^2 + Bx + c + \tfrac{1}{2}f(b - B, 2A - 2a),$$

and of this we must find the most general equation of the first order. The usual equations of the first order will be readily seen to be (A and B being the constants for elimination),

$$xy' - y = Ax^2 - c - \tfrac{1}{2}f(b - y' + 2Ax, 2A - 2a),$$

$$xy' - 2y = -Bx - 2c - f\left(b - B, \frac{y' - B}{x} - 2a\right);$$

if we throw these into the forms $U - A = 0$, $V - B = 0$, then $V - B = F(U - A)$ is the general equation of the first order.

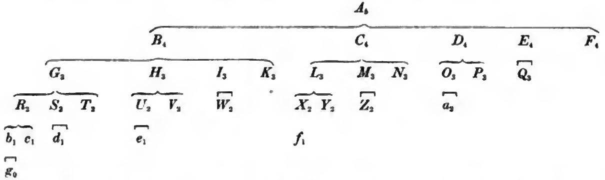
Taking a simple case of this, as $U = A$, which gives the first of the above equations again, we ought to find from it $y''' = 0$; and if we differentiate both sides, we find

$$\{x - \tfrac{1}{2}f'(b - y' + 2Ax, 2A - 2a)\}\ (y'' - 2A) = 0.$$

Neglecting the alternative offered by the first factor, which gives a singular solution, we have $y'' - 2A = 0$, $y''' = 0$.

I cannot find any condition worthy of note, except the simple directness of the mode of derivation, which distinguishes the ordinary primitives from those which I have given above. It is not even true that the ordinary primitives are the only ones which contain one arbitrary constant more than those from which they are immediately derived. Thus $a = (xy' - y)y'$ is as much a primitive of $y'' = 0$ as $a = xy' - y$ or $a = y'$. And it is to be noticed, in reference to the ordinary theory, that the primitives of the order $n - 1$ are the only ones which are independent of each other. Of those of the order $n - 2$, $\frac{1}{2}n(n - 1)$ in number, only $n - 1$ are independent of each other, the rest being deducible from them; only $n - 2$ of the order $n - 3$, and so on.

From looking at the small number of cases in which the complete series of ordinary primitives can be found, I surmise the following mode of derivation to be universal.



Each letter represents an equation with its order subscript. Of the five equations of the fourth order, F_4 produces none of the third; E_4, one only, Q_3; D_4, two only, O_3, P_3; and so on. This process is repeated on each of the groups, until the complete primitive, g_0, is attained. Looking at the fact that all in the first group are independent, it may be suspected, I submit, that any general mode of thus obtaining the successive primitives will find it convenient to treat the groups of largest number in each line as containing the independent equations, from which all the others are to be deduced.

SECTION IV.

On the condition of integrability of $\phi\,(x, y, y', y'', \ldots y^{(n)})$*.

Let the successive differential coefficients of y with respect to x be denoted by p, q, r, s, t, &c., using y', y'', &c., when convenient. Let V be a function of x, y, p, q, &c., and let V_x, V_y, V_p, &c. be its partial differential coefficients with respect to the several letters, taken independently. Let the accent be the symbol of complete differentiation with respect to x, so that we have in all cases

$$V' = V_x + V_y p + V_p q + V_q r + V_r s + \ldots$$

If V contain no letters beyond r, then s occurs only as seen in $V_r s$; and so on.

Various proofs have been given of the theorem that Vdx is integrable *per se*, independently of relation between y and x, when $V_y - V_p' + V_q'' - \ldots$ vanishes identically. Euler proved it by the calculus of variations. Lagrange gave a complicated proof involving the same principles. Lexell gave so much more complicated a proof (according to Lagrange, whose mention of it is all I know about it) that it is difficult to decide either upon its truth or generality. Poisson gave another proof derived from the calculus of variations. M. Bertrand gave two proofs, one from the calculus of variations, another from the introduction of definite integration with respect to a subsidiary variable. M. Jacques Binet extended the second proof of M. Bertrand, and the resulting form of Poisson, so as to make them apply to all cases, without any failure. (Poisson, *Mem. Inst.*, 1829, p. 260; Bertrand, *Jo. Ec. Polytech.* cah. 28; Moigno, Vol. II. p. 551.) Allusion is made to some investigations of M. Sarrus, which I have not seen.

The objection to all these proofs is that they are not fundamental: they introduce considerations the farthest from elementary of any which can now be imagined, the principles of the calculus of variations, and the use of expression by integration with respect to a variable introduced for nothing but expression. This objection, however, is of no validity until a simple and fundamental proof is given: and this I propose to do.

Again, it is to be shewn, not only that the criterion is *sufficient*, but that it is *necessary*. Some of the proofs of the latter point appear to me to fail entirely. They depend upon the reduction of $\int Vdx$ to an integrated portion together with an integral of the form

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* April 9, 1851. I picked up a memoir by M. Sarrus, which has the appearance of being part of the *Journal de l'Ecole Polytechnique*, and the signature has "Tome XIV. No. VII. ier janvier 1824." This memoir contains the proof here given, in substance, though the equations on which the criterion is founded are not demonstrated. It is singular that M. Bertrand takes no notice of it, except to observe that M. Sarrus does not use the calculus of variations. MM. Cauchy and Moigno pass it over altogether. But it must be observed that M. Sarrus establishes only the *necessity* of the criterion, and does not establish its *sufficiency*, except when the equations that give it are granted with it.

$\int(V_y - V_p' + \ldots)\, Q\,dx$. This, it is assumed, must vanish; which, though clear enough in the common case in which $Q = y$, and $V_y - \ldots$ is a function of x only, is not sufficiently supported in any other. Why may not $(V_y - \ldots)\, Q$ be itself a new integrable function?

Let V be an integrable function, say of x, y, p, q, r, s, t, which therefore can be thrown into the form

$$V = X + Yp + Pq + Qr + Rs + St,$$

where $X, Y, \ldots S$ are functions of x, y, p, q, r, s, and not of t, and are themselves differential coefficients of $\int V dx$. Consequently we must have all such relations as $P_r = R_p$, $X_q = Q_x$, &c.; or, M and N being any two of the capitals, and m and n their corresponding small letters, we must have $M_n = N_m$. If then we differentiate with respect to any one letter, say r, we have

$$\begin{aligned} V_r &= (X_r + Y_r p + P_r q + Q_r r + R_r s + S_r t) + Q, \\ &= (R_x + R_y p + R_p q + R_q r + R_r s + R_r t) + Q, \\ &= R' + Q. \end{aligned}$$

Proceeding thus, we get*

$$\begin{aligned} V_x &= X', & S &= V_t, \\ V_y &= Y' + 0, & R &= V_s - V_t', \\ V_p &= P' + Y, & Q &= V_r - V_s' + V_t'', \\ V_q &= Q' + P, & P &= V_q - V_r' + V_s'' - V_t''', \\ V_r &= R' + Q, & Y &= V_p - V_q' + V_r'' - V_s''' + V_t^{IV}, \\ V_s &= S' + R, & 0 &= V_y - V_p' + V_q'' - V_r''' + V_s^{IV} - V_t^{V}, \\ V_t &= 0 + S. \end{aligned}$$

whence the necessity of the criterion is established. (The general theorem on which it depends is

$$(W')_{y^{(n)}} = (W_{y^{(n)}})' + W_{y^{(n-1)}} \ldots\ldots (W),$$

which, we may notice in passing, gives

$$(W^{(k)})_{y^{(n)}} = (W_{y^{(n)}})^{(k)} + k\,(W_{y^{(n-1)}})^{(k-1)} + k\,\frac{k-1}{2}\,(W_{y^{(n-2)}})^{(k-2)} + \ldots$$

the last term containing $(W_y)^{(k-n)}$, whenever k is greater than n. It follows that $(W^{(k)})_{y^{(n)}}$ is always integrable *per se* $k - n$ times, whenever k is greater than n.

The last six equations give the first: for if we differentiate $V = X + Yp + Pq + \ldots$ we have

$$V_x + V_y p + V_p q + \ldots = X' + Y'p + (P' + Y)\,q + \ldots\ldots$$

which, by the last six equations, is $V_x = X'$. And thus it appears that the partial differential coefficient of $X + Yp + \ldots$ with respect to any letter (except only x) is so much of the *total* differential coefficient as is seen in the explicit coefficient of the next letter. Again, we must observe the following theorems:

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* I cannot anywhere find it noticed that the well known forms V_t, $V_s - V_t'$, &c. are nothing but the partial differential coefficients of the integral required. [Except by M. Sarrus.]

If V be integrable *per se*, so are also V_x and V_y; and moreover, V is identically equivalent with

$$\int V_x dx + (V_p - V_q' + \ldots)\, p + (V_q - V_r' + \ldots)\, q + (V_r - V_s' + \ldots)\, r + \ldots$$

If V be integrable *per se*, so also is

$$V_y W + V_p W' + V_q W'' + V_r W''' + \ldots\ldots$$

where W is any function whatsoever of $x, y, p, q, \ldots$, or of any of them. For common integration by parts shows that this is integrable or not with $\int (V_y - V_p' + \ldots)\, W dx$. This is an easy way of making complicated integrable functions from simple ones.)

I have put the preceding remarks in parentheses, because, though immediately arising out of what goes before, they are not wanted in what follows, namely, in proving the ***sufficiency*** of the criterion. For this purpose observe, first, that if the criterion exist, or if $V_y - V_p' + \ldots = 0$, we can re-establish the last six equations (and thence the first) in *form*, without any assumption as to X, Y, &c. being all partial differential coefficients of one function. For we have a right to assume $S = V_t$, $R = V_s - S' = V_s - V_t'$, &c., and proceeding thus, we obtain $Y = V_p - P' = V_p - V_q' + \ldots$. But, by the criterion, $V_y = V_p' - \ldots = Y'$, whence the second equation follows: lastly, assume X so that $V = X + Yp + \ldots$ and we have, as before shewn, $V_x = X'$.

From these forms alone, and the universal theorem (W), it follows that in every case $M_n = N_m$, or that $X + Yp + \ldots$ is a complete differential coefficient. Take any letter, say r. If the partial tests of integrability be all true whenever one of the letters m, n, is r, they are all true when one of them is q, the letter preceding r. To prove this, observe that

$$(V_q)_r = Q_r' + Q_q + P_r = R_q' + Q_q + R_p \text{ by hypothesis,}$$
$$(V_r)_q = R_q' + R_p + Q_q.$$

This proves nothing, being only an identity. Now take

$$(V_p)_r = P_r' + P_q + Y_r = R_p' + P_q + R_y,$$
$$(V_r)_p = R_p' + R_y + Q_p, \text{ whence } P_q = Q_p.$$

Again,
$$(V_y)_r = Y_r' + Y_q = R_y' + Y_q,$$
$$(V_r)_y = R_y' + Q_y; \text{ whence } Q_y = Y_q.$$

Lastly,
$$(V_x)_r = X_r' + X_q = R_x' + X_q,$$
$$(V_r)_x = R_x' + Q_x; \text{ whence } Q_x = X_q,$$

whence the theorem, if true for any case in which r enters, is true for any case in which q enters. Everything then depends upon proving the tests in which s enters, since no one of the capitals is a function of t.

$$\text{Now} \qquad V_{ts} = S_s, \quad V_{st} = 0 + S_s + 0$$

which is an identity. But $V_{tr} = S_r$, and $V_{rt} = 0 + R_s + 0$, whence $S_r = R_s$; and so on. Hence it is proved, from first principles, that the criterion is sufficient as well as necessary.

If the function $X + Yp + Pq + \ldots$ be integrable *per se*, it appears that $Xdx + Ydy + Pdp + \ldots$ is integrable *per se*, independently of the connexion between y, p, q, &c. If then

$$Ada + Bdb + Cdc + Ede + \ldots$$

be a complete differential of n variables a, b, c, &c., it follows that the partial tests $A_b = B_a$, &c. $\frac{1}{2}n(n-1)$ in number, are all contained in one, as follows. Assume b a function of a, and let c, e, &c. be the differential coefficients of b with respect to a. Then the condition that $Ada + Bdb + \ldots$ is integrable, or rather the collection of all the $\frac{1}{2}n(n-1)$ conditions, is seen in

$$0 = (A + Bc + Ce + \ldots)_b - (A + Bc + Ce + \ldots)_c' + (A + Bc + Ce + \ldots)_e'' - \ldots;$$

and as the above suppositions may be made upon every order in which the letters can be written, we have $1 \, . \, 2 \, . \, 3 \ldots n$ modes in which this reduction to one form can be applied; any one of which is sufficient.

This one criterion, then, involves many; but as we know that perfectly distinct relations cannot be reduced explicitly to one, we cannot but say we know that $V_y - V'_p + \ldots = 0$ is not merely one relation. The following mode of shewing that it involves others suggested itself to me before I was in possession of the preceding sole equation.

If we were to suppose any letters beginning with y, say y, p, q, to be constants, so that V is a function of x, r a function of x, and s, t, its first two differential coefficients, it is clear that we should still have an integrable function. For if $Ada + Bdb +$ &c. be integrable independently of relation between a, b, &c. it is still integrable if we make $da = 0$, or $db = 0$, &c. Consequently, on such a supposition, $V_r - V_s' + V_t''$ vanishes. That is, if in V_r, V_s, V_t, we write ρ, σ, τ, for r, s, t, in every place in which they appear, and then form $V_r - V_s' + V_t''$, allowing r, s, t, to appear whenever they result from y, p, q, but writing σ for ρ' &c.,—the part containing ρ, σ, τ, will vanish identically.

It appears above that the mode of actually finding $\int Vdx$ is equally applicable to the determination of $\int(Ada + Bdb \ldots)$, when a and b are unconnected. As cleared from all liability to fail by M. Jacques Binet, it is as follows.

In V write $u + zy$ for y, u being a function of x only which must be definitely assigned in any convenient way before the operation is completed; y being an indeterminate function of x; and z a variable independent of x and y. Let p, q, r, &c. be the differential coefficients of y in $u + zy$. If then we look at V, which is a function of x, $u + zy$, $u' + zp$, &c., as a function of z, say fz, we have $f(1) = f(0) + \int_0^1 f'z\,dz$,

$$\text{or } V_1 = V_0 + \int_0^1 (V_y y + V_p p + V_q q + V_r r + V_s s + V_t t)\,dz,$$

where V_1 is the function of x, $u + y$, $u' + p$, &c. and V_0 the same function of x, u, u', &c., which is a determinate function of x: and V_y, V_p, &c. have $u + zy$, $u' + zp$, &c. written for y, p, &c. Consequently

$$\int V_1 dx = \int V_0 dx + \int_0^1 dz \, . \int (V_y y + V_p p + \ldots)\,dx$$

of which the first term is determinate, while the second term is found by performing $\int_0^1 dz$ upon

$$\begin{aligned}&y(V_p - V_q' + V_r'' - V_s''' + V_t^{iv}) + p(V_q - V_r' + V_s'' - V_t''')\\ &+ q(V_r - V_s' + V_t'') + r(V_s - V_t') + sV_t\\ &+ \int y(V_y - V_p' + V_q'' - V_r''' + V_s^{iv} - V_t^{v})dx.\end{aligned}$$

The sufficiency of the criterion now appears again; and when it is satisfied, we see that we have only a determinate function of x to integrate, and after integration, and addition of V_0, to write $y - u$ for y, $p - u'$ for p, &c.

In Poisson's process, the result is that of the preceding with $u = 0$: but, as M. Moigno remarks, this supposition may make V_0 infinite or incalculable.

But the easiest way of arriving at $\int V dx$, in any case that can occur, is by observing that, when the criterion is satisfied, V_x and V_y are both integrable *per se*, and give $(\int V dx)_x$ and $(\int V dy)_y$ by the process. Again, if t be the last letter that enters, we know that $\int V dx = \int V_t ds$ (s only varying) + (a function of precedents of s): and if observation be not now sufficient to detect that function, we have* other partial differential coefficients of the integral sought in $V_s - V_t'$, &c. M. M. Bertrand and Moigno have both preferred an indirect mode, (which introduces arbitrary constants to be determined during the process) in preference to the operation $\int_0^1 dx$: and the example they have both chosen is

$$V = 2x + y^2 + 2xyp + xq + x^2r - p.$$

With this I should proceed as follows:—If $\int V dx = U$, we have

$$\begin{aligned}&V_x = 2 + 2yp + q + 2xr, &&U_x = 2x + y^2 + p + 2xq - 2p = 2x + y^2 + 2xq - p,\\ &V_y = 2y + 2xp, &&U_y = 2xy.\end{aligned}$$

and x^2q must be the only term involving q; whence the integral must be immediately seen to be

$$U = x^2 + xy^2 + x^2q - px.$$

It is sufficiently well known that, in mathematics, undefined dependence is independence, though perhaps the maxim never finds words. But it appears in the preceding that this remains true to the following extent. If b be *any* function of a, and if c, e, &c. be the successive differential coefficients of b with respect to a, then, till the connexion of b and a is defined, a, b, c, e, &c., are perfectly independent of each other. This might easily be established otherwise.

SECTION V.

On the demonstration† *of the mode of transforming multiple integrals.*

I have always been dissatisfied with the manner in which Lagrange, Legendre, and all other writers who have treated the subject, establish the mode of changing the variables in a multiple integral. In my *Differential Calculus*, (pp. 392—395,) I abandoned this mode, and substituted another, which for the case of two variables, as shewn, and of three, as might be shewn, is perfectly sufficient and not very difficult. Having thus evaded the necessity of considering the usual demonstration, I have never, until now, endeavoured fully to settle in my own mind the question whether the doubts which I entertained about it were or were not well grounded. This I have now done, and have satisfied myself that the fault is one of omission only, and that the omission can be easily supplied. It would not become me to conclude that analysts do *not* supply this omission, each for himself: and yet I am forced to suspect it by observing that the most recent writers, as M. Cauchy and M. Catalan, (cited by M. Moigno), neither mention omission, nor supply it.

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* We *have* them, because the easiest way of establishing the criterion (which must be done first) is by going through the successive formation of $S = V_t$, $R = V_s - S'$, $Q = V_r - R'$, &c.

† This section was a subsequent communication, dated February 22, 1851.

In making the summation expressed by $\iint z\,dx\,dy$, any process is sufficient which divides the whole extent (say an area), over which integration is made, into elements of the second order, n in number, n being infinite, and sums all, or neglects a number of an order lower than n. It is not necessary that to one value, or rather position and value, of dx, should correspond an infinite number of cases of dy: it is enough that the whole area signified by $\iint dx\,dy$ should be subdivided into an infinite number of rectangles, each of which has all its sides coinciding with sides, or parts of sides, of others. We have then to shew that the particular mode of treatment we propose for $\iint dx\,dy$ just, as it were, *covers the ground:* and we have then to treat $\iint z\,dx\,dy$ by that mode, *or another.* There is an unlimited number of ways of dividing a given area into rectangles of the sort of contiguity required: and it is obviously allowable to shew one way of filling up the required extent, and afterwards to use another way in summation; provided always that we assure ourselves of the equivalence of the two extents which are used. Now it is the sole defect of the usual proof, that in converting $\iint z\,dx\,dy$ into $\iint w\,du\,dv$, the mode of dividing into elements of the form $du\,dv$ which is adopted in the proof of equivalence, is *not* that which is intended to be used in the subsequent integration, and that this is not stated. Legendre's* proof, changing symbols, is literally as follows, (*Mem. Acad. Sc.* 1788, pp. 458, 459.)

Let u and v be two variables which we are to introduce in place of x and y: suppose that between these four quantities we have established two equations, from whence we have

$$dx = x_u\,du + x_v\,dv,$$
$$dy = y_u\,du + y_v\,dv,$$

the value of dx in $z\,dx\,dy$ must always suppose y constant; so that for substitution we have

$$y_u\,du + y_v\,dv = 0, \text{ or } dv = -\,y_u\,du \div y_v: \text{ whence}$$

$$dx = \frac{x_u y_v - x_v y_u}{y_v}\,du, \qquad z\,dx\,dy = \frac{x_u y_v - x_v y_u}{y_v}\,z\,du\,dy.$$

We have yet to introduce v in place of y: for this, observe that dy in the above element supposes u constant. We have then $dy = y_v\,dv$ and the final result of the transformation is

$$(x_u y_v - x_v y_u)\,z\,du\,dv.$$

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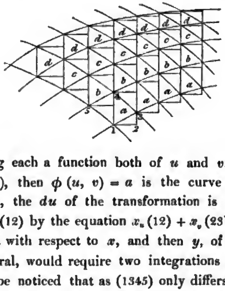
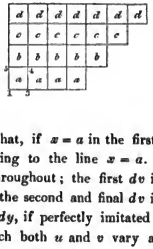
* See also Lagrange (*Mem. Acad.* 1773, p. 125). I have cited Legendre, whose case has two independent variables, in preference to Lagrange, whose case has three.

Here is then a relation instituted between *du* and *dv*, which is forthwith tacitly abandoned after being used. M. Cauchy makes the abandonment formal. He says (Moigno, Vol. II. p. 215.) "Dès lors" that is after *du* has replaced *dx*, "ces deux variables devraient être indépendantes," for which no reason is given. Now it is plain that there is no objection to our instituting a relation between *du* and *dv* in the element *dudv*. If we take *u* and *v* to be new rectangular co-ordinates, and mark out the (*uv*) area corresponding to the (*xy*) area over which we are to integrate, we may cover this area with elementary rectangles, in each of which *dv* is a function of *du*, *u*, and *v*. We may not have a *process* of integration, inverse to differentiation, by which to sum the results: but we want no such process for distinct conception either of the end to be gained, or of any approximate amount of acquisition. It is not true, indeed, that there is a relation between *du* and *dv* in the final element above constructed: the second *dv* is a different one from the first. But it is true that $\iint(x_u y_v - x_v y_u)\, z\, du\, dv$, summed in the same order of elements as $\iint z\, dx\, dy$, will not proceed first with respect to *u* and then with respect to *v*, but in another mode, in which *u* and *v* vary together, both in the first integration, and in the second.

In the plane of *xy* draw perpendicular straight lines having the equations $x = \text{const.}$, $y = \text{const.}$; and in the plane of *uv* draw the curves which answer to these straight lines,





x and *y* being each a function both of *u* and *v*. So that, if $x = a$ in the first system, and $x = \phi(u, v)$, then $\phi(u, v) = a$ is the curve answering to the line $x = a$. Taking the element 1345, the *du* of the transformation is (12) throughout; the first *dv* is (23) which is related to (12) by the equation $x_u(12) + x_v(23) = 0$; the second and final *dv* is (14). The integral, first with respect to *x*, and then *y*, of $\iint z\, dx\, dy$, if perfectly imitated in the transformed integral, would require two integrations in which both *u* and *v* vary at every step. And it may be noticed that as (1345) only differs from (24) by a quantity of the third order, the usual mode of proceeding with $\int z\, dx\, dy$ may be imitated with the curvilinear elements. But it is not intended that the transformed integral $\iint w\, du\, dv$ should be summed in this way: a different subsequent resolution of the whole (*uv*) area, by lines parallel to the axes of *u* and *v*, is always contemplated, and the absence of all notice of this step has, I dare say, made the process as unintelligible to some others as it was to me.

I need not enter upon the way in which the preceding remarks may be made to apply to triple, &c., integrals.

A. DE MORGAN.

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January 13, 1851.

ADDITIONS.

Section 1, page 8 (of the memoir). It should have been noticed that when $\chi_z = \infty$ or $\chi_y = \infty$ produces a result which does not satisfy $y' = \chi$, the equation which it *does* satisfy is $y' = \chi - X$, where X is the value of $P \div \chi_y$.

Though Clairaut was the first who entered upon the theory of the singular solution, yet it must be observed that (as noted by the late D. F. Gregory) the existence of such solutions was first seen, and the name given to them, by Brook Taylor (*Method. Increment.* p. 27). His instance is $y^2 - 2xyy' + (1 + x^2)\, y'^2 = 1$, of which his general solution is $y = a + x\sqrt{(1 - a^2)}$, and his *singularis quædam solutio* is $y^2 = 1 + x^2$, obtained by differentiation.

Section 2. The method given in my former paper for partial differential equations, and first imitated in section 2 with reference to equations between two variables, and then extended, may now have the corresponding extension made with reference to partial differential equations. Let $\phi\,(x, y, z, X, Y, Z) = 0$ be an equation connecting z with the variables x, y, and the constants X, Y, Z. Let p, q, r, s, t, have their usual significations. From $\phi = 0$, $\phi_x + \phi_z \cdot p = 0$, $\phi_y + \phi_z \cdot q = 0$, we may obtain $X = \xi\,(x, y, z, p, q)$, $Y = \eta\,(x, y, z, p, q)$, $Z = \zeta\,(x, y, z, p, q)$: from which, if Z be a function of X and Y, we obtain the partial differential equation of which $\phi = 0$, X and Y being constant, is Lagrange's *primary solution*. Instead of this, let all six letters be variables, and let z be a function of x and y understood, whether determined or undetermined: then $X = \xi$, $Y = \eta$, $Z = \zeta$ imply that Z is a conjugate function of X and Y. These last relations are substitutions for transformation, by aid of which any partial differential equation may be transformed into another of the same order. For, P, Q, R, &c. being differential coefficients of Z, $\phi = 0$ gives

$$(\phi_x + \phi_z p)\, dx + (\phi_y + \phi_z q)\, dy + (\phi_X + \phi_Z P)\, dX + (\phi_Y + \phi_Z Q)\, dY = 0,$$

whence since $\phi_x + \phi_z p = 0$ and $\phi_y + \phi_z q = 0$ even when X and Y are variable (for under these relations they were made to vary), we have $\phi_X + \phi_Z P = 0$, $\phi_Y + \phi_Z Q = 0$, that is, P and Q are expressible in terms of x, y, z, p, q. Carrying this on, we may shew that R, S, T are expressible by aid of no higher differential coefficients of z than r, s, t. And the analogies alluded to in section 2 between the differential equations in which the small letters are constants, and those in which the large letters are constants, exist also in this case. If then, obtaining from the above

$$P = \varpi\,(x, y, z, p, q), \quad Q = \kappa\,(x, y, z, p, q),$$

we have an equation to be integrated,

$$f\,\{\xi(x, y, z, p, q), \quad \eta(\,.\,.), \quad \zeta(\,.\,.), \quad \varpi(\,.\,.), \quad \kappa(\,.\,.)\,\ldots\ldots\} = 0,$$

and if we can integrate $f\,(X, Y, Z, P, Q, \ldots) = 0$, it follows that we integrate the original equation, by eliminating X, Y, p, q, between the five equations $X = \xi$, $Y = \eta$, $Z = \zeta$, $P = \varpi$, $Q = \kappa$, in which Z is now known in terms of X and Y. If we choose that X, Y, Z shall be reciprocally related to x, y, z, we have but to take a form of ϕ in which x, y, z are simultaneously interchangeable with X, Y, Z, as in $z + Z = Xx + Yy$, the case treated by M. Chasles, and by myself in my former paper.

To take another such instance, let us have $Xx + Yy + Zz = 1$, $X + Zp = 0$, $Y + Zq = 0$, giving

$$X = \frac{p}{px + qy - z}, \quad Y = \frac{q}{px + qy - z}, \quad Z = -\frac{1}{px + qy - z},$$

$$P = -\frac{x}{z}, \qquad Q = -\frac{y}{z};$$

in which the great and small letters are interchangeable.

M. Chasles himself has gone so far as to extend his case, by trial, to every instance in which X, Y, and Z are fractions having numerators and denominators which are linear functions of p, q, and $px + qy - z$. The preceding is therefore not a new case: but we might get one if we took $X^2x^2 + Y^2y^2 + Z^2z^2 = 1$, or the like, to say nothing of the unlimited number of cases in which the transforming functions are not reciprocal.

Section 3. In confirmation of the intermediate primitives of a differential equation containing arbitrary functions, it may be shown that an equation of the nth order is not distinguishable, so far as one integration is concerned, from a partial differential equation with n independent variables. Let $y''' = \phi(x, y, y', y'')$, or, making $y' = p$, $y'' = q$, &c.,

$$r = \phi(x, y, p, q). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the first integral of this must give q in terms of x, y, p. When this integral is found, it must satisfy the partial differential equation

$$\frac{dq}{dx} + \frac{dq}{dy}\cdot p + \frac{dq}{dp}\cdot q = \phi(x, y, p, q). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Obviously, then, the first integral of (1) is contained in the complete solution of (2): if my conclusion be true, the second is not more extensive than the first. The first and chief objection to this would seem to be that (1) has in it implied conditions which do not accompany (2). In (1), p is understood to be a function of x and y, and y a function of x: while in (2), though these relations are recognized in the formation of the first side from q, they do not exist as part of the connexion between the equation and its solution. To which I reply that though in the course of the progress towards the *original* primitive of (1), we recognize the connexion between p, y, and x, yet in the first step of that transition, it will be found that we do not further recognize it than was done in the substitution of $q_x + q_y\cdot p + q_p\cdot q$ instead of r. For if $q = \psi(x, y, p)$ be the first integral, we return to (1) by combining it with

$$r = \psi_x + \psi_y\cdot p + \psi_p\cdot q,$$

without any other consideration whatever.

Now let $q = \psi(x, y, p)$ satisfy (2): x, y, p, being independent. The following equation then is *identically* true,

$$\psi_x + \psi_y\cdot p + \psi_p\cdot \psi = \phi(x, y, p, \psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and being identically true, is true if for p we write y'. And in this equation is all that is necessary for deducing $r = \phi(x, y, p, q)$ from $q = \psi(x, y, p)$ on the supposition that p, q, and r, are y', y'', y''': which shows that $r = \phi(x, y, p, q)$ follows from $q = \psi(x, y, p)$, when $p = y'$, &c. provided only that $r = \phi(x, y, p, q)$ satisfies (2) when x, y, p are independent. The whole of this part of the argument may be summed up as follows:—

When we grant the absolute identity of

$$\psi_x(x,\ y,\ y') + \psi_y(x,\ y,\ y')\cdot y' + \psi_{y'}(x,\ y,\ y')\cdot\psi(x,\ y,\ y'),$$

$$\text{and}\qquad \phi\{x,\ y,\ y',\quad \psi(x,\ y,\ y')\}$$

we grant all that is necessary to prove that $y''' = \phi(x,\ y,\ y',\ y'')$ follows from $y'' = \psi(x,\ y,\ y')$, and no more.

When we grant that $q = \psi(x,\ y,\ p)$ satisfies

$$\frac{dq}{dx} + \frac{dq}{dy}p + \frac{dq}{dp}q = \phi(x,\ y,\ p,\ q)$$

in which x, y, p, are independent variables, it can be shown that we have granted the preceding identity, and no more. Consequently, every solution of the partial differential equation gives an integral of

$$y''' = \phi(x,\ y,\ y',\ y''),$$

and every integral can be thus obtained.

It can further be shewn that any solution of (1) is ordinary or singular, according as the solution of (2) from which it arises is ordinary or singular.

July 3, 1851.

ERRATUM.

At the beginning of Section 1, *for* $\phi(x + y\sqrt{-1})$, *read* $\phi(x,\ y,\ a + b\sqrt{-1})$

XII. *Of the Transformation of Hypotheses in the History of Science.*

By W. Whewell, D.D., *Master of Trinity College.*

[Read *May* 19, 1851.]

1. The history of science suggests the reflection that it is very difficult for the same person at the same time to do justice to two conflicting theories. Take for example the Cartesian hypothesis of vortices and the Newtonian doctrine of universal gravitation. The adherents of the earlier opinion resisted the evidence of the Newtonian theory with a degree of obstinacy and captiousness which now appears to us quite marvellous: while on the other hand, since the complete triumph of the Newtonians, *they* have been unwilling to allow any merit at all to the doctrine of vortices. It cannot but seem strange, to a calm observer of such changes, that in a matter which depends upon mathematical proofs, the whole body of the mathematical world should pass over, as in this and similar cases they seem to have done, from an opinion confidently held, to its opposite. No doubt this must be, in part, ascribed to the lasting effects of education and early prejudice. The old opinion passes away with the old generation: the new theory grows to its full vigour when its congenital disciples grow to be masters. John Bernoulli continues a Cartesian to the last; Daniel, his son, is a Newtonian from the first. Newton's doctrines are adopted at once in England, for they are the solution of a problem at which his contemporaries have been labouring for years. They find no adherents in France, where Descartes is supposed to have already explained the constitution of the world; and Fontenelle, the secretary of the Academy of Sciences at Paris, dies a Cartesian seventy years after the publication of Newton's *Principia*. This is, no doubt, a part of the explanation of the pertinacity with which opinions are held, both before and after a scientific revolution: but this is not the whole, nor perhaps the most instructive aspect of the subject. There is another feature in the change, which explains, in some degree, how it is possible that, in subjects, mainly at least mathematical, and therefore claiming demonstrative evidence, mathematicians should hold different and even opposite opinions. And the object of the present paper is to point out this feature in the successions of theories, and to illustrate it by some prominent examples drawn from the history of science.

2. The feature to which I refer is this; that when a prevalent theory is found to be untenable, and consequently, is succeeded by a different, or even by an opposite one, the change is not made suddenly, or completed at once, at least in the minds of the most tenacious adherents of the earlier doctrine; but is effected by a transformation, or series of transformations, of the earlier hypothesis, by means of which it is gradually brought nearer and nearer to the second; and thus, the defenders of the ancient doctrine are able to go on as if still asserting their first opinions, and to continue to press their points of advantage, if they have any, against the new theory. They borrow, or imitate, and in some way accommodate to their original hypothesis, the new explanations which the new theory gives, of the observed facts; and thus they maintain a

sort of verbal consistency; till the original hypothesis becomes inextricably confused, or breaks down under the weight of the auxiliary hypotheses thus fastened upon it, in order to make it consistent with the facts.

This often-occurring course of events might be illustrated from the history of the astronomical theory of epicycles and eccentrics, as is well known. But my present purpose is to give one or two brief illustrations of a somewhat similar tendency from other parts of scientific history; and in the first place, from that part which has already been referred to, the battle of the Cartesian and Newtonian systems.

3. The part of the Cartesian system of vortices which is most familiarly known to general readers is the explanation of the motions of the planets by supposing them carried round the sun by a kind of whirlpool of fluid matter in which they are immersed: and the explanation of the motions of the satellites round their primaries by similar subordinate whirlpools, turning round the primary, and carried, along with it, by the primary vortex. But it should be borne in mind that a part of the Cartesian hypothesis which was considered quite as important as the cosmical explanation, was the explanation which it was held to afford of terrestrial gravity. Terrestrial gravity was asserted to arise from the motion of the vortex of subtile matter which revolved round the earth's axis and filled the surrounding space. It was maintained that by the rotation of such a vortex, the particles of the subtile matter would exert a centrifugal force, and by virtue of that force, tend to recede from the center: and it was held that all bodies which were near the earth, and therefore immersed in the vortex, would be pressed towards the center by the effort of the subtile matter to recede from the center*.

These two assumed effects of the Cartesian vortices—to carry bodies in their stream, as straws are carried round by a whirlpool, and to press bodies to the center by the centrifugal effort of the whirling matter—must be considered separately, because they were modified separately, as the progress of discussion drove the Cartesians from point to point. The former effect indeed, the *dragging* force of the vortex, as we may call it, would not bear working out on mechanical principles at all; for as soon as the law of motion was acknowledged (which Descartes himself was one of the loudest in proclaiming), that a body in motion keeps all the motion which it has, and receives in addition all that is impressed upon it;—as soon, in short, as philosophers rejected the notion of an inertness in matter which constantly retards its movements,—it was plain that a planet perpetually dragged onwards in its orbit by a fluid moving quicker than itself, must be perpetually accelerated; and therefore could not follow those constantly-recurring cycles of quicker and slower motion which the planets exhibit to us.

The Cartesian mathematicians, then, left untouched the calculation of the progressive motion of the planets; and, clinging to the assumption that a vortex would produce a tendency of bodies to the center, made various successive efforts to construct their vortices in such a manner that the centripetal forces produced by them should coincide with those which the phenomena required, and therefore of course, in the end, with those which the Newtonian theory asserted.

In truth, the Cartesian vortex was a bad piece of machinery for producing a central force: from the first, objections were made to the sufficiency of its mechanism, and most of these objections were very unsatisfactorily answered, even granting the additional machinery which its defenders demanded. One formidable objection was soon started, and continued to the last to be the torment of the Cartesians. If terrestrial gravity, it was urged, arise from the centrifugal force of a vortex which revolves about the earth's axis, terrestrial gravity ought to act in planes perpendicular to the earth's axis, instead of tending to the earth's center. This objection was taken by James Bernoulli[*], and by Huyghens[†] not long after the publication of Descartes's *Principia*. Huyghens (who adopted the theory of vortices with modifications of his own) supposes that there are particles of the fluid matter which move about the earth in every possible direction, within the spherical space which includes terrestrial objects; and that the greater part of these motions being in spherical surfaces concentric with the earth, produces a tendency towards the earth's center.

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* Cartes. *Princip.* IV. 23.

This was a procedure tolerably arbitrary, but it was the best which could be done. Saurin, a little later[‡], gave nearly the same solution of this difficulty. The solution, identifying a vortex of some kind with a central force, made the hypothesis of vortices applicable wherever central forces existed; but then, in return, it deprived the image of a vortex of all that clearness and simplicity which had been its first great recommendation.

But still there remained difficulties not less formidable. According to this explanation of gravity, since the tendency of bodies to the earth's center arose from the superior centrifugal force of the whirling matter which pushed them inward as water pushes a light body upward, bodies ought to tend more strongly to the center in proportion as they are less dense. The rarest bodies should be the heaviest; contrary to what we find.

Descartes's original solution of this difficulty has a certain degree of ingenuity. According to him (*Princip.* IV. 23) a terrestrial body consists of particles of the *third element*, and the more it has of such particles, the more it excludes the parts of the *celestial matter*, from the revolution of which matter gravity arises; and therefore the denser is the terrestrial body, and the heavier it will be.

But though this might satisfy him, it could not satisfy the mathematicians who followed him, and tried to reduce his system to calculation on mechanical principles. For how could they do this, if the celestial matter, by the operation of which the phenomena of force and motion were produced, was so entirely different from ordinary matter, which alone had supplied men with experimental illustrations of mechanical principles? In order that the celestial matter, by its whirling, might produce the gravity of heavy bodies, it was mechanically necessary that it must be very dense; and *dense* in the ordinary sense of the term; for it was by regarding density in the ordinary sense of the term that the mechanical necessity had been established.

The Cartesians tried to escape this result (Huyghens, *Pesanteur*, p. 161, and John Bernoulli, *Nouvelles Pensées*, Art. 31) by saying that there were two meanings of *density* and *rarity*; that some fluids might be rare by having their particles far asunder, others, by having their particles very small though in contact. But it is difficult to think that they could, as persons well acquainted with mechanical principles, satisfy themselves with this distinction; for they could hardly fail to see that the mechanical effect of any portion of fluid depends upon the total mass moved, not on the size of its particles.

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[*] Jac. Bernoulli, *Nouvelles Pénsees sur le Systéme de M. Descartes*, Op. T. I. p. 239 (1686).

[†] *De la Cause de la Pesanteur* (1689), p. 135.

[‡] Journal des Savans, 1703. Mém. Acad. Par. 1709.

Bulfinger, in 1726 (Acad. Petrop.), conceived that by making a sphere revolve at the same time about two axes at right angles to each other, every particle would describe a great circle; but this is not so.

Attempts made to exemplify the vortices experimentally only shewed more clearly the force of this difficulty. Huyghens had found that certain bodies immersed in a whirling fluid tended to the center of the vortex. But when Saulmon* a little later made similar experiments, he had the mortification of finding that the heaviest bodies had the greatest tendency to recede from the axis of the vortex. "The result is," as the Secretary of the Academy (Fontenelle) says, "exactly the opposite of what we could have wished, for the [Cartesian] system of gravity: but we are not to despair; sometimes in such researches disappointment leads to ultimate success."

But, passing by this difficulty, and assuming that in some way or other a centripetal force arises from the centrifugal force of the vortex, the Cartesian mathematicians were naturally led to calculate the circumstances of the vortex on mechanical principles; especially Huyghens, who had successfully studied the subject of centrifugal force. Accordingly, in his little treatise on the *Cause of Gravitation*, (p. 143) he calculates the velocity of the fluid matter of the vortex, and finds that, at a point in the equator, it is 17 times the velocity of the earth's rotation.

It may naturally be asked, how it comes to pass that a stream of fluid, dense enough to produce the gravity of bodies by its centrifugal force, moving with a velocity 17 times that of the earth (and therefore moving round the earth in 85 minutes), does not sweep all terrestrial objects before it. But to this Huyghens had already replied (p. 137), that there are particles of the fluid moving *in all directions*, and therefore that they neutralize each other's action, so far as lateral motion is concerned.

And thus, as early as this treatise of Huyghens, that is, in three years from the publication of Newton's *Principia*, a vortex is made to mean nothing more than some machinery or other for producing a central force. And this is so much the case, that Huyghens commends (p. 165), as confirming his own calculation of the velocity of his vortex, Newton's proof that at the Moon's orbit the centripetal force is equal to the centrifugal; and that thus, this force is less than the centripetal force at the earth's surface in the inverse proportion of the squares of the distances.

John Bernoulli, in the same manner, but with far less clearness and less candour, has treated the hypothesis of vortices as being principally a hypothetical cause of central force. He had repeated occasions given him of propounding his inventions for propping up the Cartesian doctrine, by the subjects proposed for prizes by the Paris Academy of Sciences; in which competition Cartesian speculations were favourably received. Thus the subject of the Prize Essays for 1730 was, the explanation of the Elliptical Form of the planetary orbits and of the Motion of their Aphelia, and the prize was assigned to John Bernoulli, who gave the explanation on Cartesian principles. He explains the elliptical figure, not as Descartes himself had done, by supposing the vortex which carries the planet round the sun to be itself squeezed into an elliptical form by the pressure of contiguous vortices; but he supposes the planet, while it is carried round by the vortex, to have a limited oscillatory motion to and from the center, produced by its being originally, not at the distance at which it would float in equilibrium in the vortex, but above or below that point. On this supposition, the planet would oscillate to and from the center, Bernoulli says, like the mercury when deranged in a barometer: and it is evident that such an oscillation, combined with a motion round the center, might produce an oval curve, either with a fixed or with a moveable aphelion. All this however merely amounts to a possibility that the oval *may* be an ellipse, not to a proof that it will be so; nor does Bernoulli advance further.

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* Acad. Par. 1714. *Hist.* p. 106.

It was necessary that the vortices should be adjusted in such a manner as to account for Kepler's laws; and this was to be done by making the velocity of each stratum of the vortex depend in a suitable manner on its radius. The Abbé de Molières attempted this on the supposition of elliptical vortices, but could not reconcile Kepler's first two laws, of equal elliptical areas in equal times, with his third law, that the squares of the periodic times are as the cubes of the mean distances*. Bernoulli, with his circular vortices, could accommodate the velocities at different distances so that they should explain Kepler's laws. He pretended to prove that Newton's investigations respecting vortices (in the ninth Section of the Second Book of the *Principia*) were mechanically erroneous; and in truth, it must be allowed that, besides several arbitrary assumptions, there are some errors of reasoning in them. But for the most part, the more enlightened Cartesians were content to accept Newton's account of the motions and forces of the solar system as part of their scheme; and to say only that the hypothesis of vortices explained the origin of the Newtonian forces; and that thus theirs was a philosophy of a higher kind. Thus it is asserted (Mém. Acad. 1734), that M. de Molières retains the beautiful theory of the Newton entire, only he renders it in a sort less Newtonian, by disentangling it from attraction, and transferring it from a vacuum into a plenum. This plenum, though not its native region, frees it from the need of attraction, which is all the better for it. These points were the main charms of the Cartesian doctrine in the eyes of its followers;—the getting rid of attractions, which were represented as a revival of the Aristotelian "occult qualities," "substantial forms," or whatever else was the most disparaging way of describing the bad philosophy of the dark ages†;—and the providing some material intermedium, by means of which a body may affect another at a distance; and thus avoid the reproach urged against the Newtonians, that they made a body act where it was not. And we are the less called upon to deny that this last feature in the Newtonian theory was a difficulty, inasmuch as Newton himself was never unwilling to allow that gravity might be merely an effect produced by some ulterior cause.

With such admissions on the two sides, it is plain that the Newtonian and Cartesian systems would coincide, if the hypothesis of vortices could be modified in such a way as to produce the force of gravitation. All attempts to do this, however, failed: and even John Bernoulli, the most obstinate of the mathematical champions of the vortices, was obliged to give them up. In his Prize Essay for 1734, (on the Inclinations of the Planetary Orbits*,) he says (Art. VIII.), "The gravitation of the Planets towards the center of the Sun and the weight of bodies towards the center of the earth has not, for its cause, either the attraction of M. Newton, or the centrifugal force of the matter of the vortex according to M. Descartes;" and he then goes on to assert that these forces are produced by a perpetual torrent of matter tending to the center on all sides, and carrying all bodies with it. Such a hypothesis is very difficult to refute. It has been taken up in more modern times by Le Sage†, with some modifications; and may be made to account for the principal facts of the universal gravitation of matter. The great difficulty in the way of such a hypothesis is, the overwhelming thought of the whole universe filled with torrents of an invisible but material and tangible substance, rushing in every direction in infinitely prolonged straight lines and with immense velocity. Whence can such matter come, and whither can it go? Where can be its perpetual and infinitely distant fountain, and where the ocean into which it pours itself when its infinite course is ended? A revolving whirlpool is easily conceived and easily supplied; but the central torrent of Bernoulli, the infinite streams of particles of Le Sage, are an explanation far more inconceivable than the thing explained.

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* Acad. Par. 1733.

† Acad. Sc. 1709. If we abandon the clear principles of mechanics, the writer says, "toute la lumière que nous pouvons avoir est eteinte, et nous voilà replongés de nouveau dans les anciennes tenébres du Peripatetisme, dont le Ciel nous veuille preserver!"

It was also objected to the Newtonian system, that it did not account for the remarkable facts, that all the motions of the primary planets, all the motions of the satellites, and all the motions of rotation, including that of the sun, are in the same direction, and nearly in the same plane; facts which have been urged by Laplace as so strongly recommending the Nebular Hypothesis; and that hypothesis is, in truth, a hypothesis of vortices respecting the *origin* of the system of the world.

But however the hypothesis of vortices, or some hypothesis substituted for it, was adjusted to explain the facts of attraction to a center, this was really nearly all that was meant by a vortex or a "tourbillon," when the system was applied. Thus in the case of the last act of homage to the Cartesian theory which the French Academy rendered in the distribution of its prizes, the designation of a Cartesian Essay in 1741 (along with three Newtonian ones) as worthy of a prize for an explanation of the Tides; the difference of high and low water was not explained, as Descartes has explained it, by the pressure, on the ocean, of the terrestrial vortex, forced into a strait where it passes under the Moon; but the waters were supposed to rise towards the Moon, the terrestrial vortex being disturbed and broken by the Moon, and therefore less effective in forcing them down. And in giving an account of a Tourmaline from Ceylon, (Acad. Sc. 1717) when it has been ascertained that it attracts and repels substances, the writer adds, as a matter of course, "It would seem that it has a vortex." As another example, the elasticity of a body was ascribed to vortices between its particles: and in general, as I have said, a vortex implied what we now imply by speaking of a central force.

4. In the same manner vortices were ascribed to the Magnet, in order to account for its attractions and repulsions. But we may note a circumstance which gave a special turn to the hypothesis of vortices as applied to this subject, and which may serve as a further illustration of the manner in which a transition may be made from one to the other of two rival hypotheses.

If iron filings be brought near a magnet, in such a manner as to be at liberty to assume the position which its polar action assigns to them; (for instance, by strewing them upon a sheet of paper while the two poles of the magnet are close below the paper;)

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* *Nouvelle Physique Celeste*, Op. T. III. p. 163.

The deviation of the orbits of the planets from the plane of the sun's equator was of course a difficulty in the system which supposed that they were carried round by the vortices which the sun's rotation caused, or at least rendered evident. Bernoulli's explanation consists in supposing the planets to have a sort of *leeway* (*dérive des vaisseaux*) in the stream of the vortex.

† See *Phil. Ind. Sc.* B. III. c. ix. Art. 7.

they will arrange themselves in certain curves, each proceeding from the N. to the S. pole of the magnet, like the meridians in a map of the globe. It is easily shewn, on the supposition of magnetic attraction and repulsion, that these *magnetic curves*, as they are termed, are each a curve whose tangent at every point is the direction of a small line or particle, as determined by the attraction and repulsion of the two poles. But if we suppose a *magnetic vortex* constantly to flow out of one pole and into the other, in streams which follow such curves, it is evident that such a vortex, being supposed to exercise material pressure and impulse, would arrange the iron filings in corresponding streams, and would thus produce the phenomenon which I have described. And the hypothesis of *central torrents* of Bernoulli or Le Sage which I have referred to, would, in its application to magnets, really become this hypothesis of a magnetic vortex, if we further suppose that the matter of the torrents which proceed to one pole and from the other, mingles its streams, so as at each point to produce a stream in the resulting direction. Of course we shall have to suppose two sets of magnetic torrents;—a boreal torrent, proceeding to the north pole, and from the south pole of a magnet; and an austral torrent proceeding to the south and from the north pole:—and with these suppositions, we make a transition from the hypothesis of attraction and repulsion, to the Cartesian hypothesis of vortices, or at least, torrents, which determine bodies to their magnetic positions by impulse.

Of course it is to be expected that, in this as in the other case, when we follow the hypothesis of impulse into detail, it will need to be loaded with so many subsidiary hypotheses, in order to accommodate it to the phenomena, that it will no longer seem tenable. But the plausibility of the hypothesis in its first application cannot be denied:—for, it may be observed, the two *opposite* streams would counteract each other so as to produce no local *motion*, only *direction*. And this case may put us on our guard against other suggestions of forces acting in curve lines, which may at first sight appear to be discerned in magnetic and electric phenomena. Probably such curve lines will all be found to be only resulting lines, arising from the direct action and combination of elementary attraction and repulsion.

5. There is another case in which it would not be difficult to devise a mode of transition from one to the other of two rival theories: namely, in the case of the emission theory and the undulation theory of Light. Indeed several steps of such a transition have already appeared in the history of optical speculation; and the conclusive objection to the emission theory of light, as to the Cartesian theory of vortices, is, that no amount of additional hypotheses will reconcile it to the phenomena. Its defenders had to go on adding one piece of machinery after another, as new classes of facts came into view, till it became more complex and unmechanical than the theory of epicycles and eccentrics at its worst period. Otherwise, as I have said, there was nothing to prevent the emission theory from migrating into the undulatory theory, and as the theory of vortices did into the theory of attraction. For the emissionists allow that rays may *interfere*; and that these interferences may be modified by alternate *fits* in the rays; now these fits are already a kind of *undulation*. Then again the phenomena of polarized light shew that the fits or undulations must have a *transverse* character: and there is no reason why emitted rays should not be subject to *fits* of *transverse* modification as well as to any other fits. In short, we may add to the emitted rays of the one theory, all the properties which belong to the undulations of the other, and thus

account for all the phenomena on the emission theory; with this limitation only, that the emission will have no share in the explanation, and the undulations will have the whole. If, instead of conceiving the universe full of a *stationary* ether, we suppose it to be full of etherial particles moving in every direction; and if we suppose, in the one case and in the other, this ether to be susceptible of undulations proceeding from every luminous point; the results of the two hypotheses will be the same; and all we shall have to say is, that the supposition of the emissive motion of the particles is superfluous and useless.

6. This view of the manner in which rival theories pass into one another appears to be so unfamiliar to those who have only slightly attended to the history of science, that I have thought it might be worth while to illustrate it by a few examples.

It might be said, for instance, by such persons, "Either the planets are not moved by vortices, or they do not move by the law by which heavy bodies fall. It is impossible that both opinions can be true." But it appears, by what has been said above, that the Cartesians did hold both opinions to be true; and one with just as much reason as the other, on their assumptions. It might be said in the same manner, "Either it is false that the planets are made to describe their orbits by the above quasi-Cartesian theory of Bernoulli, or it is false that they obey the Newtonian theory of gravitation." But this would be said quite erroneously; for if the hypothesis of Bernoulli be true, it is so because it agrees in its result with the theory of Newton. It is not only possible that both opinions may be true, but it is certain that if the first be so, the second is. It might be said again, "Either the planets describe their orbits by an inherent virtue, or according to the Newton theory." But this again would be erroneous, for the Newtonian doctrine decided nothing as to whether the force of gravitation was inherent or not. Cotes held that it was, though Newton strongly protested against being supposed to hold such an opinion. The word *inherent* is no part of the physical theory, and will be asserted or denied according to our metaphysical views of the essential attributes of matter and force.

Of course, the possibility of two rival hypotheses being true, one of which takes the explanation a step higher than the other, is not affected by the impossibility of two contradictory assertions of the *same order* of generality being both true. If there be a new-discovered comet, and if one astronomer asserts that it will return once in *every* 20 years, and another, that it will return once in every 30 years, both cannot be right. But if an astronomer says that though its interval was in the last instance 30 years, it will only be 20 years to the next return, in consequence of perturbation and resistance, he may be perfectly right.

And thus, when different and rival explanations of the same phenomena are held, till one of them, though long defended by ingenious men, is at last driven out of the field by the pressure of facts, the defeated hypothesis is transformed before it is extinguished. Before it has disappeared, it has been modified so as to have all palpable falsities squeezed out of it, and subsidiary provisions added, in order to reconcile it with the phenomena. It has, in short, been penetrated, infiltrated, and metamorphosed by the surrounding medium of truth, before the merely arbitrary and erroneous residuum has been finally ejected out of the body of permanent and certain knowledge.

W. W.

Trinity Lodge,
April 15, 1851.

XIII. *On the Colours of Thick Plates. By* G. G. STOKES, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 19, 1851.]

THE expression "colours of thick plates," has been appropriated to a class of phenomena discovered by Newton, and described by him in the fourth part of the second book of his Optics. In Newton's experiment, the sun's light was admitted into a darkened room through a hole in the window-shutter, and allowed to fall perpendicularly upon a concave mirror, formed of glass quicksilvered at the back. A white opaque card pierced with a small hole being then interposed, at the distance of the centre of curvature of the mirror, so that the regularly reflected light returned by the same small hole by which it entered, a set of coloured rings was seen depicted on the card encompassing the hole. The existence of these rings was attributed by Newton to the light scattered on entering the glass, and then regularly reflected and refracted; and he succeeded in deducing from his theory of fits the laws of the rings, both as regards the relation between the diameters of successive rings, the order of the colours, the variation of the diameter of a given ring corresponding to a variation either in the radius of curvature of the surfaces or in the thickness of the glass, and even the absolute magnitude of the system formed under given circumstances. The phenomena which present themselves when the mirror is inclined a little, so as to throw the image of the hole to one side, are very curious, and have been accurately described by Newton in his tenth and eleventh Observations.

In the course of a series of experimental researches on these rings, the Duke de Chaulnes[*] discovered accidentally that their brilliancy was greatly increased by breathing on the glass. Since the moisture soon evaporated, in order to procure a permanent tarnish, he spread over the surface a small quantity of a mixture of milk and water, which on drying left a degree of dimness very suitable to the experiments. By substituting for the glass mirror a metallic speculum, in front of which there was placed a plate of tarnished mica, it was easy to observe the variation in the diameter of the rings corresponding to a variation in the distance of the mica from the speculum. In this form of the experiment the glass plate was replaced by the plate of air comprised between the mica and the speculum. Rings were also produced when the tarnished mica was replaced by a screen of fine muslin. In this case, however, according to the Duke's statement, the rings were nearly square, though rounded off a little at the angles. A set of parallel wires gave merely a bright band intersected by short bands which were vividly coloured. Even the blade of a knife produced a similar appearance, weak indeed, but sufficient to establish the identity of the effect. It is unnecessary here to discuss the theoretical views of the Duke de Chaulnes, since the progress of optical science has since led to a complete explanation of the formation of the rings.

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[*] *Mémoires de l'Académie*, 1755, p. 136.

The colours of thick plates were first explained on the undulatory theory by Dr Young[*], by whom they were attributed to the interference of two streams of light, of which one is scattered on entering the glass, and then regularly reflected and refracted, and the other regularly refracted and reflected, and then scattered on its return through the first surface. Dr Young's explanation is however excessively brief; and he has rather pointed out the application of the grand and newly-discovered principle of interference to the explanation of the phenomenon, than followed the subject into any of its details. At the same time, it appears evident, from an attentive perusal of what he has written, that at least the broad outlines of the complete explanation were clearly present to his mind.

In the course of a paper entitled "Experiments for investigating the cause of the coloured concentric rings, discovered by Sir Isaac Newton, between two object-glasses laid upon one another," Sir William Herschel mentions an experiment in which rings of the nature of those of thick plates were produced by merely strewing hair-powder in the air in front of a metallic speculum placed as the mirror in Newton's experiment[†]. The result of this experiment was justly regarded by Herschel as inexplicable on the theory of fits. It may here be remarked that it is in perfect accordance with the theory of undulations.

In the *Annales de Chimie et de Physique*[‡], will be found a report by Ampère and Poisson on a memoir by M. Pouillet, containing some experiments on the rings. The experiments were mostly the same as those of the Duke de Chaulnes, but accompanied by measures. M. Pouillet found that the rings were produced by placing in front of a metallic speculum an opaque screen containing an aperture of any form. In this case the rings were round, whatever might be the form of the aperture. The experiments are mentioned by M. Pouillet in his *Elémens de Physique*[§].

A complete explanation of the rings, according to the theory of undulations, has been given by Sir John Herschel in his treatise on Light[||]. The rings are supposed to be formed in Newton's manner with a glass mirror, the luminous point being situated in the axis. Having investigated the elementary system of rings which would be produced by the two streams scattered in passing and repassing at the point of the first surface where it is cut by the axis, Sir John Herschel shews that if the surfaces be supposed to be a pair of concentric spheres, having the luminous point in their centre, the elementary systems corresponding to the several elements of the first surface will be superposed, in such a manner that a distinct system would be thrown on a screen held at the distance of the luminous point. The laws of the rings resulting from theory are precisely those which had been discovered by Newton, and the calculated magnitudes were found to agree almost exactly with Newton's measures.

A set of coloured bands have since been observed by Dr Whewell, which are formed when the image of a candle held near the eye is viewed by reflexion in a plane mirror of quicksilvered glass, placed at the distance of some feet. This observation was communicated to M. Quetelet, by whom it has been published*. In repeating the experiment together, Dr Whewell and M. Quetelet found that it was an essential condition of success that the surface should not be perfectly bright, and that to ensure the production of the bands it was sufficient to breathe gently on the surface of a cool mirror. Instead of vapour, which soon evaporates, M. Quetelet recommends a tarnish of grease†.

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[*] *On the Theory of Light and Colours. Philosophical Transactions* for 1802, p. 41.

[†] *Philosophical Transactions* for 1807, p. 231.

[‡] Tom. I. (1816) p. 87.

[§] Tom. II. p. 476.

[||] *Encyclopædia Metropolitana*, Arts. 676, &c.

In closing this sketch of the history of the subject, I may be allowed to express my obligations to Dr Lloyd for his valuable Report on Physical Optics, which contains a brief account of all that was known about the subject, accompanied by references to the original papers.

My attention was called to the subject by the Master of Trinity College, who shewed me the bands above-mentioned, which he shortly afterwards brought before the notice of this Society‡. It seemed to me from the first that these bands were of the nature of the coloured rings of thick plates, so that the theory of the former only required to be worked out, that of the latter being known. Had I felt any doubt on the subject, it would soon have been dissipated when I came to make experiments; for by properly varying the experiments the two systems were seen to be incontestably of the same nature.

The mirrors, whether plane or curved, were prepared in the following manner, which I can recommend to any one who wishes to repeat the experiments, as being both easy and efficacious. The mirror being held horizontally, a mixture consisting of three or four parts of water to one of milk was poured on it, and allowed to spread over the surface. The mirror was then held in a vertical position in front of a fire, when the greater part of the mixture ran off, and the remainder dried in two or three minutes, when the mirror was ready for use. To prevent disappointment on the part of any one who may be looking for Dr Whewell's bands, I will here mention that in order to see them properly the image of the flame must be seen distinctly, so that a shorted-sighted person requires an eye-glass or spectacles.

A concave mirror prepared with milk and water is well adapted for performing Newton's experiment in his manner, or rather by substituting, as in the Duke de Chaulnes's experiments, the image of the sun in the focus of a convex lens for the small hole employed by Newton. The experiment may however be varied in the following manner. Whatever appearance is presented on a screen may be seen without a screen by receiving the rays directly into the eye, and adapting it for distinct vision of an object at the distance of the screen. Accordingly, in order to see the rings which in Newton's experiment were thrown on a screen, it is sufficient to place a small flame in front of the mirror, in such a position as to coincide with its inverted image, when a remarkably beautiful system of rings is seen in air, surrounding the flame. Not the least striking circumstance connected with these rings is their apparent corporeity, since they seem to have a definite position in space like an actual object. The striking and beautiful phenomena so accurately described by Newton in his 10th and 11th Observations may be seen in this manner by moving the flame sideways. By altering in various ways the positions of the flame and of the eye, both in this experiment and in that with a plane mirror, the rings or bands seen in the two cases may be perceived, independently of any theory, to be evidently of the same nature. It is unnecessary here to describe at length the various appearances presented, since they are noticed in the body of the paper, in connexion with the theory.

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* *Correspondance Mathématique et Physique*, Tom. v. (1829) p. 394.

† Tom. VI. p. 69.

‡ See the *Philosophical Magazine* for April, 1851, p. 336.

The first section contains the theory of the rings formed in Newton's manner. The investigation, though differing a little in the mode in which it is conducted, is the same in principle as that given by Sir John Herschel, but is somewhat more general, inasmuch as the curvatures of the two surfaces are supposed to be any whatsoever, and the luminous point is not supposed to be situated in the axis. The distance, too, of this point from the axis is at first supposed to be arbitrary, in order to investigate under what circumstances the rings can be formed most dictinctly on a screen. The second section contains the theory of the bands and rings formed by a plane mirror. The expression for the retardation is deduced as a particular case from the formula investigated in the first section; but on account of the interest which attaches itself to these bands, and the simplicity of their theory, a separate investigation is likewise given. The next two sections are devoted to cases of more generality, and on the whole less interest: still, a few results of some interest are obtained. The last two sections contain a closer examination of the precise mode in which the phenomena are produced.

Although the present paper is a little long, the reader must not suppose that the theory of the rings and bands is anything but simple. The length arises partly from the detail in which the subject has been considered, partly from the generality of some of the investigations, partly from the description of experiments which accompanies the theoretical investigations.

SECTION I.

Rings thrown on a screen by a concave mirror consisting of a lens dimmed at the first surface, and quicksilvered at the back. Condition of distinctness when the rings are thrown on a screen, or of fixity when they are viewed in air. Investigation of the phenomena observed when the luminous point is moved in a direction perpendicular to the axis of the mirror.

1. Let a luminous point L be situated either in or not far out of the axis of a mirror such as that just described; and let it be required to investigate the illumination, at the point M of a screen, due to two streams of light, of which one is scattered at the first surface, and then regularly reflected and refracted, and the other is regularly refracted and reflected, and then scattered in coming out, the point M being supposed to be situated not far out of the axis. Let the mirror be referred to the rectangular axes of x, y, z, the axis of z being the axis of the mirror, and the origin being situated in the first or dimmed surface. Let r be the radius of the first surface, s that of the second, t the thickness of the glass, μ its index of refraction; and suppose r and s positive when the concavities of both surfaces are turned in

the direction of z positive. Let a, b, c be the co-ordinates of L; a', b', c' those of M; and suppose a, b, a', and b' small compared with c, c', r, and s. Let x, y, z be the co-ordinates of any point P on the dimmed surface, R the retardation of the stream which was scattered at P on emergence, relatively to that which was scattered at the same point on entrance. Let L_1, L_2, L_3 be the images of L after refraction, reflexion, and second refraction, respectively; M_1, M_2, M_3, the images of M; and let a, b, c, or a', b', c', with the suffixes 1, 2, 3, denote the co-ordinates of L_1, L_2, L_3, or M_1, M_2, M_3. In approximating to the value of R, let the squares of the small quantities a, b, x, y, &c. be retained, so that the terms neglected are of the fourth order, since all the terms are of even orders, as will be immediately seen when the approximation is commenced.

2. The rays diverging from L may after refraction be supposed to diverge from L_1, notwithstanding the spherical aberration of direct pencils, and the astigmatism of oblique pencils. For, first, let L be in the axis. The supposition that the rays diverge from L_1 is equivalent to supposing that the front of a wave is a sphere having L_1 for centre, whereas it is really a surface of revolution such that L_1 is the centre of curvature of a section made by a plane through the axis. This plane cuts the sphere above mentioned in a circle, which, being a circle of curvature, cannot have with the curve a contact lower than one of the second order. But the contact is actually of the third order, since the curve and circle touch without cutting. Hence the error produced in the calculation of R by supposing the front of a wave to be a sphere, instead of that surface which it actually is, is only a small quantity of the fourth order, and quantities of this order are supposed to be neglected.

Next, consider an oblique pencil. Let L' and L'' be two points in the axis of the pencil which are the centres of curvature of its principal sections. If the distance of L' and L'' from each other, and from L_1, were not small, the front of the wave would have a contact of the first order with a sphere described round L_1, with such a radius as to pass through the point where the front is cut by the axis of the pencil; and in that case the error committed by taking the sphere for the actual front would be of the second order. But L' and L'' are situated at distances from L_1 which are small quantities of the second order, whence it will readily be seen that the actual error is only of the fourth order.

3. Let the expression (L to L_3) denote the retardation of a wave proceeding from L to L_3, or rather, in case L_3 be a virtual focus, the difference of retardations of two waves starting from L and L_3 and reaching the same given point. Then

$$R = (L \text{ to } L_3) + PM - PL_3 - \{(M \text{ to } M_3) + PL - PM_3\} = K + V,$$

where

$$K = (L \text{ to } L_3) - (M \text{ to } M_3),$$

$$V = PM - PL_3 - (PL - PM_3) = V' - V'', \text{ suppose.}$$

According to the explanation given in the preceding article, when the position of P changes K remains constant, to the degree of approximation which it is proposed to employ, but the value of V depends upon the position of P. We have

$$PM = \surd\{(a' - x)^2 + (b' - y)^2 + (c' - z)^2\}, \quad PL_3 = \surd\{(a_3 - x)^2 + (b_3 - y)^2 + (c_3 - z)^2\},$$

and $z = \frac{1}{2r}(x^2 + y^2)$, nearly. Expanding, we get

$$V' = c' - \frac{1}{2r}(x^2 + y^2) + \frac{1}{2c'}\{(a' - x)^2 + (b' - y)^2\} - c_3 + \frac{1}{2r}(x^2 + y^2) - \frac{1}{2c_3}\{(a_3 - x)^2 + (b_3 - y)^2\}.$$

Let $V' = c' - c_3 + V_x' + V_y'$, where V_x' is the sum of the terms containing a and x, and V_y' the sum of the terms containing b and y. Then

$$V_x' = \frac{(a' - x)^2}{2c'} - \frac{(a_3 - x)^2}{2c_3}.$$

There is no occasion to write down V_y', since it may be deduced from V_x' by writing b, y for a, x. Taking V_x to denote for V what V_x' denotes for V', we get by interchanging a and a', a_3 and a_3', and subtracting,

$$V_x = \frac{(a' - x)^2}{2c'} - \frac{(a_3 - x)^2}{2c_3} - \frac{(a - x)^2}{2c} + \frac{(a'_3 - x)^2}{2c'_3}. \quad . \quad . \quad . \quad (1)$$

In order that the rings may be formed on a screen with perfect distinctness, it is necessary that the difference of phase of the several pairs of streams which come from the several points of the dimmed surface should be the same; in other words, that the retardation R should be independent of x and y. Deferring for the present the investigation of the conditions of distinctness, we may observe that when these conditions are satisfied the expression for R must be the same as if x and y were each equal to zero. We have therefore

$$R = [L, M] - [M, L],$$

where

$$[L, M] = (L \text{ to } L_3) + c' - c_3 + \frac{e'^2}{2c'} - \frac{e_3^2}{2c_3},$$

and $[M, L]$ is formed from $[L, M]$ by interchanging the co-ordinates of L and M. In the above expression e^2 has been written for shortness, in place of $a^2 + b^2$. Now supposing c, c_1, c_2, c_3, to be all positive, and denoting by A, B the points in which the first and second surfaces respectively are cut by the axis of the mirror, we have

$$(L \text{ to } L_3) = AL - \mu AL_1 + \mu BL_1 + \mu BL_2 - \mu AL_2 + AL_3; \quad . \quad . \quad (2)$$

which gives, on expanding,

$$(L \text{ to } L_3) = c + \frac{e^2}{2c} - \mu c_1 - \frac{\mu e_1^2}{2c_1} + \mu(c_1 + t) + \frac{\mu e_1^2}{2(c_1 + t)} + \mu(c_2 + t) + \frac{\mu e_2^2}{2(c_2 + t)}$$

$$- \mu c_2 - \frac{\mu e_2^2}{2c_2} + c_3 + \frac{e_3^2}{2c_3}$$

$$= c + c_3 + 2\mu t + \frac{e^2}{2c} + \frac{e_3^2}{2c_3} - \frac{\mu t e_1^2}{2c_1(c_1 + t)} - \frac{\mu t e_2^2}{2c_2(c_2 + t)}.$$

We have therefore

$$R = \frac{\mu t}{2}\left\{\frac{e'^2_1}{c'_1(c'_1 + t)} + \frac{e'^2_2}{c'_2(c'_2 + t)} - \frac{e_1^2}{c_1(c_1 + t)} - \frac{e_2^2}{c_2(c_2 + t)}\right\}. \quad . \quad . \quad (3)$$

Although this formula was obtained on the supposition that the points L, L_1, L_2, L_3, M, M_1, M_2, M_3, lay on the positive side of the plane of xy, it is true independently of that restriction. For when one of the foci L, L_1, L_2, L_3, from having been real becomes virtual, or from having been virtual becomes real, the corresponding ordinate c, c_1, $c_2 + t$, or c_3 changes sign. At the same time, in the expression for the retardation *distance passed over* is converted into *distance saved*, and *vice versâ*. Hence in any such expression as (2) the sign of one or more of the lines is changed. But in the expansion of the radical by which the length of such line is expressed, the sign of c, c_1, $c_2 + t$, or c_3 must be changed at the same time, and therefore no change is required in the expanded expressions.

To eliminate e_1 and e_3 from (3), we may observe that we have very nearly

$$\frac{e_1}{e} = \frac{1}{\mu}\frac{c_1}{c}, \quad \frac{e_3}{e_1} = -\frac{c_2 + t}{c_1 + t},$$

and similar expressions hold good for e', &c. Hence

$$R = \frac{e'^2 t}{2\mu c'^2}\,\frac{c'_1}{c'_1 + t}\left(1 + \frac{c'_1}{c'_1 + t}\,\frac{c'_2 + t}{c'_2}\right) - \frac{e^2 t}{2\mu c^2}\,\frac{c_1}{c_1 + t}\left(1 + \frac{c_1}{c_1 + t}\,\frac{c_2 + t}{c_2}\right) \quad . \quad (4)$$

We might, if required, express c_1 and c_2 in terms of c by the formulæ of common optics, without making any supposition as to the magnitude of t. In practice, however, t is usually small compared with c, c_1, &c., so that we may simplify the above expression by retaining only the first power of t. We thus get

$$R = \frac{t}{\mu}\left(\frac{e'^2}{c'^2} - \frac{e^2}{c^2}\right) = \frac{t}{\mu}\left(\frac{a'^2 + b'^2}{c'^2} - \frac{a^2 + b^2}{c^2}\right) \quad . \; . \; . \; . \; . \quad (5)$$

4. Before proceeding to apply this expression, let us investigate the conditions of distinctness. Denoting by $\Delta_x R$, $\Delta_y R$ the additions to R on account of the terms involving x, y, we get from (1)

$$\Delta_x R = \tfrac{1}{2}\left(\frac{1}{c'} + \frac{1}{c'_3}\right)x^2 - \left(\frac{a'}{c'} + \frac{a'_3}{c'_3}\right)x$$

$$-\left\{\frac{1}{2}\left(\frac{1}{c} + \frac{1}{c_3}\right)x^2 - \left(\frac{a}{c} + \frac{a_3}{c_3}\right)x\right\}; \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

and $\Delta_y R$ may be obtained by interchanging a and b, x and y.

We have by the formulæ of common optics

$$\frac{\mu}{c_1} = \frac{\mu - 1}{r} + \frac{1}{c}, \qquad \frac{1}{c_2 + t} = \frac{2}{s} - \frac{1}{c_1 + t}, \qquad \frac{\mu}{c_2} = \frac{\mu - 1}{r} + \frac{1}{c_3}; \quad . \; . \; . \quad (7)$$

whence, supposing t small, expanding as far as the first power of t, and putting for shortness

$$\frac{\mu}{s} - \frac{\mu - 1}{r} = \frac{1}{s} - (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{1}{\rho}, \quad . \; . \; . \; . \; . \; . \quad (8)$$

so that ρ is the radius of a speculum having the same focal length as the mirror, we obtain

$$\frac{1}{c} + \frac{1}{c_3} = \frac{2}{\rho} + \frac{2t}{\mu}\left\{\frac{2}{\rho^2} + \frac{2(\mu - 1)}{\rho r} + \frac{(\mu - 1)^2}{r^2} - \frac{2}{\rho c} + \frac{1}{c^2}\right\};$$

and therefore

$$\frac{1}{2}\left(\frac{1}{c'}+\frac{1}{c'_3}-\frac{1}{c}-\frac{1}{c_3}\right)=\frac{t}{\mu}\left(\frac{1}{c'}-\frac{1}{c}\right)\left(\frac{1}{c'}+\frac{1}{c}-\frac{2}{\rho}\right). \quad . \quad . \quad . \quad . \quad (9)$$

Again

$$a_1=\frac{a\,c_1}{\mu\,c},\quad a_2=-a_1\frac{c_2+t}{c_1+t},\quad a_3=\mu\,a_2\frac{c_3}{c_2};$$

whence

$$\frac{a}{c}+\frac{a_3}{c_3}=\frac{a}{c}\left\{1-\frac{c_1\,(c_2+t)}{(c_1+t)\,c_2}\right\}=\frac{a\,t}{c}\left(\frac{1}{c_1}-\frac{1}{c_2}\right),\text{ nearly.}$$

Neglecting t altogether in the formulæ (7), we get

$$\frac{1}{c_1}-\frac{1}{c_2}=\frac{2}{\mu}\left(\frac{1}{c}-\frac{1}{\rho}\right);$$

and therefore

$$\frac{a'}{c'}+\frac{a'_3}{c'_3}-\frac{a}{c}-\frac{a_3}{c_3}=\frac{2t}{\mu}\left\{\frac{a'}{c'}\left(\frac{1}{c'}-\frac{1}{\rho}\right)-\frac{a}{c}\left(\frac{1}{c}-\frac{1}{\rho}\right)\right\}. \quad . \quad . \quad . \quad (10)$$

Let $\Delta_x R+\Delta_y R=\Delta R$. The formulæ (6), (9), (10), and the corresponding formulæ relating to $\Delta_y R$, give

$$\Delta R=\frac{t}{\mu}\left(\frac{1}{c'}-\frac{1}{c}\right)\left(\frac{1}{c'}+\frac{1}{c}-\frac{2}{\rho}\right)(x^2+y^2)$$

$$-\frac{2t}{\mu}\left\{\frac{1}{c'}\left(\frac{1}{c'}-\frac{1}{\rho}\right)(a'x+b'y)-\frac{1}{c}\left(\frac{1}{c}-\frac{1}{\rho}\right)(ax+by)\right\}. \quad . \quad . \quad . \quad (11)$$

The condition of distinctness, as has been already observed, is that ΔR shall vanish independently of x and y, in which case the elementary systems of rings corresponding to the several elements of the dimmed surface will be superimposed on the screen. The coefficient of x^2+y^2 will vanish when either of the following equations is satisfied:

$$c'=c,\quad\text{or }\frac{1}{c'}+\frac{1}{c}=\frac{2}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In order that the coefficients of x and y may vanish independently of particular values of a' and b' we must have

$$c'=c=\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which equations satisfy at the same time both of the equations (12), of which it would have been enough that one should have been satisfied. Hence the rings are formed most distinctly when the luminous point and the screen are both at a distance from the mirror equal to that of coincident conjugate foci. This agrees with observation*.

Whatever be the position of the luminous point, if a', b', c' be the co-ordinates of its image, we have

$$\frac{a'}{c'}=-\frac{a}{c},\quad\frac{b'}{c'}=-\frac{b}{c};$$

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* See Newton's *Optics*, Book II., Part IV., Obs. 1, for the case in which the curvatures of the two surfaces are alike, and an experiment by the Duke de Chaulnes (*Mém. de l'Académie*, 1755, p. 141) for the case in which they are unlike. In this experiment a plano-convex lens was employed. Each face in succession, after having been tarnished, was turned towards the incident light. It appears from a passage at the end of Newton's twelfth Observation that he had himself made experiments of a similar nature, the results of which however are not described.

and the second line in the expression for ΔR becomes

$$\frac{2t}{\mu c}\left(\frac{1}{c'}+\frac{1}{c}-\frac{2}{\rho}\right)(ax+by),$$

which vanishes since c' satisfies the second of equations (12). If a screen be held in a direction perpendicular to the axis of the mirror, at such a distance as to receive a distinct image of the luminous point, and if a', b', c' be now taken to denote the co-ordinates, not of the image itself, but of a point of the screen very near the image, the part of ΔR which involves the squares of x and y will continue to vanish, inasmuch as c' remains the same as before, and the part which contains first powers, though not absolutely evanescent, will be very small; and therefore a portion of the system of rings in the neighbourhood of the image will be formed distinctly.

5. This agrees with observation. In repeating Newton's experiment in his way, except that a lens of short focus was employed instead of a small hole, and that the surface of the glass was purposely dimmed with milk and water, I found that when the mirror was placed at a distance from the luminous point widely different from its radius of curvature, and inclined a little, so as to allow of receiving the image on a sheet of paper without stopping the incident light, and when the paper was held at such a distance from the mirror as to receive a distinct image of the luminous point, the image was accompanied by very distinct arcs of rings.

Whatever appearance is presented on a screen may be seen without a screen, by placing the eye in such a position as to receive the rays, and adapting it to distinct vision of an object at the distance of the screen in its former position. It is found universally that when the image of the luminous point is seen distinctly it is accompanied by a portion, more or less extensive, of a system of coloured rings or bands. In this way the rings may be seen when the image is virtual, in which case they cannot, of course, be thrown on a screen.

In the experiment described in the introduction, in which a small flame is placed in such a position as to coincide with its inverted image, and viewed directly, the rings seen are remarkable for their fixity, appearing like a bodily object surrounding the flame, and having a definite parallax, whether judged of by the motion of the head, or by the convergence of the axes of the two eyes. The same is true of the system of rings formed when the flame is moved sideways out of the position above mentioned. The reason of this fixity is, that inasmuch as the retardation is independent of x and y, a given point of an imaginary plane drawn through the flame perpendicular to the axis of the mirror belongs to a ring of the same order, whatever be the point of the mirror against which it is seen projected.

6. Having investigated the conditions of distinctness, let us now proceed to consider the magnitude and character of the rings, supposing the luminous point to be situated at a distance ρ from the mirror, and the rings to be thrown on a screen at the same distance, or else viewed in air. In this case $c = c' = \rho$; and if the luminous point be in the axis $e = 0$, which reduces (5) to

$$R = \frac{te'^2}{\mu c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It readily follows from this expression that a system of rings is formed similar to the transmitted rings of the system to which Newton's name is especially attached. The rings in the present case, however, especially when viewed in air, are far more brilliant, and in this respect more resemble the reflected system. If e_1 be the radius of the first bright ring, for which $R = \lambda$, the length of a wave of light,

$$e_1 = \sqrt{\frac{\mu\lambda}{t}} \, . \, c \, ;$$

and for the bright ring of the order n, $e' = \sqrt{n} \, . \, e_1$. The formula (14) has already been discussed by Sir John Herschel, and compared with Newton's measures, with which it manifests a very close agreement.

With white light, only a moderate number of rings can be seen, on account of the variation of the scale of the system depending on a variation in the refrangibility of the component parts of which white light is made up. When the rings were formed in air, and the source of light was the flame of an oil-lamp with a small wick, I have counted seven or eight surrounding the central bright spot. But when the system is viewed through a prism, or when the flame of a spirit-lamp is used, an immense number of rings may be seen.

7. Next, suppose the luminous point out of the axis. Referring to the formula (5), we see that the retardation is not now equal to zero at the axis, but throughout a circle whose radius e' is equal to e. Hence the *achromatic line** of the system, which was formerly reduced to a point, is now a circle having its centre in the axis, and passing through the luminous point and its image, which are situated at the opposite extremities of a diameter. The fringes of the first order will be a pair of circles having their centre in the axis, and lying, one outside, and the other inside the central fringe: the fringes of the second order will be another pair of circles lying, one outside the larger, and the other inside the smaller fringe of the first order, and so on. It is to be remarked, however, that only a finite number of fringes are formed inside the central white fringe. If the value of R when $e' = 0$ be denoted by $-n_0\lambda$, n_0 will be a numerical quantity, a function of λ, which determines the number of fringes and the fraction of a fringe, belonging to the light of which the wave-length is λ, which are formed inside the central white fringe. The value of n_0 may be got from equation (5) on putting $e' = 0$, which gives

$$n_0 = \frac{te^2}{\mu\lambda c^2}.$$

If white light be used, and if n_0 exceed 8 or thereabouts for rays of mean refrangibility, all the fringes which the overlapping of the different colours allows to be visible are formed inside as well as outside the central white fringe; and if the luminous point be moved still further from the axis, a portion of the field of view around the axis will appear free from rings. If the radius of the central white fringe, or, which is the same, the distance of the luminous point from the axis, be denoted by $\sqrt{n_0}$, the radii of the bright fringes of the first, second,... orders will be denoted, on the same scale, by $\sqrt{(n_0 \pm 1)}$, $\sqrt{(n_0 \pm 2)}$ ... and those of the dark rings by $\sqrt{(n_0 \pm \frac{1}{2})}$, $\sqrt{n_0 \pm \frac{3}{2})}$ ...

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* I use this term to denote the locus of the points for which the retardation is equal to zero, which forms a curve on either side of which the colours are arranged in descending order.

The manner in which the rings open out from the centre as the luminous point is moved sideways out of the axis is very striking, and has been accurately described by Newton. The explanation of it is obvious. It may be remarked that the system of rings, regarded as indefinite, is formed on the same scale whatever be the distance of the luminous point from the axis, but the portion of the indefinite system which alone is visible, in consequence of the coincidence or approximate coincidence of the maxima and minima of intensity corresponding to the several colours, depends altogether upon that distance. Since in passing from the interior to the exterior boundary of a given fringe the square of the radius receives a given increase, it follows that the area of the fringe is constant, that is, independent of the perpendicular distance of the luminous point from the axis. Hence the breadth of the fringe continually decreases as the diameter of the circle which forms either boundary increases. When a small flame is used for the source of light, and is moved sideways from the axis, the fringes soon become confused, because a flame which does very well for forming the broad fringes of comparatively small radius seen near the axis, will not answer for the fine fringes of large radius which are formed at a distance from the axis. But on using for the source of light the image of the sun in the focus of a small concave mirror belonging to a microscope apparatus, I found that the fringes were formed quite distinctly even when their diameters became very large and consequently their breadths very small.

SECTION II.

Bands formed by a plane mirror, and viewed directly by the eye.

8. In the case of a plane mirror $\rho = \infty$; and if R be the retardation of the stream scattered at emergence relatively to the stream scattered at entrance, R will be obtained by adding together the second members of equations (5) and (11). Hence we have

$$R = \frac{t}{\mu}\left[\frac{1}{c'^2}\left\{(x-a')^2 + (y-b')^2\right\} - \frac{1}{c^2}\left\{(x-a)^2 + (y-b)^2\right\}\right] \quad . \quad . \quad . \quad . \quad (15)$$

It is to be remembered that in this formula a, b, c denote the co-ordinates of the luminous point; x, y those of any point in the dimmed surface; a', b', c' those of any point M of space towards which the eye is directed, and for distinct vision of which it is adapted; and that the formula is only approximate, the approximation depending both upon the smallness of the obliquity, and upon the smallness of the thickness t of the glass in comparison with the distances of the luminous point and the point M from the mirror.

As regards the illumination at a given point M, we are evidently concerned with so much only of the dimmed surface as lies within a cone having M for vertex and the pupil of the eye for base; and the bands will be seen distinctly if R do not alter by more than a small fraction of λ when x, y alter from one point to another of the portion of the dimmed surface which lies within this cone. Now we have seen already that the bands are in all cases seen distinctly in the neighbourhood of the image when the image itself is seen distinctly, so that when the image is real the bands may even be thrown on a screen, in which case a comparatively large portion of the dimmed surface is concerned in their formation. We may conclude that in the present case the bands will be seen with sufficient distinctness throughout, provided the image of the luminous point be seen distinctly.

9. In considering the distinctness or indistinctness of the bands, we are concerned with the finite size of the pupil of the eye; but in investigating only their form and magnitude we may suppose the pupil a point, and reduce each pencil entering it to a single ray, which forms the axis of the pencil. Let E be the eye, or rather the centre of the pupil, h its perpendicular distance from the mirror, and suppose the axis of z to pass through E. The ray by which a portion of a band is seen as if at M cuts the mirror in a point whose co-ordinates x, y are equal to a', b' altered in the ratio of h to $h - c'$, so that

$$a' = \left(1 - \frac{c'}{h}\right) x, \qquad b' = \left(1 - \frac{c'}{h}\right) y.$$

Substituting in (15), we get

$$R = \frac{t}{\mu}\left[\frac{1}{h^2}(x^2 + y^2) - \frac{1}{c^2}\{(x-a)^2 + (y-b)^2\}\right], \quad . \quad . \quad . \quad (16)$$

from which it may be observed that c' has disappeared, as evidently ought to be the case. The expression (16) might have been at once deduced from (15) by putting the co-ordinates of the eye in place of a', b', c'. The reason of this is evident, because the retardation is constant for the same ray, and a ray may be defined by the positions of any two points through which it passes. We may therefore employ the points E and P, instead of M and P, to define the ray, and may therefore at once substitute the co-ordinates of E for those of M in the expression for the retardation.

10. To determine the forms, &c. of the bands, nothing more will be requisite than to discuss the formula (16). As however this formula was obtained as a particular case from a very general, and consequently rather complicated investigation, in which the curvatures of the surfaces were supposed to have any values, and as the bands to which it relates are of great interest, the reader may be pleased to see a special investigation of the formula for the case of a plane mirror.

Retaining the same notation as before, except where the contrary is specified, let L_0, E_0 be the feet of the perpendiculars let fall from L, E on the plane of the dimmed surface, and let $L_0P = s$, $E_0P = u$. Let R_1 be the retardation of a ray regularly refracted and reflected, scattered at emergence at P, and so reaching E; R_2 the retardation of a ray reaching E after having been scattered at P on entering into the glass, and let $R_1 - R_2 = R$. Let $LSTPE$ be the course of the first ray, which emanates from L, is regularly refracted at S, regularly reflected at T, and scattered on emergence at P. The lines LS, ST, TP will evidently lie in the plane LL_0P. Let ϕ, ϕ' be the angles of incidence and refraction at S. We have

$$R_1 = LS + 2\mu ST + PE,$$
$$= c \sec\phi + 2\mu t \sec\phi' + \sqrt{(h^2 + u^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$c \tan\phi + 2t \tan\phi' = s, \qquad \sin\phi = \mu \sin\phi'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The obliquity being supposed small, we may expand, and retain only the squares of small quantities, the terms thus neglected involving only fourth and higher powers. We get in the first place from (17)

$$R_1 = c + 2\mu t + h + \tfrac{1}{2}\left(c\phi^2 + 2\mu t\phi'^2 + \frac{u^2}{h}\right). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

But equations (18) give

$$\phi = \mu\phi' = \frac{\mu s}{\mu c + 2t};$$

and substituting in (19) we get

$$R_1 = c + 2\mu t + h + \frac{\mu s^2}{2(\mu c + 2t)} + \frac{u^2}{2h}.$$

To obtain R_2, it will be sufficient to interchange c and h, s and u, since if we supposed the course of the ray reversed it would emanate from E, be regularly refracted and reflected, then scattered on emergence at P, and so would reach L. Interchanging, subtracting, and reducing, we obtain

$$R = t\left\{\frac{u^2}{h(\mu h + 2t)} - \frac{s^2}{c(\mu c + 2t)}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This formula is more general than (16), since no approximation has yet been made depending on the magnitude of t. In practice, however, t is actually small compared with c and h, so that we may simplify the formula by retaining only the first power of t, which reduces (20) to (16), inasmuch as $u^2 = x^2 + y^2$, and $s^2 = (x - a)^2 + (y - b)^2$.

11. Let us now proceed to apply the formula (16) to the explanation of the phenomena. In discussing this formula, it is to be remembered that x, y are the co-ordinates of the point of the mirror on which a fringe is seen projected. Since the direction of the axis of y is disposable, we may make the plane of y, z pass through the luminous point, in which case $b = 0$, and

$$R = \frac{t}{\mu}\left\{\left(\frac{1}{h^2} - \frac{1}{c^2}\right)(x^2 + y^2) + \frac{2ax}{c^2} - \frac{a^2}{c^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

For a given fringe R is constant. Hence the fringes form a system of concentric circles, the centre of the system lying in the axis of x. If α be the abscissa of the centre

$$\alpha = \frac{ah^2}{h^2 - c^2} = \tfrac{1}{2}\left(\frac{ah}{h + c} + \frac{ah}{h - c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Now $ah(h - c)^{-1}$ and $ah(h + c)^{-1}$ are the abscissæ of the points in which the plane of the mirror is cut by two lines drawn through the eye, one to the luminous point, and the other to its image. Hence we have the following construction: join the eye with the luminous point and with its image, and produce the former line to meet the mirror; the middle point of the line joining the two points in which the mirror is cut by the two lines drawn from the eye will be the centre of the system.

Hence if the luminous point be placed to the right of the perpendicular let fall from the eye on the plane of the mirror, and between the mirror and the eye, the concavity of the fringes will be turned to the right. If the luminous point, lying still on the right, be now drawn backwards, so as to come beside the eye, and ultimately fall behind it, the curvature will decrease till the fringes become straight, after which it will increase in the contrary direction, the convexity being now turned towards the right. This agrees with observation.

12. The expression for R shews that the circle which forms the achromatic line of the system passes through the two points mentioned in the last paragraph but one. This is always observed to be true in experiment as far as regards the image, and is found to be true of the luminous point also when it is in front of the eye, so as to be seen along with the fringes, provided the fringes reach so far.

Denoting by $n_0\lambda$ the value of R at the centre of the system of circles, taken positively, we get from (21) and (22)

$$n_0 = \frac{a^2 t}{\mu\lambda\,(h^2 - c^2)}. \qquad \dots\dots\dots \qquad (23)$$

The numerical quantity n_0 may conveniently be called the *central order*, since when it lies between $i - \frac{1}{2}$ and $i + \frac{1}{2}$, where i is any integer, the colour at the centre belongs to the bright ring of the i^{th} order. If v be the radius of the central fringe, v will be equal to the semi-difference of the quantities $ah\,(h + c)^{-1}$ and $ah\,(h - c)^{-1}$, whence

$$v = \frac{ach}{h^2 - c^2}. \qquad \dots\dots\dots \qquad (24)$$

Having found the centre of the system of circles and the projection of the image, or the point where the line joining the eye and the image cuts the mirror, describe a circle passing through this projection. This will be the central line of the bright fringe of the order 0, and its radius will be equal to v. Now describe a pair of circles whose radii are to v as $\sqrt{(n_0 \pm 1)}$ to $\sqrt{n_0}$. These will be the central lines of the two bright fringes of the first order, for the particular colour to which the assumed value of λ relates. The central lines of the two bright fringes of the second order will be a pair of circles with radii proportional to $\sqrt{(n_0 \pm 2)}$, and so on. The fringes will be broader on the concave than on the convex side of the central white fringe. When the fringes become straight, n_0 becomes infinite, and the system becomes symmetrical with respect to the central fringe. This agrees with observation.

13. When the luminous point is situated in a line drawn through the eye perpendicular to the mirror $a = 0$, and we have simply

$$R = \frac{(c^2 - h^2)\,t}{\mu c^2 h^2}\,(x^2 + y^2).$$

In this case the achromatic line of the system is reduced to a point, and the rings are analogous in every respect to the transmitted system of Newton's rings. For the bright ring of the first order $R = \pm \lambda$, and therefore the radius of the ring is equal to

$$\frac{\sqrt{\mu\lambda} \, . \, ch}{\sqrt{\{(c^2 - h^2)\, t\}}},$$

which becomes infinite when $c = h$. Hence if the luminous point be at first situated in front of the eye, and be then conceived to move backwards through the eye till it passes behind it, the rings will expand indefinitely, and so disappear, and will reappear again when the luminous point has passed the eye.

14. All the preceding conclusions agree perfectly with experiment, so far as qualitative results are concerned; for I have not taken any measures. The change in the direction of curvature, which I had not noticed till it was pointed out by the formula, may be readily seen, when the flame of a candle is the source of light, by holding the candle at arm's length nearly in front of the eye, but a little to the right, then drawing it back beside the eye, and finally holding it at arm's length behind the head, and as nearly in a line drawn through the eye perpendicular to the mirror as the shadow of the head will allow.

When the candle is held near the eye, a portion only of the circles can be seen; the circles are in fact reduced to circular arcs, and these arcs may even become perfectly straight. But when the candle is placed at a good distance from the eye, suppose half-way between the eye and the mirror, and a small piece of card is placed as a screen in front of the flame to keep off the glare of the direct light, the circles, or at least several of them, may be seen complete, except that it must be left to the imagination to fill them up where they are hid by the screen. In this way the manner in which the rings open out from the centre of the circles may be observed, though not for the first ring or two, which open out while the centre is hid by the screen. Instead of a candle with a screen, it is better to use the image of the sun in the focus of a small convex or concave mirror.

15. The conclusion deduced from theory which was mentioned in Art. 13 cannot, of course, be compared with experiment directly. But the experiment may be successfully performed by substituting for either the luminous point or the eye a virtual image. Using for the luminous point the image of the sun in the focus of a small concave mirror, at the distance of some feet in front of a plane mirror of which the surface had been prepared with milk and water, I placed a piece of plate glass between the mirrors, inclined at an angle of about 45°. The greater part of the light coming from the image of the sun was transmitted through the plate of glass; and on returning from the large mirror a portion of this light was reflected sideways, so that the rings could be seen by reflexion in the plate of glass without obstructing the incident light. The system of rings thus seen was very beautiful, and there was no direct light glaring in the eye, and yet no screen to hide any part of the system. It was easy to know when the image of the eye in the inclined plate lay in a line drawn through the luminous point perpendicular to the plane mirror, by observing when the image of the luminous point seen by reflexion, first at the plane mirror, and then at the plate

of glass, lay in the centre of the system of rings. Supposing the image to have this position, on moving the head sideways the opening out of the rings could be traced from its very commencement. By moving back the head so as to keep the image of the luminous point in the centre of the system of rings, it was easy to try the experiment to which Art. 13 relates, the virtual image of the eye being thus kept in a line drawn through the luminous point perpendicular to the mirror, and the eye moving relatively to the luminous point, which is as good as if the luminous point had moved while the eye remained fixed. I found, in fact, that on moving back the head the rings expanded till the bright central patch surrounding the image filled the whole field of view, and on continuing to move back the head the rings appeared again. In the position in which the central patch filled the whole field of view, the least motion of the eye sideways was sufficient to bring into the field portions of excessively broad coloured bands.

Between the system of rings seen when the eye was respectively nearer to and further from the mirror than when in the position in which the rings became infinite, there was one difference which may here be mentioned. So long as the image occupied the centre of the systems, they were similar to each other; but when the head was moved sideways, the centre of the circles passed in the first case to the side of the image towards which the head was moved, and in the second case to the contrary side. This affords another way of comparing with experiment the result of theory already mentioned relating to the direction of curvature, and it will be readily seen that the result of experiment agrees with the prediction of theory. For, suppose the distance of the eye from the oblique plate less than that of the luminous point, so that the virtual image of the eye lies between the luminous point and the mirror, and let the eye move to the right. Then its virtual image moves to the left, and therefore, according to theory, the centre of curvature ought to fall to the left of the image of the luminous point, right and left being estimated with reference to an eye supposed to occupy the position of the virtual image of the actual eye. But on account of reflexion at the glass plate, there is reversion from right to left, and therefore to the eye in its actual position the centre of curvature falls to the right of the image of the luminous point, which agrees with observation. The experiments described in this article may be tried very well with the flame of a taper, but in examining what becomes of the rings when they expand it is more satisfactory to use sun-light.

16. In describing the disappearance of the rings, I said that the central spot expanded till it filled the whole field. In truth, when the rings had expanded a faint luminous central spot of finite size remained visible, which was surrounded by a dark ring, and then a faint luminous ring. It would have been more correct to speak of the *dark* ring as faint, since these rings consisted merely in slight alternations of intensity in a generally bright field. These rings, however, had evidently nothing to do with the former rings, which had disappeared; for they continued to have the image of the luminous point for their centre when the head was moved to one side. They were doubtless of the same nature as those which are seen when a luminous point, or the flame of a candle, is viewed through a piece of glass powdered with lycopodium seed, and arose from the interference of pairs of streams of light which passed on opposite sides of the globules of dried milk. I merely mention these rings

lest any one in repeating the experiment should observe them, and mistake them for something relating to the colours of thick plates.

17. The formula (21) determines the breadths of the several fringes, which are unequal, except in the case in which the eye and the luminous point are at the same distance from the mirror. It will be convenient, however, to investigate a simple formula to express what may be regarded as a sort of mean breadth. Let the mean breadth be defined to be that which would be the breadth of one fringe if the rate of variation of the order of a fringe, for variation of position in a direction perpendicular to the length of a fringe, were constant, and equal to the rate in the neighbourhood of the projection of the image, and let β be this mean breadth.

Putting $y = 0$ in (21), differentiating on the supposition that R and x vary together, and after differentiation putting $ah\,(h+c)^{-1}$ for x, we find

$$dR = \frac{2ta}{\mu ch}\,dx;$$

and since, according to the definition of β, $\lambda^{-1}dR = \beta^{-1}dx$, we have

$$\beta = \frac{\mu ch\lambda}{2ta}. \qquad \qquad (25)$$

When $c = h$ the bands are straight, and of uniform breadth, that breadth being equal to β; and when the bands are not very much curved β may still be taken as a convenient measure of the scale of the system; but the formula (25) is not meant to be applied to cases in which the projection of the image of the luminous point falls at all near the centre of the circles.

SECTION III.

Rings formed by a curved mirror, and viewed directly by the eye, when the luminous point and its image are not in the same plane perpendicular to the axis.

18. The rings and bands of which the theory has been considered in the two preceding sections may be regarded as forming the two extreme cases of the general system. In the first case, the rings appear to have a definite position in space; in the second case, every thing depends upon the position of the eye. These are the cases of most interest, but there are some properties of the general system which deserve notice.

In order that rings may be thrown on a screen, it is necessary that the retardation of one of the interfering streams relatively to the other should be sensibly constant over the whole of the dimmed surface, or at least over a large portion of it. But when the rings are viewed directly by the eye, we are concerned with so small a portion of the dimmed surface, in viewing a given point of a ring, that the rings may be seen very well in cases in which they could not be thrown on a screen. Moreover, we have seen that even independently of the small size of the pupil, a portion at least of the system is seen distinctly when the image of the luminous point is seen distinctly. Omitting further consideration of the conditions of distinctness, let us regard the eye as a point, and investigate the form and character of the rings.

19. Let f, g, h be the co-ordinates of the eye. To find the retardation, it will be sufficient, as in Art. 9, to write f, g, h for a', b', c', and take x, y to denote the co-ordinates of that point of the mirror on which a given point of a ring is seen projected. The whole retardation is the sum of the expressions in (5) and (11); and making the above substitution we find

$$R = \frac{t}{\mu}\left(\frac{f^2+g^2}{h^2} - \frac{a^2+b^2}{c^2}\right) + \frac{t}{\mu}\left(\frac{1}{h}-\frac{1}{c}\right)\left(\frac{1}{h}+\frac{1}{c}-\frac{2}{\rho}\right)(x^2+y^2)$$
$$- \frac{2t}{\mu}\left\{\frac{1}{h}\left(\frac{1}{h}-\frac{1}{\rho}\right)(fx+gy) - \frac{1}{c}\left(\frac{1}{c}-\frac{1}{\rho}\right)(ax+by)\right\}. \quad . \quad . \quad . \quad (26)$$

Hence the bands still form a system of concentric circles. If X, Y be the co-ordinates of the centre of the system,

$$X = \frac{\frac{f}{h}\left(\frac{1}{h}-\frac{1}{\rho}\right) - \frac{a}{c}\left(\frac{1}{c}-\frac{1}{\rho}\right)}{\left(\frac{1}{h}-\frac{1}{c}\right)\left(\frac{1}{h}+\frac{1}{c}-\frac{2}{\rho}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and Y may be written down from symmetry.

The equations of a line joining the eye and the luminous point are

$$\frac{\xi - a}{f - a} = \frac{\eta - b}{g - b} = \frac{\zeta - c}{h - c}.$$

At the point in which this line cuts the mirror $\zeta = 0$, or at least is a very small quantity, which may be neglected. Hence we have

$$\xi = \frac{\frac{a}{c} - \frac{f}{h}}{\frac{1}{c} - \frac{1}{h}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

from whence η may be written down if required. If ξ_1, η_1 denote the co-ordinates of the point in which the line joining the eye and the image meets the mirror, ξ_1, η_1 may be obtained from ξ, η by writing a_1, b_1, c_1 for a, b, c, where a_1, b_1, c_1 denote the co-ordinates of the image. Observing that

$$\frac{a_1}{c_1} = -\frac{a}{c}, \qquad \frac{1}{c_1} = \frac{2}{\rho} - \frac{1}{c},$$

we find

$$\xi_1 = \frac{\frac{a}{c} + \frac{f}{h}}{\frac{1}{h} + \frac{1}{c} - \frac{2}{\rho}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The formulæ (27), (28), and (29) shew that X is equal to the semi-sum of ξ and ξ_1, and for the same reason Y is equal to the semi-sum of η and η_1. Hence the geometrical construction given in Art. 11 for finding the centre of the system in the case of a plane mirror applies equally to a curved mirror, even when the curvatures of the two surfaces are different. Since the retardation vanishes for the image itself, it follows that the achromatic line is a circle having the two points of intersection above mentioned for opposite extremities of a diameter.

20. It follows from the expressions for X and Y, or from the geometrical construction to which they lead, that if the eye be not in the line joining the luminous point and its image, whenever it crosses either of two planes drawn perpendicular to the axis, and passing, one through the luminous point, and the other through its image, the centre of curvature of the bands moves off to an infinite distance, and the bands become straight, and then bend round the other way.

When the eye coincides with the luminous point, f, g, h become equal to a, b, c, respectively, and R vanishes independently of x and y. The same takes place when the eye coincides with the image, since in this case

$$\frac{f}{h} = -\frac{a}{c}, \quad \frac{g}{h} = -\frac{b}{c}, \quad \frac{1}{h} + \frac{1}{c} = \frac{2}{\rho}.$$

Hence, when the eye crosses either of the planes above mentioned, remaining in the line joining the luminous point and its image, instead of bands which become straight and then change curvature, we have rings which disappear by moving off to infinity, and then appear again.

I have verified these conclusions by experiment, substituting when necessary a virtual image of the eye for the eye itself, in the manner explained in Art. 15. The experiments embraced the following cases, in the description of which O will be used to denote the centre of curvature of the mirror, F its principal focus, L the luminous point, and L_3 its image.

Concave mirror: L beyond O. Eye (1) beyond L; (2) passing L; (3) between L and L_3; (4) passing L_3; (5) between L_3 and the mirror.—Concave mirror: L between O and F. Eye (1) beyond L_3; (2) passing L_3; (3) between L_3 and L; (4) passing L; (5) between L and the mirror.—Concave mirror: L between F and the mirror. Eye (1) beyond L; (2) passing L; (3) between L and the mirror.—Convex mirror. Eye (1) beyond L; (2) passing L; (3) between L and the mirror.

The mirrors employed were formed, as usual, with surfaces of equal curvature. When the observation was made directly, there was no difficulty in determining at which side of the line joining the luminous point and its image the eye lay, and consequently in deciding whether the direction of curvature agreed with theory or not. When the observation was made by reflexion in a plate of glass, the eye was placed so that its virtual image fell in the line LL_3 by moving the head till the image of the luminous point was seen in the centre of the system of rings. The radii of the two surfaces of the mirror being the same, or only differing by a small quantity comparable with the thickness of the glass, the surfaces may be regarded as forming a pair of concentric spheres; and therefore, everything being symmetrical with respect to the line joining L and L_3, when the image of the eye is in this line the bands necessarily become rings, having the image for their centre. Hence the theory of the rings or bands, which it is the object of the experiment to compare with observation, is not involved in the assumption that the image of the eye was in the line LL_3 when the system of rings appeared arranged symmetrically around the image of the luminous point. By moving the head a little to one side, and observing whether the centre of the system of rings then lay to the right or left of the image, it was easy to compare theory with observation as to the direction of curvature.

There was no difficulty in telling when the virtual image of the eye coincided with the image of the luminous point, since in that case the latter image expanded indefinitely. The phenomena observed offered no direct test of the coincidence of the virtual image of the eye with the luminous point, except what arose from the appearance of the bands themselves. I did not think it worth while to take any measures, but contented myself with observing that when the eye was in the expected position, or thereabouts, the rings expanded indefinitely when the image was kept in the centre of the system, and the bands formed when the image was allowed to pass to one side changed curvature as the head moved backwards and forwards.

21. The bands may be considered as completely characterized by the position and magnitude of the achromatic line, and by the value of the numerical quantity which has been already defined as the central order. A simple geometrical construction has already been given for determining the achromatic circle. Substituting X, Y for x, y in (26), and denoting the resulting value of R by $-n_0\lambda$, we find

$$n_0 = \frac{t}{\mu\lambda} \frac{\left\{\frac{f}{h}\left(\frac{1}{c}-\frac{1}{\rho}\right) - \frac{a}{c}\left(\frac{1}{h}-\frac{1}{\rho}\right)\right\}^2 + \left\{\frac{g}{h}\left(\frac{1}{c}-\frac{1}{\rho}\right) - \frac{b}{c}\left(\frac{1}{h}-\frac{1}{\rho}\right)\right\}^2}{\left(\frac{1}{h}-\frac{1}{c}\right)\left(\frac{1}{h}+\frac{1}{c}-\frac{2}{\rho}\right)}. \quad . \quad (30)$$

In the application of this formula n_0 is to be taken positively.

Denoting as before the radius of the achromatic circle by v, we find from (28), (29), and the formulæ thence derived which give η, η_1,

$$v^2 = \frac{\left\{\frac{f}{h}\left(\frac{1}{c}-\frac{1}{\rho}\right) - \frac{a}{c}\left(\frac{1}{h}-\frac{1}{\rho}\right)\right\}^2 + \left\{\frac{g}{h}\left(\frac{1}{c}-\frac{1}{\rho}\right) - \frac{b}{c}\left(\frac{1}{h}-\frac{1}{\rho}\right)\right\}^2}{\left(\frac{1}{h}-\frac{1}{c}\right)^2\left(\frac{1}{h}+\frac{1}{c}-\frac{2}{\rho}\right)^2} \quad . \quad . \quad . \quad (31)$$

When the bands are nearly straight, instead of the central order it is more convenient to consider the mean breadth of a fringe. According to the definition of β,

$$v + \beta dn_0 : v :: \sqrt{(n_0 + dn_0)} : \sqrt{n_0},$$

since the radii of the several rings are as the square roots of their orders. We have therefore

$$\beta = \frac{v}{2n_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

22. In the case of a concave mirror, if a small flame be so placed as to coincide with its image, and be then moved a little towards the mirror, or from it, it is possible to see a single

system of rings with both eyes at once, if the eyes be situated symmetrically with respect to the flame and its image. The rings so seen appear to be situated between the flame and its image. Let E be the right and E' the left eye, and suppose the head so placed that the line LL_3 bisects EE' at right angles. On account of the similarity of position of the two eyes, the system of rings seen with one eye must be exactly like the system seen with the other, and therefore, in order that a single system may be seen with both eyes at once, it is necessary and sufficient that the axes of the eyes be directed to the centres of the respective systems. It has been shewn already that the projected place on the mirror of the centre of the system seen with either eye, suppose the right eye, bisects the line joining the projected places of the flame and its image. On account of the supposed smallness of the obliquities, this is the same thing as saying that the centre of the system seen by the right eye appears in the direction of a line bisecting the angle LEL_3. Similarly, the centre of the system seen by the left eye appears in the direction of a line bisecting the angle $LE'L_3$. In order therefore that a single system only may be seen, the axes of the eyes must be made to converge to the point in which the bisecting lines intersect LL_3, and therefore the system of rings will appear to be situated between the flame and its image.

Since the angles LEL_3, $LE'L_3$, are bisected by the axes of the eyes when the system of rings is seen single, it follows that the flame and its inverted image are each seen double, in such a manner that the erect flame seen by either eye is superposed on the inverted flame seen by the other. This agrees with observation: in fact, I was led by experiment to the above rule for determining the apparent position of the rings before I had deduced it from theory. The observation was made when the flame was in front of its image, in which case the position of the rings in space appears more definite than when the image is in front of the flame.

SECTION IV.

Straight bands formed by a plane mirror at a considerable angle of incidence, and viewed by the eye, either directly, or through a telescope.

23. As the angle of incidence increases, the bands become finer and finer, and after they have become too fine to be distinguished by the naked eye they may still be seen through a telescope, provided the source of light be sufficiently small. When the source of light was the image of the sun in a lens of short focus, I saw traces of the bands when the angle of incidence was about 24° 50′, but they were not at all well formed beyond an angle of about 10° 40′, after which they began to be confounded with rays which shot in all directions from the image of the luminous point. With a mirror made of thinner glass they would probably have been visible at a still larger angle of incidence. The theory of Section II. sufficiently explains their origin and general character; but inasmuch as the obliquity was supposed small in investigating the formulæ of that section, it may be desirable to obtain an expression for their breadth, in which no approximation shall be made depending on the smallness of the obliquity, in order to meet the case of any future measures which may be taken at a large angle of incidence.

24. The notation being the same as in Art. 10, we have merely to employ the equations (17) and (18), without making any approximation depending on the smallness of ϕ and ϕ'. The thickness t may still be supposed small compared with c and h. Neglecting t for a first approximation, and then substituting in the small terms the values of $\tan\phi'$ and $\sec\phi'$ got from the first approximation, we find

$$R_1 = \sqrt{c^2 + s^2} + \sqrt{h^2 + u^2} + 2t\sqrt{\frac{(\mu^2 - 1)s^2 + \mu^2 c^2}{s^2 + c^2}}.$$

Interchanging c and h, s and u, and subtracting, we get finally

$$R = 2t\left(\sqrt{\mu^2 - \frac{s^2}{s^2 + c^2}} - \sqrt{\mu^2 - \frac{u^2}{u^2 + h^2}}\right). \quad . \quad . \quad . \quad . \quad (33)$$

25. For the achromatic line $R = 0$, and therefore $s : c :: u : h$. Hence s is to u in the constant ratio of c to h, and therefore, by a well-known geometrical theorem, the achromatic curve is a circle, having its centre in the line L_0E_0 produced, and cutting this line in the two points in which it is divided internally and externally in the given ratio. The latter of these points may be formed by producing LE to meet L_0E_0 produced, and the former by producing LL_0 till the produced part is equal to the line itself, and then joining the extremity of the produced part with E. Hence the construction given in Art. 11 for determining the bright band of the order zero continues to hold good whatever be the angle of incidence.

26. In the neighbourhood of the image the bands are sensibly straight, being arcs of circles of very large radius. To find the mean breadth of a band, it will be sufficient to suppose the point P to lie in the line L_0E_0, to differentiate equation (33) making R, s, and u vary together, while $s + u$ remains constant, replace $\frac{dR}{du}$ by $\frac{\lambda}{\beta}$, and after differentiation take u and s to refer to the small pencil, regarded as a ray, by which the image is seen. If i be the angle of incidence, we may put after differentiation $s = c\tan i$, $u = h\tan i$. We thus find

$$\beta = \frac{\lambda c h}{2t(c + h)}\,\frac{\sqrt{\mu^2 - \sin^2 i}}{\sin i\cos^3 i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

On account of the largeness of the angle of incidence, the breadth of the bands is sensibly uniform, and therefore β may be regarded as the breadth of any one band. It is to be remembered that β denotes the linear breadth of a band as seen projected on the mirror. If we denote the angular breadth by ϖ, we have on account of the smallness of ϖ

$$\varpi = \frac{\beta\cos i}{h\sec i} = \frac{\lambda c}{t(c + h)}\,\frac{\sqrt{\mu^2 - \sin^2 i}}{\sin 2i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

SECTION V.

On the nature of the deflection of the two interfering streams from the course of the regularly refracted light.

27. It was suggested to me by a friend, to whom I was shewing some of the experiments described in this paper, that in order to see bands or rings of the same nature it would probably be sufficient to dim the two faces of a plate of glass, and view a luminous point through it. But having prepared the two faces of a piece of plate glass with milk and water in the same manner as for mirrors, taking care to treat the two faces as nearly as possible alike, on viewing a luminous point through the plate I found not the slightest trace of the rings or bands, whatever were the distance of the eye from the plate. There were indeed one or two indistinct rings surrounding the luminous point; but these were of a totally different nature, being analogous to the rings seen with lycopodium seed, and arising from the interference of pairs of streams which passed on opposite sides of the milk globules. There was no difficulty in distinguishing them from the system sought for, since they continued to have the luminous point for their centre when the plate was inclined to the line joining that point with the eye. The absence of rings or bands indicates therefore that the streams scattered at the opposite sides of a plate are incapable of interfering.

The rays scattered so as to make infinitely small angles with the regularly refracted rays belong to a point in the bright band of the order zero, and are therefore brought to a focus on the retina when the luminous point is seen distinctly. The same must be at least very approximately true for neighbouring points of the system of rings, did any such exist, and therefore a portion at least of the system would be seen distinctly when the luminous point was seen distinctly. The distances of the luminous point from the glass plate, and of the glass plate from the eye, were comparable with the corresponding distances in the experiment with a plane mirror, and the thickness of glass was comparable likewise; and with a mirror the bands are seen with the utmost facility within wide limits of the thickness of the glass, and of the distances of the luminous point and of the eye from the mirror. But to prevent any doubt as to whether the bands might not have been too small to be seen when formed by transmission, I have calculated the retardation in the same manner as in Art. 10. The result is

$$R = -\frac{t}{2\mu}\left(\frac{1}{c} + \frac{1}{h}\right)^2 (x^2 + y^2),$$

where R is the retardation of the stream scattered at emergence relatively to that scattered at entrance, c is the distance of the luminous point from the plate, h that of the plate from the eye, t the thickness, and μ the refractive index of the plate, and x, y are the co-ordinates of the point in which the plate is cut by any small pencil (regarded as a ray) which enters the eye, and are measured from the point in which the plate is cut by a line joining the luminous point and the eye, a line to which the plane of the plate is supposed to be perpendicular. On substituting numerical values in the above formulæ, it appeared that the dimensions of the rings were such that they could not possibly have escaped notice had they really been formed. The non-appearance of the rings leads to the following law.

In order that two streams of scattered light may be capable of interfering, it is necessary that they should be scattered, in passing and repassing, by the same set of particles. Two streams which have been scattered by two different sets of particles, although they may have come originally from the same source, behave with respect to each other like two streams coming from different sources.

According to this law, in all calculations relating to the colours of thick plates, we must consider the elementary system of rings or bands corresponding to each element of the dimmed surface of the mirror, and then conceive these elementary systems superposed. We must not compound the vibrations corresponding to streams which have been scattered by different elements, and then find the resulting illumination.

28. The reason of this law will be apparent if it be considered that particles of dust, &c. small as they may be, are usually large in comparison with waves of light, so that the light scattered at entrance, taken as a whole, is most irregular; and the only reason why regular interference is possible at all is, that each particle of dust acts twice in a similar manner, once when the general wave is going, and again when it is returning.

To examine more particularly the mode of action, let P be any particle of dust, and consider a wave of light which emanates from any particular element of the flame or source of light whatever it be. When this wave reaches P and proceeds along it, a portion is reflected externally in all directions, and with this we have nothing more to do. When the wave has just passed P, we may conceive it as having in a certain sense *a hole* in its front, corresponding in size to P, that is to say, there will be a certain portion of the surface forming the general front of the wave where the ether is quiescent. As the wave proceeds, the disturbance diverges from the neighbourhood of this hole by regular diffraction, and when the disturbance reaches the quicksilvered surface the general wave suffers reflection, as well as the secondary waves, which, having diverged from the neighbourhood of P, do not constitute a wave with an unruffled front, in consequence of the absence of secondary waves diverging from the hole, which would be necessary to complete a wave with a front similar to that of the original wave. If we consider any particular diffracted ray, the chances are that on its return it will get out by regular refraction, since the dust is supposed to cover a moderate portion only of the first surface of the mirror. A portion of the original wave which entered the glass by regular refraction at a certain distance from P, after regular reflexion is incident on P from within. The chances are that the portion thus incident on P does not correspond to a spot where the front of the wave is materially ruffled by diffraction at entrance, so that in considering the wave incident on P we may neglect the previous diffraction. The wave, then, just after refraction, is incident on P, by which a portion is reflected back again in various directions, with which we have nothing to do, a portion, it may be, is refracted or absorbed by P, and the remainder passes on. The wave so passing on diverges from the neighbourhood of P by ordinary diffraction, and the two diffracted streams, having been diffracted in a similar manner by the same particle, are in a condition to interfere. The similarity of the two diffractions will be considered in more detail in the next section.

Now while the light is still in the glass conceive the particles of dust removed, and then replaced at random. The chances are that no particle will now occupy the position formerly occupied by P. Let P' be the particle nearest to the former position of P; and, to make a supposition as favourable as possible to interference, let P' be the very particle P moved a little along the surface without rotation. Although the interval of retardation R of the two streams diffracted by P in its first position, and reaching a given point of space, is sensibly the same as the interval of retardation of the two streams diffracted by P in its second position, and reaching the same point, yet this interval would be changed altogether were the transference of position to take place during the interval of time which elapses between the departure of the wave from P and its return after reflexion, as may very readily be seen. The amount, too, by which the interval of retardation would be changed would vary in an irregular manner from one particle to another, and therefore no regular interference would take place. Now the purely ideal case just considered is precisely analogous to the case of actual experiment when a luminous point is viewed through a plate of glass with both faces dimmed, since the particles on one face have no relation to those on the other. We ought not therefore in such a case as this to expect to see rings or bands.

29. According to the formula (35), the angular breadth of one of the bands formed by a plane mirror becomes considerable when i becomes nearly equal to 90°, so that, apparently, bands ought to be visible at a large angle of incidence. But if the courses of the two streams scattered by the same set of particles be traced, it will be found that they are so widely separated that, for various reasons, no regular interference can be expected to take place. Accordingly, the bands are not seen at a large angle of incidence.

30. In the preceding sections I have spoken of the light by which the rings are formed as having been *scattered* at the dimmed surface. And so it really is, if by that term we merely understand deflected from the course it would have followed according to the regular law of refraction. But according to the explanation given in the preceding article the light is not scattered, in the strict sense of the term, but regularly diffracted. Scattered light is, strictly speaking, such as that by which objects are commonly seen, or again, such as that which is transmitted through white paper and similar substances. The preceding view of the nature of the light by which the rings are formed is confirmed by the results of several experiments.

In the experiments of Sir William Herschel and M. Pouillet mentioned in the introduction, as well as in some of those of the Duke de Chaulnes, rings of the same nature as those formed by a tarnished mirror of quicksilvered glass were produced in cases in which the deflection of the light from its regular course was incontestibly of the nature of diffraction. From the similarity of effect we have a right to infer a probable similarity of cause, unless such a supposition should entail some peculiar difficulty, which does not seem to be the case in the present instance, but quite the contrary.

31. Having cleaned the surface of a concave mirror which had been employed in forming the bands, I rubbed a little tallow on it, and then wiped the mirror in one direction with a handkerchief, so as to leave a finely striated tarnish on it. The tarnish was not sufficient to cause much obscurity; but the image of a candle seen in the mirror was accompanied, as is usual in such cases, by two tails of light, which ran out in a direction perpendicular to the striæ. Having placed a small flame in the centre of curvature of the mirror, I found that the rings were formed with great brilliancy where they were intersected by the tails of light, but elsewhere they were almost wholly invisible.

Now the tails of light are known to be a phenomenon of diffraction: the striated tarnish may in fact be regarded as a sort of irregular grating, and the tails of light are of the nature of Fraunhofer's spectra. If a tarnish in general were capable of producing rings independently of diffraction, there appears no reason why a tarnish of tallow should not be capable; for the particles of tallow are as fine as those of most other kinds of tarnish. But if in the case of a tarnish of tallow the deflection of the light from its regular course be not a phenomenon of diffraction, there appears no reason why the rings should be confined to the tails of light in the experiment described above.

32. The phenomena of polarization seem however to lead to a crucial experiment for deciding whether the deflection of the light from its regular course, which enables the rings to be formed, be a phenomenon of diffraction, or of scattering in the strict sense of the term. When polarized light is scattered, as for example when it is reflected from or transmitted through white paper, it loses its polarization, but when polarized light suffers regular diffraction it retains its polarization.

Having placed a small flame near the centre of curvature of a concave mirror, of which the surface had been prepared with milk and water, I placed a Nicol's prism close to the flame, so as to polarize the light incident on the mirror. On examining the rings with another Nicol's prism, they proved to be perfectly polarized.

33. It may not be considered out of place here to point out what appears to be the cause of a phenomenon observed by M. Pouillet. In an experiment in which rings were occasioned simply by the straight edge of an opaque body held in front of a metallic speculum, it was found that they were formed distinctly in only one half of their circumference. The reason of this appears to be simply as follows. As the waves of light pass the diffracting edge in their progress towards the mirror, those rays which are diffracted inwards, so as to enter the geometrical shadow, after being regularly reflected at the mirror fall upon the opaque body, by which they are stopped. As these rays are required for the formation of that half of the system which lies on the same side as the opaque body, the other half only is well formed. The first half may be formed obscurely by a few rays which are diffracted in the required direction at such a distance from the edge that on their return they pass clear of the edge, and so proceed to interfere with other rays diffracted by the edge on the return of the general wave.

SECTION VI.

Investigation of the angles of diffraction.

34. Something yet remains to be done in order to complete the theory of these rings and bands, namely, to compare the two diffractions which a wave of light experiences at its entrance into the glass and on its return, respectively. For the phase and intensity of a ray diffracted in a given direction depend altogether on the circumstances under which the diffraction takes place; and were these circumstances materially different in the case of the two diffractions abovementioned, the rings might be modified, or might even disappear altogether.

Let us consider first the case of a concave mirror when the luminous point and its image are in the same plane perpendicular to the axis. In this case, if we consider any point P on the dimmed surface, and any point M in the plane of the rings, the angle of diffraction for the ray diffracted at emergence will be L_3PM*. For the ray diffracted at entrance, the angle of diffraction *measured in air* will be LPM_3, that is to say, M_3P is the course of a ray in air which by regular refraction into glass would be brought into the direction of the ray diffracted at P. If C be the intersection of the axis and the plane of the rings, C will be the centre of the system, and the middle point of both the lines LL_3 and MM_3, and therefore LM_3 will be equal and parallel to ML_3. Hence, on account of the smallness of the obliquities, the angles of diffraction LPM_3, L_3PM are sensibly equal, and their planes sensibly coincident, but the deviations take place in opposite directions. But between the two diffractions the light undergoes reflexion; and since the mutual inclination of two rays is reversed by reflexion, we must conceive the direction of deviation reversed in the first diffraction, in order to compare the circumstances of the two diffractions. Allowing for this reversion, we see that not only are the angles of diffraction sensibly coincident, but the directions of deviation are the same.

Accordingly, the interference connected with diffraction, and the interference which gives rise to the colours of thick plates, take place independently of each other. For, let I, I' denote the vibrations at M due to two streams of light diffracted by any particle of dust P on entering the glass, and passing on opposite sides of P; let J, J' denote the vibrations due to two streams diffracted at emergence, and passing on the same sides of P as I, I', respectively; and let $I + I'$ denote the resultant of I and I', and similarly in other cases. Let χ be the difference of phase corresponding to the retardation R, and ω the difference of phase of I, I', and therefore also of J, J', on account of the similarity of the two diffractions. We may represent the phases of the four vibrations by $\theta + \chi + \omega$, $\theta + \chi$, $\theta + \omega$, θ, respectively. Writing down for greater clearness the phases along with the symbols of the vibrations, we may express the resultant of the whole four vibrations by

$$(I_{\theta+\chi+\omega} + I'_{\theta+\chi}) + (J_{\theta+\omega} + J_{\theta}).$$

Moreover, on account of the similarity of the two diffractions, the coefficients of the two vibrations I, J may be supposed equal to each other, and likewise those of the vibrations I', J'. It is true that the diffractions take place at different distances from the source of light, on account of the finite thickness of the glass, but the difference of distance compared with either of the absolute distances is a small quantity of the order t, which may be neglected. Hence the two resultants $I + I'$, $J + J'$ belong to a diffraction ring of the same kind, and in fact differ in nothing but in phase; the phase of the former exceeding that of the latter by χ. Hence the two kinds of interference go on independently of each other. It is true that in the preceding reasoning we have considered only two interfering streams I, I', and that in calculations of diffraction we have to consider the resultant of an infinite number of streams. But the same reasoning would evidently hold good whatever were the number of streams I, I', I''... with their correspondents J, J', J''...

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* In speaking of angles of diffraction, such as L_3PM, I shall distinguish between L_3PM and MPL_3, using the former notation to denote that the deviation takes place from PL_3 to PM, and the latter to denote that it takes place from PM to PL_3.

35. When an irregular powder, or anything of the kind, is used to scatter the light, no diffraction rings are visible, because a given point M in the plane of the rings would belong to a diffraction ring of one kind so far as one particle of dust was concerned, and to a diffraction ring of another kind so far as another such particle was concerned; and therefore nothing is seen but the interference rings belonging to thick plates. But when lycopodium seed is used the lycopodium rings and the interference rings are seen together. The former are always arranged symmetrically around the image, as ought to be the case, since they depend only on the angle of diffraction, which is the same for all points of a circle described round L_3 as a centre. By this circumstance they are at once distinguished from the latter, the centre of which falls half way between the luminous point and its image. On scattering some lycopodium seed on a concave mirror, and placing a small flame near the centre of curvature, at such a distance laterally that the two systems of rings intersected each other, I found in fact that whatever colour appeared in that part of a lycopodium ring which lay outside the interference system was predominant in the latter system throughout the remaining part of a circle described round the image. When the flame was placed in the axis, an abnormal inequality in the brilliancy of the rings of the interference system became very apparent. This inequality was easily seen to correspond to the alternations of intensity in the lycopodium system.

36. Let us now turn to the general case, in which the luminous point and the eye are supposed to have any positions, either in the axis or not far out of it.

The equations of the lines PL_3, PE are

$$\frac{\xi - a_3}{a_3 - x} = \frac{\eta - b_3}{b_3 - y} = \frac{\zeta - c_3}{c_3},$$

$$\frac{\xi - f}{f - x} = \frac{\eta - g}{g - y} = \frac{\xi - h}{h}.$$

Let the small angle L_3PE be projected on the planes of zx and zy, and let α, β be the projections, measured positively towards x, y, and from PL_3 towards PE. The preceding equations give

$$\alpha = \frac{x - a_3}{c_3} - \frac{x - f}{h},$$

which becomes, when a_3 and c_3 are expressed in terms of a and c,

$$\alpha = \left(\frac{2}{\rho} - \frac{1}{c} - \frac{1}{h}\right)x + \frac{a}{c} + \frac{f}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If α', β' be the projections of the angle of diffraction LPE_3, (where E_3 denotes the image of the eye,) we may find α', β' from α, β by interchanging a, b, c, and f, g, h, and changing the sign. If now we change the signs of the resulting expressions, in order to allow for reflexion at the back, and so compare the circumstances of the two diffractions, we shall obtain the very same expressions as at first, since (36) and the corresponding expression which gives β remain unchanged when a, b, c, and f, g, h are interchanged. Hence in the general case, as well as in the particular case first considered, the two diffractions take place under the same circumstances, and therefore the interference rings are not affected by any irregularities which may attend the mode of diffraction. Furthermore, should the diffraction take place with a certain degree of regularity, as in the case of lycopodium seed, so as to exhibit rings or fringes in the aggregate effect of all the particles which send light into the eye in such a direction as to be brought to a given point on the retina, the diffraction rings and the interference rings are seen independent of each other*.

37. If δ be the small angle of diffraction, $\delta^2 = \alpha^2 + \beta^2$, whence from (36) and the other equation which may be written down from symmetry,

$$\delta^2 = \left\{\left(\frac{2}{\rho} - \frac{1}{c} - \frac{1}{h}\right)x + \frac{a}{c} + \frac{f}{h}\right\}^2 + \left\{\left(\frac{2}{\rho} - \frac{1}{c} - \frac{1}{h}\right)y + \frac{b}{c} + \frac{g}{h}\right\}^2. \quad . \quad (37)$$

Hence the loci of the points for which the angle of diffraction has given values form a system of concentric circles. Referring to (29), we see that the co-ordinates of the centre of the system are ξ_1, η_1, so that the centre is situated in the point in which the mirror is cut by the line joining the eye and the image of the luminous point. This result might have been foreseen, since δ vanishes only for the regularly refracted light, and this enters the eye only in the direction of the line joining the eye and the image. By introducing the co-ordinates ξ_1, η_1, the equation (37) may be put under the form

$$\delta^2 = \left(\frac{1}{h} + \frac{1}{c} - \frac{2}{\rho}\right)^2 \left\{(x - \xi_1)^2 + (y - \eta_1)^2\right\}. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Since the diffraction becomes very sparing when the angle of diffraction becomes at all considerable, it follows that the interference rings are but weak at a considerable angular distance from the image of the luminous point. This agrees with observation. In the experiment in which a flame is placed in the centre of curvature of a concave mirror, and is then moved to one side, although the rings are symmetrical with respect to the flame and its image, so far as regards their forms and colours, they are not symmetrical so far as regards their intensities, but are decidedly more brilliant on the side of the image than on the side of the flame itself. That this is not due merely to the glare of the direct light, may be proved by holding a small object in front of the flame, so as to screen the eye from the direct light, when the rings, though better seen than before in the neighbourhood of the flame, are still much weaker than on the opposite side, if the distance of the flame from the axis is at all considerable. For the same reason, in the case of a plane mirror, when the luminous point is placed a good distance in front of the eye, so that the rings do not run out of the field of view, they cannot be traced throughout their whole extent if the angular distance between the luminous point and its image be too great, but only throughout a portion, more or less considerable, on the side of the image.

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* From some rough experiments which I have myself made with gauze stretched in front of a concave glass mirror, of which the surface was clean, I am inclined to think that the squarish rings observed by the Duke de Chaulnes in the experiment with muslin, already mentioned, were due to a combination of the coloured rings of thick plates, and of the appearance produced by a cross-bar grating. If so, the independence of the two systems would have been rendered evident by slightly inclining the mirror, when the latter system would have had the image for its centre, whereas the former would have had for its centre a point situated midway between the luminous point and the image.

38. In the case of a concave mirror when the luminous point is not far from the centre of curvature, and the rings are viewed by an eye placed at no great distance off, the first factor in the expression for δ^2 (equation 38) is not large, and the angle of diffraction does not increase rapidly in passing away from the image. In the case of a plane mirror $\rho = \infty$, and if we suppose c and h equal to what they were in the former case, or thereabouts, in order to make the two cases comparable in every respect except the curvature of the mirror, the factor in question, though larger than before, is still sufficiently small to prevent δ from increasing very rapidly on receding from the image. Accordingly, in both these cases, the rings and bands are seen with brilliancy at a considerable angular distance from the image. But in the case of a convex mirror of considerable curvature ρ is negative, and not large, so that the factor in the expression for δ^2 becomes considerable, and accordingly the angle of diffraction increases rapidly on receding from the image. I found, in fact, that such a mirror was peculiarly ill suited for producing rings or bands, inasmuch as only a comparatively small portion of the system usually seen was visible, namely, the portion which lay in the immediate neighbourhood of the image.

G. G. STOKES.

XIV. *The Deflection of imperfectly Elastic Beams and the Hyperbolic Law of Elasticity.* By HOMERSHAM COX, B.A., *Jesus College, Cambridge.*

[Read in part *March* 11, 1850, and in part *October* 1850.]

THE equation to the curve of an elastic deflected beam is usually deduced from the assumptions: 1, that the longitudinal compression, or extension, of an elastic filament is proportional to the compressing or extending force; 2, that for equal extension and compression the compressing and extending forces are equal to each other.

It appears, however, from experiment that these hypotheses are not exact. All substances appear to be subject to a *defect of elasticity*, *i. e.* their elastic forces of restitution increase in a somewhat less degree than in proportion to the extension or compression. The first of the assumptions above mentioned bears in England the name of "Dr Hooke's Law," and its inexactness was noticed very soon after he proposed it, by Leibnitz, James Bernouilli, and others. In the *Acta Eruditorum* of Leipsic for 1694, Bernouilli gives certain investigations respecting the Elastic Curve: 1, generally when the elastic forces follow any law whatever; 2, when they vary as any power of the extension; 3, when they are directly proportional to the extension. He states that it is worth while to examine the results of the latter hypothesis which had been employed by Leibnitz in his treatise *de Resistentia Solidorum*, but deems it certain that the law is different in different bodies. "Id quod experimenta, tum nostra tum aliorum, abunde confirmare videntur, quorum plurima prælaudatus auctor [Franciscus Tertius de Lanis] industrius *magisterii naturæ et artis* loco cit. recenset."

The real law of elasticity of any material can be known only by direct experiments on the material itself, and it seems nearly certain that even for two different specimens of the same metal, the laws would be in some measure different. All therefore that can be done by formulæ is to represent approximately the results of experiments.

If e be a fraction expressing the extension of a rod, and w the direct force producing that extension, w may be put equal to the sum of a series of terms involving constant coefficients and ascending integral powers of e. Such a series will be convergent. But to represent exactly by such means the results of a single set of experiments very complicated formulæ would be required. If, for instance, 20 experiments were made with different weights stretching a rod of iron, the formula must generally contain 20 terms; for the experimental results would probably give equations determining 20 different independent coefficients of such terms.

The law of Dr Hooke stops at the first term of such a series of ascending algebraical powers of e. The idea readily suggests itself, that by extending the series one term further, its errors may be corrected. The formula so modified would become

$$w = Ae - Be^2 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The coefficients A and B would be found from experiment. That B must be affected by the negative sign appears from the consideration that dividing by e we have

$$\frac{w}{e} = A - Be. \qquad (2)$$

And as the ratio $w : e$ decreases as e increases the second term on the second side of equation (2) must be negative.

From the experiments recorded in the Report (1849) of the Royal Commission "appointed to inquire into the Application of Iron to Railway Structures," it is manifest that the formula (1) there adopted, is subject to unavoidable inaccuracies. A large number of results from it are compared in the Report with the results of experiments. The bars were subjected to extension by weights regularly increased up to the breaking weights. The differences between the results of experiment and formulæ are not + and − promiscuously, but their signs observe a certain order; that is, they are negative for several terms together, and positive for several terms together.

Eight formulæ are given for extension of different kinds of cast iron, and in every case, without exception, at least one half of all the results of each set come together in the middle of the series, with errors in excess, and are preceded and followed by results in which the errors are in defect. The formulæ cannot however be considered satisfactory until the errors be affected by the + and − signs promiscuously and without regular sequence. For while the errors observe any general law, they are themselves capable of being represented by an algebraical expression, which may be added to the original formula by way of correction.

The general character of the errors is this: they are at first negative, then positive, and increasing up to some term near the middle of the series; they then decrease till they become negative again. Now since the sign of the error undergoes in general two changes, first from − to +, secondly from + to −, the algebraical expression for it passes through zero at each of these changes. Therefore if c be the correction which is zero when e, the extension, is zero, and changes sign when $e = a$, and $e = b$, the equation

$$c = A'e(a - e)(b - e),$$

where A' is constant, would express these characteristics of c.

Effecting the multiplication of the quantities in the brackets, and adding the resulting expression for the correction to the original formula we obtain a formula involving the first three powers of the extension, which if correctly applied will be found to be much more accurate than the formula involving the first and second powers only.

In the Report above referred to, the weight producing an assigned extension of iron is in one case computed by a formula involving the first four integral powers of the extension. But in this biquadratic formula, as well as in the quadratic formulæ, the numerical coefficients are obtained by substitution in experimental results selected at random, and by taking the mean of the coefficients so computed. This process being immethodic, is not likely to produce the most accurate results. The mathematical laws of combination of observations, in which several results are to be represented by a formula which on the whole shall give the least possible error, have been laid down by Gauss in the *Theoria Combinationis Observationum*, by Legendre, Laplace, Poisson, Ivory, and others.

Mr J. C. Adams, Fellow of St John's College, has by the method of Least Squares computed for the author of the present paper numerical formulæ from the experiments here considered, and has shewn that when the cube of the extension is included, the coincidence of the formulæ with the results of experiments on extension may be made almost absolutely exact.

According to such a formula, if e be the extension of a uniform rod of a unit of length, and a unit of sectional area, the longitudinal force producing that extension is

$$\alpha e + \beta e^2 + \beta' e^3,$$

where α, β, β' are empirical constants.

Similarly, if c be the compression of a similar rod, the force producing that compression is

$$\gamma c + \delta c^2 + \delta' c^3,$$

where γ, δ, δ' are three other empirical constants.

The following is a brief abstract of the method of applying these formulæ to compute small deflections of a uniform beam of rectangular section, resting horizontally on supports at its extremities and deflected by a weight midway between them. First, the compression and extension of every filament of the beam are expressed in terms of its radius of curvature and distance from the neutral axis. From these expressions the sum of the forces of compression above, and of extension below, the neutral axis, are obtained by integration. But the conditions of equilibrium require that the horizontal elastic forces developed in any section of the beam, above the neutral axis, be equal and opposite to those below the axis. Therefore by equating the two integrals just referred to, the position of the neutral axis is obtained.

Next the sum of the moments of the elastic forces about the neutral axis are obtained, and the sums are equated to the moment about that axis of the pressure (P) of the nearest fulcrum, the latter moment being the product of half the deflecting weight by the distance (x) of the fulcrum from the axis about which moments are taken. This equation involves the radius of curvature and is solved, by approximation, with respect to the reciprocal of that quantity.

Let the fulcrum be the origin of co-ordinates; x, as above defined, the horizontal, and y the vertical co-ordinate of a point in the elastic curve. The reciprocal of the radius of curvature at the point (x, y) is equal to (the second differential coefficient of y with respect to x) ÷ (a quantity which becomes equal to unity, when, as here, the horizontal inclination of the tangent at any point of the curve is very small).

It has not been considered necessary to give here the steps of the investigation, as the analysis though tedious and involved is of an ordinary kind. Making the substitution indicated, and integrating twice the equation last obtained; it is found finally that, taking the origin at the fulcrum, the equation to the elastic curve or neutral line of a rectangular beam of depth d, horizontal breadth μ and length $2a$ is

$$y = \frac{\kappa x^3}{2.3} - \frac{b\kappa^2 x^4}{3.4} + \frac{(2b^2 - c)\kappa^3 x^5}{4.5} - \left\{\frac{\kappa a^2}{2} - \frac{b\kappa^2 a^3}{3} + \frac{(2b^2 - c)a^4\kappa^3}{4}\right\} x,$$

where

$$b = \frac{3}{4} d (\beta + \delta a^2 \gamma^{-2}) (1 + a^{\frac{1}{2}} \gamma^{-\frac{1}{2}})^{-2},$$

$$c = \frac{3}{5} \frac{d^2}{a} (1 + a^{\frac{1}{2}} \gamma^{-\frac{1}{2}})^{-3} (\beta' + \delta' a^{\frac{3}{2}} \gamma^{-\frac{3}{2}}),$$

$$\kappa = \frac{P}{\mu} \frac{3}{d^3 a} (1 + a^{\frac{1}{2}} \gamma^{-\frac{1}{2}})^2.$$

The constants in the equation to the curve are determined by the conditions that it passes through the origin and that $\frac{dy}{dx} = 0$, when $x = a$.

If according to the ordinary hypotheses we put $a = \gamma$, and neglect terms depending on β, β', δ, δ' (or put those coefficients equal to zero) b and c become zero.

The equation to the curve then coincides with ordinary equation to the elastic curve given by Poisson, (*Traité de Mécanique*, 324) and others.

If in the equation to the elastic curve here investigated, x be put $= a$, the value of the deflection at the centre of the beam is

$$\frac{\kappa a^3}{3} - \frac{b \kappa^2 a^4}{4} + \frac{(2b^2 - c) \kappa^2 a^5}{5},$$

whence it may be seen that the deflection is greater than it would be if the elasticity were perfect.

All these formulæ however lead to very complicated results when applied to investigations respecting the deflection of beams. The cubic and quadratic formulæ have moreover the serious inconvenience that from the expression for the weight in terms of the extension, to find conversely the extension in terms of the weight involves the difficulty of solving in one case a cubic, in the other, a quadratic equation with large numerical coefficients.

Hyperbolic Law of Elasticity.

THE formula about to be proposed[*] is far more accurate than the formula (1), and has greatly the advantage in point of simplicity of computation from it. It possesses the practical advantage of great facility in calculating either the extension in terms of the weight or the weight in terms of the extension. When applied to the theory of beams it leads to an expression, *similar in form to itself*, from which with the utmost readiness the deflection can be computed from the transverse pressure or the transverse pressure from the deflection.

If e be the extension of a rod produced by a stretching weight w, it will be found by examination of the experiments in the Report above referred to, that as w is increased by equal successive increments up to the breaking weight, the ratio $\frac{w}{e}$ decreases in such a manner that it may be nearly expressed by the formula

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[*] A notice of this formula was read by Prof. Stokes on behalf of the author at the meeting of the British Association for the Advancement of Science, 1850, and is printed with the last Annual Report of the Association.

$$\frac{w}{e} = \alpha - \beta w, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

α and β being empirical constants.

Comparing this with the formula (2) above, we observe that the only difference is in the second term on the right hand side of the equation which is here made proportional to the stretching weight instead of as before to the extension.

It may be here remarked that the comparison of the formulæ (2) and (3) for the longitudinal extension of iron applies also to the similar formulæ for the deflection of bars by transverse pressure at their centre. The formulæ for extension are identical with those for deflection, except that in the latter, d the deflection is substituted for e and w represents the transverse pressure.

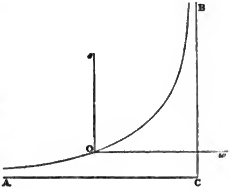
From (3) may be obtained

$$w = \frac{\alpha e}{1 + \beta e} \text{ and } e = \frac{w}{\alpha - \beta w}.$$

These expressions give w in terms of e, and e in terms of w respectively. They may also be put in the form,

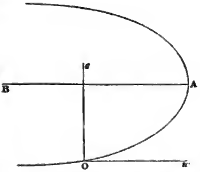
$$\left(\frac{1}{\beta} + e\right)\left(\frac{\alpha}{\beta} - w\right) = \frac{\alpha}{\beta^2},$$

which is the equation to a rectangular hyperbola of which e and w are the co-ordinates. Let CA, CB be the asymptotes of the curve; then referring to the last equation e and w will be measured parallel to the axes Oe, Ow respectively; the origin O being above one asymptote and to the left of the other.



The proposed formula, therefore, exhibits the **Hyperbolic Law of Elasticity** according to the nomenclature of James Bernouilli who (*loc. cit.*) represents the relation of the tension to the extension by a curve which he calls the ***linea tensionum***.

Similarly the formula $w = Ae - Be^2$ may be termed the *parabolic formula*, for it is the equation to a parabola of which w and e are the co-ordinates. Let A be the vertex, AB the axis of the curve, then Oe, Ow are the axes of e and w respectively. O the origin being a point in the curve below the axis.



Now it has been already observed that in comparison with a series of experimental results the parabolic formula gave the weight too small at the commencement of the series, too large towards the middle of it, and subsequently too small again. This amounts to the same thing as saying that as the experiment progresses, w increases in comparison with e too fast at first, and afterwards not fast enough. At points of the curve from O towards A the vertex of the parabola, the tangent to the curve becomes rapidly more and more nearly perpendicular to the axis. In the hyperbola, on the contrary, the direction of the curve does not change so rapidly and is not perpendicular till it meets the asymptote at an infinite distance from the origin. In cases of tranverse pressure, as we shall see, $\frac{w}{e}$ is actually reduced not much less than one-half, and in all cases the greater this reduction, the greater will be the errors of the parabolic formula.

In the following tables the Parabolic and Hyperbolic formulæ are compared with experiments upon extension and deflection. The first table contains the mean results on four different sorts of cast iron, shewing the corresponding extensions and weights for a bar 1 inch square in section and 10 feet long, together with the weight computed from the extensions by the parabolic formula

$$w = 116117e - 201905e^2$$

and the hyperbolic formula

$$w = 118156.424 \div \left(\frac{1}{e} + 2.41\right).$$

The second table contains the deflections and corresponding deflecting pressures applied horizontally at the centre of a bar of Blaenavon iron 1.522 inches deep in the direction of the pressure, and 3.066 inches broad, supported at points 13½ feet apart, with the weights computed by the parabolic formula

$$w = 149.9d - 7.204\,d^2$$

and the hyperbolic formula

$$w = 155.64 \div \left(\frac{1}{d} + .081\right).$$

The results of the parabolic formulæ are copied from the Report of the Iron Commission, pages 59 and 70. It will be observed that in each case the mean error of the parabolic formula is between 3 and 4 times as great as that of the hyperbolic formula.

EXTENSION.

(*e*) Extension in parts of an inch.	(*w*) Corresponding weight by experiment.	Weight computed by hyperbolic formula.	Difference; weight by hyperbolic formula minus real weight.	Weight computed by parabolic formula.	Difference; weight by parabolic formula minus real weight.
·0090	1053·77	1040·84	−12·83	1028·70	−25·07
·0137	1580·65	1567·06	−13·59	1552·91	−27·74
·0186	2107·54	2103·43	− 4·11	2089·92	−17·62
·0287	3161·31	3171·73	+10·42	3166·25	+ 4·94
·0391	4215·08	4222·23	+ 7·15	4231·46	+16·38
·0500	5268·85	5272·51	+ 3·66	5301·09	+32·24
·0613	6322·62	6310·84	−11·78	6358·27	+35·55
·0734	7376·39	7369·11	− 7·28	7435·22	+58·83
·0859	8430·16	8409·10	−21·06	8484·63	+54·47
·0995	9483·94	9482·86	− 1·08	9554·78	+70·74
·1136	10537·71	10538·28	+ ·57	10585·31	+47·60
·1283	11591·48	11579·18	−12·30	11574·28	−17·20
·1448	12645·25	12683·16	+37·91	12580·39	−64·86
			13)143·74		13)473·24
			Mean Error. 11·05		Mean Error. 36·40

Parabolic formula $w = 116117e - 201905e^2$. Hyperbolic formula $w = 118156{\cdot}424 \div \left(\frac{1}{e} + 2{\cdot}41\right)$.

DEFLECTION.

(*d*) Deflection in inches and parts of an inch.	(*w*) Corresponding weight by experiment.	Weight computed by hyperbolic formula.	Difference; weight by hyperbolic formula minus real weight.	Weight computed by parabolic formula.	Difference; weight by parabolic formula minus real weight.
·181	28	27·7	− ·3	26·9	− 1·1
·3754	56	56·6	+ ·6	55·2	− ·8
·7686	112	112·5	+ ·5	110·9	− 1·1
1·184	168	167·9	− ·1	167·4	− ·6
1·632	224	224·9	+ ·9	225·4	+ 1·4
2·105	280	280·4	+ ·4	283·6	+ 3·6
2·604	336	334·5	− 1·5	341·5	+ 5·5
3·169	392	393·0	+ 1	402·7	+10·7
3·756	448	448·5	+ ·5	461·4	+13·4
4·402	504	503·3	− ·7	520·3	+16·3
5·035	560	557·8	− 2·2	572·1	+12·1
5·777	616	612·7	− 3·3	625·5	+ 9·5
6·565	672	666·7	− 5·3	673·6	+ 1·6
7·610	728	734·1	+ 6·1	723·5	− 4·5
8·730	784	798·1	+14·1	759·6	−24·4
9·887	840	854·0	+14·0	777·8	−62·2
			16)51·5		16)168·8
			Mean Error. 3·2		Mean Error. 10·5

Parabolic formula $w = 149{\cdot}9d - 7{\cdot}204d^2$. Hyperbolic formula $w = 155{\cdot}64 \div \left(\frac{1}{d} + {\cdot}081\right)$.

The Curvature of Deflected Beams.

We shall now proceed to connect the formulæ for direct extension and compression with that for the deflection of an elastic beam, of which the transverse section is a rectangle with vertical sides, by transverse pressure at its centre.

If for a rod of one unit of length and one unit of sectional area

$$w = \frac{\alpha}{\frac{1}{e} + \beta}.$$

For a rod of length l, the extension by the same weight will be l times as great: therefore if e be the extension of such a rod, we must have in order that w may remain unaltered,

$$w = \frac{\alpha}{\frac{l}{e} + \beta}.$$

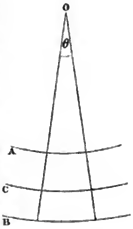
Also, if the sectional area of the rod be a units of area, the weight to produce a given extension in such a rod will be a times as great as before, or,

$$w = \frac{\alpha a}{\frac{l}{e} + \beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A).$$

Which is the general formula for the extension of a rod l feet long, and having a sectional area a. Similarly, let the formula for the compression d of a similar rod be

$$w = \frac{\gamma a}{\frac{l}{d} + \delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

where γ and δ are empirical contents.



To apply these formulæ to determine the extension or compression of a deflected beam, let A, B be the sections of parts of the upper and lower surfaces of the beam, made by a vertical plane passing through the points of support. Let C also be a part of the intersection of that plane with the neutral surface, and θ the small angle made by the intersection of contiguous normals of three lines A, C, B which are assumed to have the same centre of curvature O.

Then if R be the radius of curvature of an element of the curve C, and $r_{,}$ of an element of a parallel curve below C, $R\theta$ will be the length of the former, $r_{,}\theta$ of the latter. If the beam be subject to no strains of torsion, it may be assumed that the neutral surface is a cylindrical surface, and that the material above and below it is bent in cylindrical laminæ parallel to that surface. So that if $dr_{,}$ be the depth of the filament measured along its radius of curvature, μ the transverse thickness of the beam at the part where it is situated, $\mu dr_{,}$ will be the area of the section subject to the extension $(r_{,} - R)\theta$.

Substituting then for the area and extension in the formula (A), the total tension at a distance $r_{,} - R$ below the neutral surface is

$$\frac{\mu\, d r_{,}\, \alpha}{\dfrac{R}{r_{,} - R} + \beta}.$$

Similarly substituting in (B), the total compression at a distance $R - r$ above the neutral surface, supposing μ to remain constant (as the transverse section of the beam is rectangular) is

$$\frac{\mu\, d r\, \gamma}{\dfrac{R}{R - r} + \delta},$$

where r is the radius of curvature of a filament above the neutral surface.

The total moment about the neutral axis of the tension developed in any transverse vertical section of the beam is found by multiplying the first of the two preceding expressions by $r_{,} - R$ and integrating between the limits defined by the lower surface of the beam and the neutral surface. In this way it will be found that if $r_{,}$ be now taken for the radius of curvature at the lower surface of the beam, the total moment of tension at the vertical section here considered is

$$\mu\alpha\left\{\frac{(r_{,} - R)^3}{3R} - \frac{\beta(r_{,} - R)^4}{4R^2} + \frac{\beta^2(r_{,} - R)^5}{5R^3} - \ldots.\right\} \quad . \quad . \quad . \quad . \quad . \quad (A').$$

And similarly the total moment of compression is, r being the radius of curvature of the upper surface of the beam,

$$\mu\gamma\left\{\frac{(R - r)^3}{3R} - \frac{\delta(R - r)^4}{4R^2} + \frac{\delta^2(R - r)^5}{5R^3} - \ldots.\right\} \quad . \quad . \quad . \quad . \quad . \quad (B').$$

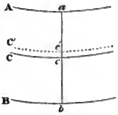
Equating moments about the neutral axis for the equilibrium of the portion of the beam between the transverse section in which the above moments are developed and the nearest fulcrum, we have the sum of these moments equal to the moment of the pressure on the fulcrum. This pressure is equal to half the deflecting pressure applied at the centre of the beam. Also, if x be the perpendicular distance of the centre of moments from the direction of P, the moment of P is Px and

$$Px = (A') + (B').$$

The relation of $r_{,} - R$ to $R - r$ determines the position of the neutral axis. Now, if for equal degrees of longitudinal compression and extension the elastic forces of a rod of iron were the same, we should have, by known principles, the neutral axis at the centre of gravity of the transverse vertical section of the beam. But that the neutral axis is near the centre of gravity is evident from the fact that the ordinary formulæ for the deflection of beams, in which the position of the neutral axis is so assumed, agree well with experiment. The same hypothesis also renders the numerical results accompanying this paper consistent with themselves.

It is important, however, to shew that a small error with respect to the position of the neutral axis does not induce a large error in the equation of moments, but, on the contrary, produces an error which is small in comparison with the original error, and is therefore of a second order of small quantities.

For, as in the last diagram, let A, B represent the positions of the upper and lower surfaces of the beam respectively, C of the neutral surface, ab of a transverse vertical section, so that the sum of the tensions developed between c and b is equal to the sum of the compressions between c and a. Also, at a small distance above the real neutral surface, let another, $C'c'$, be taken. Let T be the sum of the forces of tension developed between c and b, and C the sum of the forces of compression developed between c' and a. Then C is nearly equal to T, and $C - T$ is a small quantity.



Also, let mT be the moment of the forces T about c; then it is clear that their moment about c' is $(m + cc')T$. If nC be the moment of the forces C about c, $(n - cc')C$ is their moment about c'. Hence the difference of the sum the moments of T and C taken about c and c' respectively is $(C - T')cc'$. Also cc' is a small quantity; hence $(C - T)cc'$ is the product of two small quantities.

Next with respect to the moment about c' of the *tension* which must be supposed to exist between c and c', if c' be taken as the neutral axis, it is obvious that the moment of this tension is still smaller than the above.

Hence, on the whole, the difference of moments due to a small error in taking the position of the neutral axis too high is two quantities small with respect to that error, and therefore small in a higher degree with respect to the total moment. A similar conclusion would be arrived at if $C'c'$ were taken a little below the real neutral axis.

Assuming then for the purpose of this computation the neutral axis to coincide with the centre of gravity, we have for a rectangular beam $R - r = r_{,} - R = d$, where $2d$ is the thickness of the beam.

Substituting for the values of (A') and (B') in the equation of moments

$$\frac{Px}{\mu} = \frac{d^3}{3R}(\alpha + \gamma) - \frac{d^4}{4R^2}(\alpha\beta + \gamma\delta) + \frac{d^5}{5R^3}(\alpha\beta^2 + \gamma\delta^2) - \&c.$$

$$= \frac{d^3}{R}\,\frac{\alpha + \gamma}{3}\left\{1 - \frac{d}{R}\cdot\frac{3}{4}\cdot\frac{\alpha\beta + \gamma\delta}{\alpha + \beta} + \frac{d^2}{R^2}\,\frac{3}{5}\cdot\frac{\alpha\beta^2 + \gamma\delta^2}{\alpha + \gamma} - \dots\right\}.$$

Now this series is always very convergent, both because the fraction $\frac{d}{R}$ is small, and because β and δ are small in comparison with α and β respectively. Hence, we may substitute for the above equation the following, without considerable error,

$$\frac{Px}{\mu} = \frac{d^3}{R}\,\frac{\alpha+\gamma}{3}\left\{1 - \left(\frac{3}{4}\cdot\frac{d}{R}\cdot\frac{\alpha\beta+\gamma\delta}{\alpha+\gamma}\right) + \left(\frac{3}{4}\cdot\frac{d}{R}\cdot\frac{\alpha\beta+\gamma\delta}{\alpha+\gamma}\right)^2 - \dots\right\} = \frac{d^3}{R}\,\frac{\alpha+\gamma}{3}\left(1 + \frac{3}{4}\cdot\frac{d}{R}\cdot\frac{\alpha\beta+\gamma\delta}{\alpha+\gamma}\right)^{-1}.$$

The last series coincides with the former in its first two terms, and differs from it by small fractions of each of the subsequent terms, if to α, β, γ, and δ be given the values hereafter determined from experiments.

Solving the equation with respect to $\frac{1}{R}$,

$$\frac{1}{R}=\frac{Px}{\mu d^3\frac{\alpha+\gamma}{3}-d\frac{3}{4}\frac{\alpha\beta+\gamma\delta}{\alpha+\gamma}Px}$$

$$=\frac{x}{p^{-1}-nx}\quad\text{if } p^{-1}=\mu d^3\frac{\alpha+\gamma}{3P}\quad\text{and } n=d\,.\frac{3}{4}.\frac{\alpha\beta+\gamma\delta}{\alpha+\gamma}.$$

Now if y be the co-ordinate of the curve perpendicular to x and referred to the same origin of co-ordinates, namely, the extremity of the neutral axis,

$$\frac{1}{R}=\frac{d^2y}{dx^2}\left(1+\frac{dy^2}{dx^2}\right)^{-\frac{3}{2}}.$$

Also, since $\frac{dy}{dx}$ is always very small, $\frac{dy^2}{dx^2}$ may be neglected as compared with unity, and then $\frac{1}{R}=\frac{d^2y}{dx^2}$.

Substituting $\frac{d^2y}{dx^2}$ for $\frac{1}{R}$ in the above equation, and expanding the right side of it,

$$\frac{d^2y}{dx^2}=px+p^2x^2n+p^3x^3n^2+\ldots$$

Integrating twice and remembering that when $x=a$ (the half length of the beam) $\frac{dy}{dx}=0$; and $y=0$ when $x=0$,

$$\frac{dy}{dx}=p\,\frac{a^2-x^2}{2}+p^2n\,\frac{a^3-x^3}{3}+p^3n^2\,\frac{a^4-x^4}{4}+\ldots$$

$$y=\frac{p}{2}\left(a^2x-\frac{x^3}{3}\right)+\frac{p^2n}{3}\left(a^3x-\frac{x^4}{4}\right)+\frac{p^3n^2}{4}\left(a^4x-\frac{x^5}{5}\right)+\ldots$$

The integration might also be easily effected without expanding the expression for $\frac{1}{R}$, in which case the equation to the elastic curve would be exhibited in finite terms.

If f be the central deflection or the value of y when $x=a$,

$$f=\frac{pa^3}{3}\left(1+\frac{3}{4}pna+\frac{3}{5}p^2n^2a^2+\ldots\right)$$

This series is rapidly convergent, and if we assume

$$f=\frac{pa^3}{3}\left\{1+\left(\frac{3}{4}pna\right)+\left(\frac{3}{4}pna\right)^2+\ldots\right\}$$

The latter series will coincide with the former for the first two terms, and differ from it by about $\frac{1}{26}$th part of the third term, $\frac{1}{38}$th part of the fourth term, &c. We may therefore consider the last equation very nearly accurate: and we have

$$f=\frac{pa^3}{3}\left(1-\frac{3apn}{4}\right)^{-1},$$

or, giving p and n their values,

$$f=pa^3\left\{\mu d^3(\alpha+\gamma)-\frac{27}{16}Pd\frac{\alpha\beta+\gamma\delta}{\alpha+\gamma}\right\}^{-1};$$

which is the required equation connecting the formulæ for direct extension and compression with that for the deflection of a beam of rectangular section.

On the hypothesis of perfect elasticity β and $\delta = 0$, and $\alpha = \gamma$;

$$\therefore f = \frac{Pa^3}{2\mu d^3 \alpha}:$$

which coincides with Poisson's formula for the deflection of a perfectly elastic beam of rectangular section (*Traité de Mécanique*, No. 324).

The Breaking Weight of Rectangular Beams.

Cast iron is much stronger to resist rupture by compression than by extension. Consequently a rectangular bar of that metal when deflected is always broken by the extension of its convex side.

Suppose the ultimate extension of a rod of cast iron, or the greatest extent to which it can be stretched before breaking, to be the n^{th} part of its length. We have shewn that if R be the radius of curvature of the neutral surface and d half the depth of a deflected bar, $\frac{d}{R}$ is the ratio of the extension of a filament on the convex side of the bar to its unstretched length; therefore when the bar breaks, d is the n^{th} part of R or $d = nR$.

Also we have found that

$$R = \frac{1}{px}\mu\delta^3 . \frac{\alpha + \gamma}{3} - \frac{3}{4} d . \frac{\alpha\beta + \gamma\delta}{\alpha + \gamma}.$$

At the centre of the beam $x = a$. Let B be the breaking weight, or twice the value of P the pressure on the fulcrum. Then putting $d = nR$,

$$\frac{d}{n} = 2\mu d^3 \frac{\alpha + \gamma}{3Ba} - \frac{3}{4} d \frac{\alpha\beta + \gamma\delta}{\alpha + \gamma},$$

$$B = 2\mu d^3 \frac{\alpha + \gamma}{3a} \left(\frac{1}{u} + \frac{3}{4} . \frac{\alpha\beta + \gamma\delta}{\alpha + \gamma}\right)^{-1};$$

or the breaking weight varies as the thickness and the cube of the depth directly and as the length inversely. In respect to this law the preceding formula agrees with that hitherto used and is confirmed by experiment. It is observed indeed in the Report of the Iron Commission that the law is not accurately true for bars differing considerably in magnitude, but the anomaly is satisfactorily accounted for (p. 111) as arising "principally from the superior hardness of smaller castings."

Direct Compression of Cast Iron.

From the formulæ for direct tension and for deflection, by substituting numerical values of the coefficients deduced from experiment, may be obtained the numerical values of the coefficients in the formula for direct compression.

This method seems to give more accurate values than the experiments on direct compression detailed in the Report. For those experiments were made on bars enclosed in tubes of which the sides resisted the flexure of the bars; but this resistance had of necessity great effect in sustaining the external force applied longitudinally at the ends of the bars. Also part of the quantities set down for the compression of each bar is probably due to the two ends approaching each other by its flexure. Moreover the numerical results are very irregular, for instead of exhibiting the ratio of the weight to the compression as constantly diminishing as the weight increases, they represent that ratio as alternately increasing and diminishing several times.

In the following computations, the coefficients γ and δ in the hyperbolic formula for compression are deduced by three independent calculations from experiments on three bars of Blaenavon iron of different sizes compared with experiments on the direct extension of the same material.

We have seen that for the mean results of the direct extension of four different sorts of iron in bars 10 feet long and 1 square inch in section

$$w = 118156.424 \div \left(\frac{1}{e} + 2.41\right).$$

For Blaenavon iron alone the formula is nearly the same, and from the experiments will be found to be

$$w = 117106.5 \div \left(\frac{1}{e} + 2.47\right).$$

Or if the length of the rod and the extension be both measured by the same unit of length, 1 inch,

$$w = 120 \times 117106.5 \div \left(\frac{1}{e} + 2.47 \times 120\right).$$

Hence $\alpha = 120 \times 117106.5$ and $\beta = 2.47 \times 120$.

We have seen that for the deflection (f) of a bar of the same material 3.066 inches by 1.0522 inches in section, and supported on points $13\frac{1}{2}$ feet asunder,

$$w = 155.64 \div \left(\frac{1}{f} + .081\right), \quad \text{or } f = w \div (155.64 - {\cdot}081\, w).$$

The general formula for deflection was found to be

$$f = Pa^3 \div \left\{\mu d^3 (\alpha + \gamma) - \frac{27}{16} P d a \, . \, \frac{\alpha\beta + \gamma\delta}{\alpha + \gamma}\right\}.$$

Comparing this with the above value of f, and remembering that $w = 2P$, we have

$$155.64 = \frac{2\mu d^3}{a^3}(\alpha + \gamma) \quad . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

$$.081 = \frac{27}{16}\frac{d}{a^2}\frac{\alpha\beta + \gamma\delta}{\alpha + \gamma} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

Here $\mu = 3.066$, $d = .761$, and $a = 81$ inches: effecting the computation and substituting the values of α and β above given, we find from (1) $\gamma = 120 \times 144892$, and from (2) $\delta = 4.2$. Hence for a bar 10 feet long and 1 inch square the hyperbolic formula for the compression (d) is

$$w = 144892 \div \left(\frac{1}{d} + 4.2\right).$$

For another bar of the same iron 3.1 inches by 3.05 inches in section supported on points 13½ feet asunder, of which the experimental results are given in the Report, p. 75, we find that

$$f = w \div (1206.24 - .1933\,w).$$

Hence by the same method of computation, as in the preceding case, the corresponding formula for direct compression of a bar 10 feet long by 1 inch square, will be found to be

$$w = 118927 \div \left(\frac{1}{d} + 5.6\right).$$

For a bar of the same iron 1.05 inches in the direction of pressure by 2.05 inches broad, on supports 9 feet asunder, of which the experimental results are given in the Report, p. 72, it is found that

$$f = w \div (120.481 - 08875\,w).$$

Hence the corresponding formula for compression of a bar 10 feet long by one inch square is found to be

$$w = 157069 \div \left(\frac{1}{d} + 2.5\right).$$

The several formulæ for compression agree quite as closely as can be expected, when it is considered that the size of the bar influences the hardness of the metal. The last of the preceding formulæ indicates that the metal is hardest in the smallest casting, a result confirmed by practical experience. To obtain complete accuracy α and β should have had different values assigned to them in each of the above computations, but there are at present no data for ascertaining how far the tensile strength of a cast iron bar depends on its bulk.

The great desideratum for perfecting the Hyperbolic or any other hypothetical law of elasticity, is the want of knowledge of these variations of the strength and elasticity of the material, which depend on the magnitude of the castings. It is greatly to be desired that this defect of experimental data may not long continue unsupplied*.

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* The data of the numerical results of this paper are the valuable and careful experiments of Eaton Hodgkinson, Esq., tabulated in Appendix (*A*) to the Report above referred to.

July, 1851.

XV. *On the Oscillations of a Suspension Chain.* *By* J. H. Rohrs, M.A., *late Fellow of Jesus College, Cambridge.*

[Read *Dec.* 8, 1851.]

1. In the following paper I propose to determine the tension and other circumstances of motion at any point of a disturbed chain; principally with a view to exhibit the causes of fracture by vibration in suspension bridges, and to suggest means for obviating as much as possible the danger arising from such vibrations. I shall consider the curvature of the chain to be but small, as it is in practice, as also the extensibility of the material to be zero. The linear expansion of wrought iron bars being only $\frac{1}{10,000}$th for a strain of a ton to the square inch, would introduce additional terms into the equation for the motion of inextensible chains too minute to be important. To apply the results of my investigation to the case of a suspension bridge, I shall suppose the platform perfectly flexible, and the weight of the platform and load uniformly distributed throughout the chain. I shall likewise suppose the links of the chain to be so numerous, that it may be considered simply as a flexible material line; of course these conditions are never entirely fulfilled in suspension chains; but the object I propose to myself is not so much to calculate exactly the amount of strain, to which any one particular bridge is liable, but how far the suspension principle (supposing it could be perfectly carried out) would give rise to strains from vibration, and hence to infer what we might expect would take place, during the oscillations of bridges constructed mainly or partially on that principle. Before I enter upon the analysis of the question, I shall prove a property of $f(s)$ when expanded by Fourier's method in a series of cosines of the form

$$\Sigma\left\{A_n \cos.\left(\frac{2n+1}{2}\,\frac{\pi s}{a}\right)\right\}.$$

The mode of proving this property, and the propriety of doing so, were suggested to me by the paper of Professor Stokes on the "Critical Values of the Sums of Periodic series." I may observe at once, that the effect of the majority of the practical causes of disturbance in suspension chains, such as a moving load, variable pressure of wind sweeping rapidly across the platform, and the tramping of columns of troops moving across the bridge, are all easily and rapidly expressed by series of sines

$$\Sigma\,.\,A_n\,.\,\sin\left(\frac{2n+1}{2}\,\frac{\pi s}{a}\right),$$

n being any positive integer; at all events by far the greater part of the effect of such disturbing causes can be most easily exhibited in a series of this description, as we shall hereafter see.

2. Let then

$$\begin{array}{lr} f(s) \text{ vanish when} & s=a, \\ f'(s) \quad . \quad . \quad . \quad . & s=0, \\ f''(s) \quad . \quad . \quad . \quad . & s=a, \\ f'''(s) \quad . \quad . \quad . \quad . & s=0 \end{array}$$

Then if $f(s)$ be expanded in a Fourier's series of cosines of the form

$$\Sigma . A_n \cos \left(\frac{2n+1}{2} \frac{\pi s}{a} \right),$$

n being any positive integer, $f'(s)$, $f''(s)$, $f'''(s)$ obtained by differentiating the series for $f(s)$, will be the same as if obtained by direct expansion in a sum of definite integrals.

For $\int f(s) \cos \left(\frac{2n+1}{2} \frac{\pi s}{a} \right) ds$

$$= \frac{2a}{(2n+1)\pi} . f(s) \sin \frac{2n+1}{2} \frac{\pi s}{a} - \frac{2a}{(2n+1)\pi} \int f's . \sin \frac{2n+1}{2} \frac{\pi s}{a} ds$$

$$= \frac{2a}{(2n+1)\pi} . f(s) \sin \frac{2n+1}{2} \frac{\pi s}{a} + \frac{4a^2}{(2n+1)^2 \pi^2} f's \cos \frac{2n+1}{2} \frac{\pi s}{a}$$

$$- \frac{4a^2}{(2n+1)^2 \pi^2} \int f''s \cos \frac{2n+1}{2} \frac{\pi s}{a} ds$$

$$= \frac{2a}{(2n+1)\pi} . f(s) \sin \frac{2n+1}{2} \frac{\pi s}{a}$$

$$+ \frac{4a^2}{(2n+1)^2 \pi^2} f's \cos \frac{2n+1}{2} \frac{\pi s}{a} - \frac{8a^3}{(2n+1)^3 \pi^3} . f''(s) \sin \frac{2n+1}{2} \frac{\pi s}{a}$$

$$+ \frac{8a^3}{(2n+1)^3 \pi^3} \int f'''(s) \sin \frac{2n+1}{2} \frac{\pi s}{a} ds.$$

Taking the integrals between the limits $s = 0$ and $s = a$ we find in all these three equations, that all the terms on the right hand side of the equation vanish, with the exception of the last term; hence calling the n^{th} coefficient in the direct expansions of $f(s)$, $f'(s)$, $f''(s)$, $f'''(s)$, A_n, A'_n, A''_n, A'''_n, respectively, we have

$$A_n = - \frac{2a}{(2n \div 1)\pi} A'_n, \quad A_n = - \frac{4a^2}{(2n+1)^2 \pi^2} . A''_n = \frac{8a^3}{(2n+1)^3 \pi^3} . A'''_n,$$

which proves the property in question.

It is quite possible that even if $f'(s)$ vanishes when $s = a$, and $f(s)$ did *not* vanish when $s = a$, that $f'(s)$ could be expanded in a series of sines of $\frac{2n+1}{2} \frac{\pi s}{a}$, with advantage of calculation, just as Professor Stokes has shewn, that $f(s)$ can be expanded in a series of sines of $\left(\pi \frac{ns}{a} \right)$ when $f(s)$ does not vanish, when $s = a$, frequently with gain in labour of numerical details; but in the cases in which I have employed $\Sigma An \sin \frac{2n+1}{2} \frac{\pi s}{a}$ to represent $f(s)$, $f(s)$ has always had its greatest value when $s = a$. I shall proceed now to form the partial differential equation for the oscillations of a flexible chain disturbed only in a vertical plane, small quantities of the second and higher orders being at first neglected.

3. Let a tangent to the lowest point of the chain at rest and parallel to the horizon be taken for the axis of x, a normal and vertical at that point for the axis of y.

Let a mass of an units length of chain = 1.

The area of a section of chain = 1.

$$\left.\begin{array}{l} x', y' \text{ co-ordinates of a point in disturbed chain} \\ x, y, \text{ .... of same point in chain at rest} \end{array}\right\}.$$

Let $x' = x + u, \quad y' = y + v$.

Let T' = tension at (x', y') in disturbed chain,

T = ... (x, y) in chain at rest.

Let $T' = T + w$.

c = length of chain the weight of which equals the equilibrium tension at the lowest point of the chain at rest.

g the accelerating force of gravity.

s = length of chain from origin to (x', y').

Then
$$dx'^2 + dy'^2 = ds^2,$$
$$dx^2 + dy^2 = ds^2;$$
$$\therefore \frac{dx}{ds} \cdot \frac{du}{ds} + \frac{dy}{ds} \frac{dv}{ds} = 0, \text{ as far as the first order.}$$

This may be called the equation of continuity of the chain. As I have found that all circumstances of importance are equally well developed by oscillations in which the chain is always symmetrical about the vertical axis, as by oscillations in which this is not the case, I shall consider only oscillations of the former description, since the number of arbitrary functions is diminished and the labor lessened, on this hypothesis; which necessitates u and $\frac{dv}{ds}$ being each = 0 when $s = 0$. It may be observed that this condition does not at all affect the kind of analysis employed, it only shortens the arithmetic. For the equations of motion we shall have

$$\frac{d^2y'}{dt^2} = \frac{d}{ds}\left(T' \frac{dy'}{ds}\right) - g,$$
$$\frac{d^2x'}{dt^2} = \frac{d}{ds}\left(T' \frac{dx'}{ds}\right);$$

$$\text{or, } \left.\begin{array}{l} \dfrac{d^2v}{dt^2} = \dfrac{d}{ds}\left(T\dfrac{dv}{ds} + w\dfrac{dy}{ds}\right) \text{ to first order} \\ \dfrac{d^2u}{dt^2} = \dfrac{d}{ds}\left(T\dfrac{du}{ds} + w\dfrac{dx}{ds}\right) \text{ . . . .} \end{array}\right\} \text{ . . . . (1),}$$

Let $\frac{dv}{ds} = \frac{d^3V}{ds^3}$;

$$\frac{du}{ds} = -\frac{dy}{dx} \cdot \frac{dv}{ds} = -\frac{s}{c} \frac{d^3V}{ds^3} \text{ remembering that}$$

$\frac{dy}{dx} = \frac{s}{c}$ in the catenary

$$u = -\frac{s}{c}\frac{d^2V}{ds^2} + \frac{1}{c}\frac{dV}{ds} + M_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

$$\int u\,ds = -\frac{s}{c}\frac{dV}{ds} + \frac{2}{c}(V - M_0) + M_1 s\,;$$

M_1 and M_0 being arbitrary functions of t only introduced by the integration.

M_0 will be so assumed that V may vanish when $s = a$, or $V_a = 0$.

$v = \frac{d^2V}{ds^2} + L_1$, L_1 an arbitrary function of t, will be assumed to have such a value that v may $= 0$ when $s = a$.

$\int v\,ds = \frac{dV}{ds} + L_1 s + L_0$, ($L_0$ being another arbitrary function of t).

Now $\frac{dv}{ds} = 0$, $\frac{dy}{ds} = 0$ when $s = 0$, by hypothesis;

$$\therefore\; 0 = \frac{dV}{ds_{s=0}} + L_0, \text{ and if } L_0 = 0\; \frac{dV}{ds_{s=0}} \text{ must } = 0.$$

Eliminating w by integration from our two fundamental equations, we have

$$\int\frac{d^2v}{dt^2}\,ds \times \frac{dx}{ds} - \int\frac{d^2u}{dt^2}\,ds \times \frac{dy}{ds} = T\left(\frac{dv}{ds}\frac{dx}{ds} - \frac{du}{ds}\frac{dy}{ds}\right)$$
$$= cg\left(1 + \frac{s^2}{c^2}\right) \cdot \frac{d^3V}{ds^3}\,.$$

Hence,

$$\frac{d^2}{dt^2}\left[\left(\frac{dV}{ds} + L_1 s + L_0\right)\frac{c}{\sqrt{c^2+s^2}} + \frac{s^2}{c\sqrt{c^2+s^2}}\frac{dV}{ds} - \frac{s}{c\sqrt{c^2+s^2}}\left\{2\,(V - M_0) + cM_1\right\}\right]$$
$$= cg\left(1 + \frac{s^2}{c^2}\right) \cdot \frac{d^3V}{ds^3}\,.$$

The greatest value of $\frac{s}{c}$ will, throughout this investigation, be supposed to be $\frac{1}{4}$, as it is nearly in most suspension bridges of wide span, consequently powers of $\frac{s}{c}$ above the second will be rejected as too small to materially modify the result, we shall then have

$$\frac{d^2}{dt^2}\cdot\frac{dV}{ds} + \frac{d^2}{dt^2}(L_1 s + L_0)\,.\left(1 - \frac{s^2}{c^2}\right) - \frac{2s}{c^2}\frac{d^2}{dt^2}(V - M_0)$$
$$-\frac{s^2}{c}\frac{d^2}{dt^2}M_1 = cg\left(1 + \frac{s^2}{2c^2}\right)\frac{d^3V}{ds^3} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

If, as a first approximation we neglect the term $\frac{2s}{c^2}\frac{d^2}{dt^2}\,.\,V$, and assume for $\frac{dV}{ds}$ a sum

$$\Sigma\left(A_n \sin \frac{2n+1}{2}\frac{\pi s}{a}\right). \quad \text{Then } L_0 = 0,$$

$$\frac{d^2 V}{ds^2} = \Sigma\left(A_n \frac{2n+1}{2}\frac{\pi}{a} \cos \frac{2n+1}{2} . \frac{\pi s}{a}\right), \text{ and } L_1 = 0.$$

At least when $\Sigma A_n \frac{2n+1}{2}\frac{\pi}{a}$ is convergent as it is in all the cases I have considered.

Neglecting also $\frac{s^3}{2c}$, the value of which lies between 0 and $\frac{1}{32}$, and is therefore insignificant, also u must $= 0$ when $s = 0$; $\therefore M_1 = 0$;

$$\therefore \frac{d^2}{dt^2}\left(\frac{dV}{ds}\right) + \frac{2s}{c^2}\frac{d^2}{dt^2}(M_0) = cg\left(\frac{d^3 V}{ds^3}\right) \quad . \quad . \quad . \quad . \quad . \quad (4).$$

We shall afterwards approximate to the solution of the entire equation, including $\frac{d^2}{dt^2}\frac{2s}{c^2} V$, by substituting for V in that term its value found from (4), and working the equations over again,

s can be expanded in a rapidly converging series,

$$\Sigma\left\{\frac{8a(-1)^{n+1}}{(2n+1)^2\pi^2} \sin \frac{2n+1}{2}\frac{\pi s}{a}\right\},$$

between the limits $s = 0$ and $s = a$, where a is the length of the half chain, or the value of s from the lowest point, to the point of suspension.

Substituting then in (4), and equating to zero the coefficients of like sines, we have

$$\begin{aligned} \frac{d^2 A_1}{dt^2} + \frac{2}{c^2} . \frac{8a}{\pi^2}\frac{d^2}{dt^2} M_0 &= -cg\frac{\pi^2}{4a^2} A_1, \\ \frac{d^2}{dt^2} A_2 - \frac{2}{9 . c^2} . \frac{8a}{\pi^2}\frac{d^2}{dt^2} M_0 &= -cg . \frac{9\pi^2}{4a^2} A_2 \quad . \quad . \quad . \quad . \quad . \quad (5), \\ \frac{d^2}{dt^2} A_3 + \frac{2}{25 . c^2} . \frac{8a}{\pi^2}\frac{d^2}{dt^2} M_0 &= -cg . \frac{25 . \pi^2}{4a^2} A_3, \\ \&c &= \&c. \end{aligned}$$

The condition that u may $= 0$ when $s = a$ renders it necessary that

$$\frac{A_1 - A_2 + A_3 - \&c.}{c} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If, however, we assume only $A_1 - A_2 + A_3 = 0$, and suppose the initial values of the constants in the equations for A_4, &c. to be $= 0$, A_4, A_5, &c. will form an exceeding rapidly convergent series, the sum of which will be very small, so that $\frac{A_4 - A_5 + A_6}{c}$ may be neglected; we shall thus have a vibration of three terms, and two types of vibration, we shall only retain the type with the higher coefficient, and in this way we shall find

$$A_3 = h \cos \sqrt{cg . 24 . 2}\, \frac{\pi}{2a}\, t,$$

$$A_2 = .15h \cos \sqrt{cg . 24 . 2}\, \frac{\pi}{2a}\, t,$$

$$A_1 = -.85h\cos\sqrt{cg.24.2}\,\frac{\pi}{2a}\,t.$$

The chain being supposed started from instantaneous rest.

$$\left(\text{The other type is } \sqrt{cg.8.2}\,\frac{\pi}{2a}\right).$$

4. A far better way, however, to get at the types of vibration is the following, which was pointed out to me by Professor Stokes. Since it is clear that if we were to eliminate $\frac{d^2}{dt^2}(M_0)$, from the infinite number of equations (5), we should, (with the condition $A_1 - A_2 + A_3 - \&c. = 0$), have the same types in every term A_1, A_2, &c.; let $q\sqrt{cg}\,\frac{\pi}{2a}$ be a type common to all the terms, substituting for

$$\begin{array}{ll} \frac{d^2}{dt^2}(A_1) & -q^2{}_1A, \\ \frac{d^2}{dt^2}(A_2) & -q^2A_2, \\ \&c. & \&c. \end{array}$$

And eliminating $\frac{d^2}{dt^2}M_0$ between the first and each succeeding set of equations, we have,

$$A_1(1-q^2) + A_2\{3^2(3^2-q^2) = 0,$$

$$A_1(1-q^2) - A_3\{5^2(5^2-q^2) = 0,$$

$$\&c. \qquad \&c. \qquad = 0.$$

$$\text{And } A_1 - A_2 + A_3 - \&c. = 0;$$

$$\therefore\ \frac{1}{1-q^2} + \frac{1}{3^2(3^2-q^2)} + \frac{1}{5^2(5^2-q^2)} + \&c. = 0 \quad . \quad . \quad . \quad . \quad (7).$$

5. There is another yet simpler mode of determining q, for which I am also indebted to a hint of Professor Stokes', who advised me to attack equation (4) at once, without the use of series, which (when it is not required to have $\frac{dV}{ds}$ expressed in a series of a particular form,) is by far the shortest plan.

For $\frac{d^2}{dt^2}\left(\frac{dV}{ds}\right) + \frac{2s}{c^2}\cdot\frac{d^2}{dt^2}M_0 = cg\left(\frac{d^3V}{ds^3}\right)$ may be written

$$\frac{d^2}{dt^2}\left(\frac{dV}{ds} + \frac{2s}{c^2}M_0\right) = cg\,\frac{d^2}{ds^2}\left(\frac{dV}{ds} + \frac{2s}{c^2}M_0\right);$$

$$\therefore\ \frac{dV}{ds} + \frac{2s}{c^2}M_0 = h\sin(p\sqrt{cg}\,t)\sin(ps+\alpha).$$

Now $\frac{d^3V}{ds^3}_{s=0} = 0$; $\therefore\ \alpha = 0$;

and $\frac{dV}{ds}_{s=a} = 0$, in order that u_a may $= 0$;

$$\therefore\ 0 = -2\frac{a}{c^2}M_0 + h\sin p\sqrt{cgt}\sin pa.$$

$$\text{Also}\ \frac{d^2V}{ds^2}_{s=a} = 0;$$

($\because$ by hypothesis $\frac{d^2V}{ds^2}$ can be expanded in a series of $\Sigma . An\frac{2n+1\pi}{2a}\cos\frac{2n+1\pi s}{2a}$,

and that $\Sigma An\frac{2n+1.\pi}{2a}$ is convergent);

$$\therefore\ 0 = -\frac{2}{c^2}M_0 + hp\cos pa\sin p\sqrt{cgt};$$

$$\therefore\ 0 = \sin pa - pa\cos pa, \qquad \therefore\ \tan pa = pa.$$

This condition is easily shewn to be the same as (7), for putting

$$\cos\theta = D\left(\theta^2 - \frac{\pi^2}{4}\right)\left(\theta^2 - \frac{3^2.\pi^2}{4}\right)\left(\theta^2 - \frac{5^2.\pi^2}{4}\right)\ldots\ldots$$

Taking logarithms and differentiating

$$\tan\theta = \frac{2\theta}{\frac{\pi^2}{4} - \theta^2} + \frac{2\theta}{\frac{3^2\pi^2}{4} - \theta^2} + \ldots\ldots$$

Let $2\theta = \pi q$;

$$\therefore\ \tan\frac{\pi q}{2} = \frac{\pi q}{\frac{\pi^2}{4} - \frac{\pi^2 q^2}{4}} + \ldots\ldots \text{ \&c.}$$

$$= \frac{4q}{\pi}\left(\frac{1}{1-q^2} + \frac{1}{3^2-q^2} + \ldots\ldots\right).$$

And $\frac{1}{1^2-q^2} + \frac{1}{3^2.(3^2-q^2)} + \frac{1}{5^2(5^2-q^2)} + \ldots\ldots = 0$ may be written

$$\frac{1}{q^2}\left\{\Sigma.\frac{1}{(2n+1)^2} - \left\{\frac{1}{1-q^2} + \frac{1}{3^2-q^2} + \text{\&c.}\right\}\right. = 0;$$

$$\therefore\ \frac{1}{1-q^2} + \frac{1}{3^2-q^2} + \text{\&c.} = \frac{\pi^2}{8} = \frac{\pi}{4q}\tan\frac{\pi q}{q};$$

$$\therefore\ \tan\frac{\pi q}{2} = \frac{\pi q}{2},$$

which is identical with $pa = \tan pa$, since $pa = \frac{\pi q}{2}$.

The types of vibration I had found for A_3 were $\sqrt{8.2.cg}.\frac{\pi}{2a}$ and $\sqrt{24.2.cg}\frac{\pi}{2a}$, and I proved that the greatest amount of "play" at the end from the neglected terms A_4, A_5, &c. would be little more than $\frac{1}{1000}$th the maximum disturbance at the centre of the chain, of course this small quantity might be safely set aside, and we shall now see that the accurate values of the types, corresponding to those obtained at first, differ imperceptibly from the above values.

Professor Stokes was kind enough to work out for me the first two or three roots of the equation $\tan\left(\frac{\pi q}{2}\right) = \left(\frac{\pi q}{2}\right)$; which with his method I shall here transcribe. It is clear that q will always be nearly expressed by $(2i + 1)$, i being a positive integer.

$$\text{Let } \therefore \frac{\pi q}{2} = \left(\frac{2i+1}{2}\right)\pi - \theta;$$

$$\therefore \cot\theta = (2i+1)\frac{\pi}{2} - \theta.$$

This can be solved readily by successive substitution, or by trial and error, the first two roots give

$$\theta = 12^{\circ}.\ 34',$$
$$\theta = 7^{\circ}.\ 23'.$$

The second of these values corresponds to $\sqrt{24.2cg}\,\frac{\pi}{2a}$ and would give us $\sqrt{24.18cg}\,\frac{\pi}{2a}$, which only differs by 2 in the fourth figure of the root from the value we have already found.

The reason why I have chosen vibrations in which only one type appears will be given when we consider the action of small periodic forces on the chains; it will then be seen that in bridges of wide span terms containing the types $\sqrt{24.2cg}\,\frac{\pi}{2a}$ are very likely to be those which express the disturbance occasioned by soldiers marching in time along the platform; and, consequently, that I could scarcely have selected a better example to illustrate the subject, generally, and the effect of the small terms I shall hereafter include—in particular—than the one we are now discussing.

To determine the tension we have

$$\frac{d^2}{dt^2}\frac{dV}{ds} = cg\left(1 + \frac{s^2}{2c^2}\right)\frac{d^3 V}{ds^3} + w\frac{s}{c} \text{ from (1)},$$

$$\text{writing for } \frac{dy}{ds} \text{ its value } \frac{s}{c}, \left(\frac{s}{\sqrt{c^2+s^2}} \text{ exactly}\right);$$

$$\therefore\ w\frac{s}{c} = -\frac{2s}{c^2}\frac{d^2}{dt}(V - M_0) \text{ and when } s = a,$$

$$w = \frac{2}{c}\frac{d^2}{dt^2}M_0,$$

$$\text{and } \frac{2}{c^2}\cdot\frac{8a}{\pi^2}\frac{d^2}{dt^2}M_0 = -23.2cg\frac{\pi^2}{4a^2}.85h \cos\sqrt{cg.24.2}\,\frac{\pi t}{2a}.$$

The term in v or $\frac{d^2 V}{ds^2}$ corresponding to A_3 is

$$\frac{5\pi}{2a}h\cos\sqrt{cg.24.2}\,\frac{\pi}{2a}t, \quad \text{let therefore } \frac{5\pi}{2a}h = h',$$

and we shall find the tension $-122.h'g\cos\left(\sqrt{cg.24.2}\,\frac{\pi}{2a}t\right)$ nearly.

6. To include the term $\frac{2s}{c^2}\frac{d^2}{dt^2}(V)$, let us assume for $\frac{dV}{ds}$ an additional series of sines $\Sigma . B_n \sin\frac{\pi s n}{a}$ representing a function which vanishes when $s = 0$ and $s = a$.

In $\frac{2s}{c^2}$ write for s $\frac{8a}{\pi^2}\sin\frac{\pi s}{2a}$, which will be near enough for our purpose without taking in more terms of the series

$$\frac{d^2 V}{dt^2} = -\left(\frac{d^2 A_1}{dt^2}\cos\frac{\pi s}{2a}\times\frac{2a}{\pi} + \frac{d^2}{dt^2}A_2\cos\frac{3\pi s}{2a}\times\frac{2a}{3\pi} + \frac{d^2}{dt^2}A_3\cos\frac{5\pi s}{2a}\times\frac{2a}{5\pi}\right)$$

$$\frac{2s}{c^2}\frac{d^2 V}{dt^2} = -\frac{1}{\pi^3}\left(\frac{d^2 A_1}{dt^2}\sin\frac{\pi s}{a} - \frac{1}{3}\frac{d^2 A_2}{dt^2}\sin\frac{\pi s}{a}\right)$$

$$-\frac{1}{\pi^3}\left(\frac{1}{3}\frac{d^2 A_2}{dt^2}\sin\frac{2\pi s}{a} - \frac{1}{5}\frac{d^2 A_3}{dt^2}\sin\frac{2\pi s}{a}\right)$$

$$-\frac{1}{\pi^3}\left(\frac{1}{5}\frac{d^2 A_3}{dt^2}\sin\frac{3\pi s}{a}\right),\ \text{remembering that}\ \frac{16a^2}{c^2} = 1,$$

$$\frac{d^2 B_1}{dt^2} = -cg\frac{\pi^2}{a^2}B_1 - \frac{1}{\pi_3}\frac{d^2}{dt^2}A_1 + \frac{1}{3\pi^3}\frac{d^2}{dt^2}A_2$$

$$\frac{d^2}{dt^2}B_2 = -cg\frac{4\pi^2}{a^2}B_2 - \frac{1}{3\pi^3}\frac{d^2}{dt^2}A_2 + \frac{1}{5\pi^3}\frac{d^2}{dt^2}A_3$$

$$\frac{d^2}{dt^2}B_3 = -cg\frac{9\pi^2}{a^2}B_3 - \frac{1}{5\pi^3}\frac{d^2}{dt^2}A_3,$$

$$\text{also}\ A_1 - \frac{A_2}{3} = -\frac{4}{5}h\cos\sqrt{cg.24.2}\,\frac{\pi}{2a}t,\ \text{nearly},$$

$$\frac{A_2}{} - \frac{A_3}{5} = -\frac{3h}{20}.T,\ \text{(writing } T \text{ for the periodic function)},$$

$$\frac{A_3}{5} \qquad = \frac{h}{5}T.$$

Retaining only the periodic T, and putting the coefficients of $\cos\sqrt{cg}\,\frac{\pi}{a}t$ &c. $= 0$, we have

$$B_1 = \frac{1}{\pi^3}(1.07hT)\ \text{nearly},$$

$$B_2 = \frac{1}{\pi^3}(.45hT)\ \ .\ \ .\ \ .$$

$$B_3 = \frac{1}{\pi^3}(.42hT)\ \ .\ \ .\ \ .$$

The value of v when $s = a$ derived from this series is $.018h'$ nearly at its maximum.

Consequently as far as terms involving $\frac{s^2}{c^2}$ the chain is not quite fastened at its ends, in order to render it so, we must return to our original equation,

$$u = -\frac{s}{c}\frac{d^2 V}{ds^2} + \frac{1}{c}\frac{dV}{ds},$$

$$\text{where } \frac{dV}{ds} = \Sigma . A_n \sin \frac{2n+1}{2} \frac{\pi s}{a} + \Sigma . B_n \sin \frac{n \pi s}{a},$$

$$\frac{d^2 V}{ds^2} = \Sigma . B_n \frac{n \pi}{a} \cos n \pi, \text{ when } s = a;$$

$$\therefore\ 0 = \pi (B_1 - 2B_2 + 3B_3) + A_1 - A_2 + A_3.$$

$$\text{Again, } L_1 + \frac{d^2 V}{ds^2} = 0, \quad s = a_1;$$

$$\therefore\ L_1 - (B_1 - 2B_2 + 3B_3)\frac{\pi}{a} = 0.$$

$$\text{And } \pi (B_1 - 2B_2 + 3B_3) = .14hT = .14A_3; \quad \therefore\ A_1 - A_2 + 1.14A_3 = 0;$$

$$\therefore\ \frac{d^2}{dt^2} A_1 + \frac{2}{c^2} \cdot \frac{8a}{\pi^2}\frac{d^2}{dt^2} M_0 = -cg\frac{\pi^2}{4a^2} A_1 - \frac{8a}{\pi^2}\frac{d^2}{dt^2} \cdot \frac{\pi}{a}(B_1 - 2B_2 + 3B_3).$$

&c. = &c.

&c. = &c.

$$\text{Whence } \frac{d^2}{dt^2}(9A_2 + A_1) = -cg\frac{\pi^2}{4a^2}(81A_2 + A_1),$$

$$\frac{d^2}{dt^2}(A_1 - 25A_3) = -cg\frac{\pi^2}{4a^2}(A_1 - 625A_3).$$

$$\text{Whence } A_3 = h \cos \sqrt{24.13.cg}\,\frac{\pi}{2a} t,$$

$$A_2 = .165h\,(T),$$

$$A_1 = -.975h\,(T),$$

writing T as before for the periodic function. Maximum disturbance at centre from $L + \Sigma . A_n$ series (taking in the two small terms A_1, A_3) is $.92h'$.

From $\Sigma . B_n$ series is $.03h'$;

$\therefore$ $.95h'$ is the maximum disturbance nearly.

The maximum increase of tension at the ends is $124.h'gT$ nearly.

If we compare this with the results obtained by neglecting the play " $.018h'$ we shall find $.96h'$* nearly for the maximum disturbance, and $122.h'gT$ for the maximum increase of tension; the difference therefore is imperceptible. We might include the term $\frac{d^3 V}{ds^3}\frac{s^2}{2c^2}$, although it obviously produces scarcely any effect; we should then write for s^2 the following series:

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* Including the part derived from $\Sigma . B_n$ series.

$$\frac{a^2}{3} + \frac{4a^2}{\pi^2}\left(-\cos\frac{\pi s}{a} + \frac{1}{4}\cos\frac{2\pi s}{a} - \&c.\right)$$

If we take the first term only, we shall find the types increased in the ratio of $\frac{\sqrt{101}}{\sqrt{100}}$ nearly, and the tension at the end will be increased about $\frac{1}{100}$th.

If $a = 300$ feet, $h' = 1$ foot, it hence appears that for a disturbance (A_3 being the leading term, and $\sqrt{24.2cg}\,\frac{\pi}{2a}$ the type) of $\pm .95$ foot at the lowest point of the chain, the increase of tension would be more than $120g$, more than $\frac{1}{10}$th the equilibrium tension at the lowest point.

7. Let us now suppose $M_0 = 0$, or the increase of tension to be 0 at the points where the chain is supported; in this case

$$u = -\frac{s}{c}\frac{d^2V}{ds^2} + \frac{1}{c}\frac{dV}{ds} \text{ will } = \frac{1}{c}(A_1 - A_2 + A_3 - \&c.) \text{ when } s = a.$$

And if we take one term only,

$$\frac{d^2}{dt^2}A_3 = -cg\frac{25\pi^2}{4a^2}A_3,$$

$$A_3 = h\cos\sqrt{cg}\,.\,\frac{5\pi}{2a}\,t.$$

The maximum value of $u = \frac{h}{c} = \frac{2ah'}{5\pi c}$

$$= \frac{h'}{30} \text{ nearly.}$$

The greatest value of $w = \frac{2}{c}\frac{d^2}{dt^2}V = 2h'g.$

We have obtained then this remarkable result, that a "play" of $\frac{1}{30}$th the maximum value of v in u at the points of suspension is sufficient to reduce the increase of tension to an inappreciable quantity. This fact would seem to suggest the employment of buffers or saddles, or some kind of yielding elastic mode of attachment for the ends of the chain. For it is clear that if M_0 be always zero, or at all events small, w will never be considerable.

8. I shall now proceed to consider the effect of the transit of loads, and the marching of troops along the platform, and also of the variable pressure of gusts of wind moving at a high velocity. I shall suppose the load to be travelling at an uniform rate p, and to be equally distributed throughout the portion of the platform it covers; so that if b were the length of the part of the platform covered by the load, μ the ratio of the weight of the load to that of the chains and platform immediately above and below it; the effect of the load would be nearly the same as though the force of gravity were increased in the ratio of $1 + \mu : 1$, upon the chains and platform just above and below the load; (that is, neglecting terms multiplied by

$\frac{s^2}{c^2}$ which would but slightly modify the result). Of course μ is supposed small compared with 1. We should then have an additional term $f'(s)$ on the right hand side of the equation (1), which would have one of these three sets of values.

(1) 0 till $s = a - pt$ and thence $= -\mu g$ till $s = a$.

(2) 0 till $s = a - (pt + b)$ – thence $= -\mu g$ till $s = a - pt$ and thence $= 0$ till $s = a$.

(3) $-\mu g$ till $s = b - pt$ and thence $= 0$ till $s = a$.

And on the right hand side of equation (4) we shall have a term $\int f'(s)\,ds$, or $f(s)$, which will have one of these three sets of values.

(1) 0 till $s = a - pt$, and thence $-\mu g\,\{s - (a - pt)\}$ till $s = a$.

(2) 0 till $s = \{a - (pt + b)\}$, and thence $-\mu g\,\{s - a + (pt + b)\}$ till $s = a - pt$, and thence $= -\mu g b$ till $s = a$.

(3) $-\mu g s$ till $s = b - pt$, and thence $= -\mu g\,(b - pt)$ till $s = a$.

In accordance with these conditions $f(s)$ will be represented by

$$(1)\quad -\mu g\,\Sigma\,.\,\frac{8a}{\pi^2}\,.\,\frac{1}{(2n+1)^2}\left\{\sin\frac{2n+1}{2}\pi - \sin\frac{2n+1}{2}\pi\left(\frac{a-pt}{a}\right)\right\}\sin\frac{2n+1}{2}\,\frac{\pi s}{a}.\Big\}$$

$$(2)\quad -\mu g\,\Sigma\,.\,\frac{8a}{\pi^2}\,\frac{1}{(2n+1)^2}\left\{\sin\frac{2n+1}{2}\pi\left(\frac{a-pt}{a}\right)\right.$$

$$\left. - \sin\frac{2n+1}{2}\pi\left(\frac{a-pt-b}{a}\right)\sin\frac{2n+1}{2}\,\frac{\pi s}{a}\right\}\,\ldots\ldots\ldots\ldots\,(8),$$

$$(3)\quad -\mu g\,\Sigma\,.\,\frac{8a}{\pi^2}\,\frac{1}{(2n+1)^2}\left\{\sin\frac{2n+1}{2}\pi\left(\frac{b-pt}{a}\right)\right\}\sin\frac{2n+1}{2}\,\frac{\pi s}{a}.\Big\}$$

As will immediately appear if we perform the operation $\frac{2}{a}\int_a^0 fs\,\sin\frac{2n+1}{2}\,\frac{\pi s}{a}\,ds$ remembering the values of fs and $f's$.

The first set of values being employed till the load has entirely cleared the piers, the second till the formost part of the load has reached the middle of the platform, and the last from the time it has reached the middle till it has cleared the first half of the platform.

If we would calculate the exact amount of disturbance due to the passage of a single load at a given velocity, we must not of course suppose the chain to be always symmetrical about the axis, but must include the functions L_0 and M_1, and compute for the two sides of the chain at the same time. Denoting by $u'\ v'\ \frac{dV'}{ds}$, &c. the corresponding values on the one side of the chain to $u\ v\ \frac{dV}{ds}$, &c. on the other we should have to determine

$$L_0,\ M_1,\ L'_0 \text{ and } M'_1,$$

$$u_0 = -u'_0,\ v_0 = v'_0,\ \frac{d^3V}{ds^3_0} = -\frac{d^3V'}{ds^3_0}\ ;$$

and because the tension at the lowest point of the chain must be the same derived from either set of equations,

$$\frac{2}{c}(V_0 - M_0) = \frac{2}{c}(V_0' - M_0').$$

It is of course not necessary that a' should $= a$, for the chain might have been suspended from two points not in the same horizontal line. I shall however suppose that the chain is kept always symmetrical, either by the passage of another similar load from the opposite end of the platform, or by any other means, and that $a' = a$, as the laws I would arrive at are as I before stated, quite independent of the form of the disturbed chain being symmetrical or unsymmetrical about the vertical axis.

As a first example of the effect of a moving load, I shall suppose the bridge disturbed by two equal loads moving from opposite directions with the same velocity and tending to meet in the centre. I shall assume b to be $> a$, and shall consider only what happens till the load reaches the centre. In this case it will be sufficient to employ the first of the three sets of formulæ. It is clear that according to these suppositions, the chain will be symmetrical about the vertical axis. I shall suppose also that by means of saddles or some other contrivance, the tension at the ends is kept constant, so that $M_0 = 0$.

Then we should have

$$\frac{d^2}{dt^2}(A_1) = -cg\frac{\pi^2}{4a^2}A_1 - \mu g\frac{8a}{\pi^2} + \mu g\frac{8a}{\pi^2}\sin\frac{\pi}{2a}(a - pt);$$

$$\frac{d^2}{dt^2}(A_2) = -9cg\frac{\pi^2}{4a^2}A_2 + \mu g\frac{8a}{9\pi^2} + \mu g\frac{8a}{9\pi^2}\sin\frac{3\pi}{2a}(a - pt),$$

&c. = &c.

$$\therefore\ A_1\left(1 + \mu\,.\,\frac{32a^3}{\pi^4 cg}\right) = h_1\sin\left(\sqrt{cg}\,.\,\frac{\pi}{2a}t + \alpha\right) + \frac{\mu\,.\,32a^3 g}{\pi^4(cg - p^2)}\sin\frac{\pi}{2a}(a - pt),$$

&c. = &c.

If p^2 be very small compared with cg, the condition that A_1 and $\frac{dA_1}{dt}$ may $= 0$ when $t = 0$, will make $h_1 = 0$; that is to say when we put $p^2 = 0$ in the denominator of the last term;

$$\therefore\ A_1 = -\mu\,.\,\frac{32a^3}{\pi^4 c} + \mu\,.\,\frac{32a^3}{\pi^4 c}\sin\frac{\pi}{2a}(a - pt),$$

$$A_2 = \quad \mu\,.\,\frac{32a^3}{81\,.\,\pi^4 c} + \mu\frac{32a^3}{81\,.\,\pi^4 c}\sin\frac{3\pi}{2a}(a - pt).$$

Now when $pt = a$ since p has been assumed to be indefinitely small; it is clear that the position of the chain ought to coincide with its statical position, on the hypothesis, that while the tension at the ends is the same as before, the density has been increased throughout from 1 to $1 + \mu$.

And therefore the deflection in the centre as furnished by our equations ought to be the same as the statical deflection, or

$$\frac{\mu a^2}{2c}, \text{ or } \frac{d^2 V}{ds^2_a} = -\frac{\mu a^2}{2c}.$$

$$\text{Now } \frac{d^2 V}{ds^2} = \frac{\pi}{2a} A_1 + \frac{3\pi}{2a} A_2 + \&\text{c. at the centre,}$$

$$= -\frac{\pi}{2}\left(\frac{32a^2}{\pi^3 c} - \frac{32a^2}{81\pi^3 c} + \&\text{c.}\right).$$

If we take only the first term ($\pi^3 = 31$ nearly ;)

$$\frac{d^2 V}{ds^2_a} = -\frac{\mu}{2} \cdot \left(\frac{32}{31}\frac{a^2}{c}\right), \text{ a result remarkably close to the true } -\frac{\mu}{2} \cdot \frac{a^2}{c}.$$

In practice p^2 always will be small compared with cg, for taking a at only 75 feet, $c = 300$, and cg exceeds 9000; while p^2 would rarely exceed 400, and for persons walking would not probably be more than 50 (or about 4 miles an hour). In the same manner we may shew that if the chain be tied at its ends, the mere walking, without *stamping*, across the platform of a body of troops would produce an inappreciable vibration.

In the case of a gust of wind, however, sweeping across the platform, and exerting a variable pressure upon it, rapidly changing from point to point; p^2 might $= cg$, or if the chain were fastened at its ends, p^2, or $9h^2$, or $25p^2$ might $= \frac{4a^2}{\pi^2} \times$ the square of some one of the types of A_1 &c.; of course then terms not periodic would be introduced, and with this condition, a succession of gusts having the same velocity might do much mischief.

9. To estimate the effect of marching in time, we observe that if when the platform is either in its highest or lowest position, it receives an impulse from the foot, the extent of vibration will be increased each time. Such an effect may be expressed by a periodic function, which goes through all its values in the time occupied by a single step, or perhaps by two steps; certainly by two steps if the platform be in its highest or lowest position at the epoch of two consecutive steps. Let ∴ P be such a periodic function, then formulæ (8) must be multiplied by $(1 + EP)$ where E is a given constant.

To take a simple example, let us suppose the bridge covered with people, and that half of them, half way from either end to the middle are stamping in time, it is required to estimate the effect.

$$\text{Then } f(s) = 0 \text{ till } s = \frac{a}{2}, \text{ and } = -fP\left(s - \frac{a}{2}\right) \text{ till } s = a.$$

Where fP is the periodic force, P the periodic part of it, and f a coefficient.

Let us suppose the chain to have play at its ends so that M_0 can be kept $= 0$,

$$f(s) = -\Sigma . \frac{8a}{(2n+1)^2} . \pi^2 fP . \left(\sin\frac{2n+1}{2}\pi - \sin\frac{2n+1}{4}\pi\right) . \sin\frac{2n+1}{2}\frac{\pi s}{a}.$$

Let now $P = \sin\left(\sqrt{cg} . \frac{\pi}{2a} t\right)$.

Then writing k for $\frac{8a}{\pi^2}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right) . f$

$$\frac{d^2 A_1}{dt^2} = -cg\frac{\pi^2}{4a^2} A_1 - k \sin\sqrt{cg} . \frac{\pi}{2a} t,$$

$$\&\text{c.} = \qquad \&\text{c.}$$

Whence when t is considerably great

$$A_1 = \frac{kt \cos \sqrt{cg}\, \frac{\pi}{2a}\, t}{2\sqrt{cg} . \frac{\pi}{2a}}.$$

Neglecting periodic terms, in comparison with the large non-periodic term.

10. If the ends of the chain be fastened and P have for its type the same value as any one of those which we have found for a chain tied at its ends, and if the disturbing force continue for a considerable time and then cease, the disturbance will have but one type, the same as the type of P.

For if, as before, we consider equations (5), and assume ($\sin qt$) for one of the periodic functions which satisfy these equations, as a term of M_0, q being the type of P, after a long time M_0 will be nearly expressed by one term $C_0 t \cos qt$.

If then we assume for it A_1, A_2, A_3, &c. values $C_1 t \cos qt$, $C_2 t \cos qt$, &c., and equate coefficients we shall find that

$$\frac{C_1}{C_0},\ \frac{C_2}{C_0},\ \frac{C_3}{C_0}\ \text{\&c.}$$

have the same values as though we had assumed simply, $M_0 = C_0 \cos qt$.

Now if at the time τ the disturbing force ceases, $C_1\tau$, $C_2\tau$, &c. are the coefficients of A_1, A_2, &c.

And these coefficients are in the same ratio as though we had assumed for M_0, $C_0 \cos qt$, a vibration of one type only (q) and $\frac{dA_1}{dt}$, &c. are the same therefore the disturbance will be the same. In this case if we put $v = 0$ the periodic term divides out, consequently there will be nodal points at which there will be no vertical motion. The reason why the type $\left(\sqrt{24.2cg}\, \frac{\pi}{2a}\right)$ has been so prominently brought forward will now appear; suppose troops marching at the rate of 4 miles an hour; and let the length of a step be one yard; then the time of a single step is

$$\frac{3600''}{4 \times 1760} = \frac{1''}{2} \text{ nearly};$$

$$\text{if therefore, } \sin\left(\sqrt{24.2cg}\, \frac{\pi}{2a}\, t\right) = \sin .5 \sqrt{cg}\, \frac{\pi}{2a}\, t \text{ nearly,}$$

have a period of 1″ it is very likely that the impulse of the feet may be communicated to the platform at its highest and lowest positions, and therefore it may be represented by a periodic force P which goes through its periods in 1″.

And P and $\sin 5\pi \frac{\sqrt{cg}}{2a}\, t$ will have the same period

$$\text{if } \sin\left\{\frac{5\pi}{2a} . \sqrt{cg}\,(t + 1'')\right\} = \sin \frac{5\pi}{2a} \sqrt{cg}\, t;$$

$$\therefore\ \sqrt{cg} = \frac{4}{5}\, a;$$

$$\therefore\ \sqrt{ag} = \frac{2}{5}a\,; \quad \therefore\ a = \frac{25}{4}g = 200 \text{ feet nearly:}$$

for a span therefore of 400 feet, $\therefore\ \sqrt{24.2cg}\,\frac{\pi}{2a}$ is a likely type to become of importance; for a span of about 150 feet, under the same circumstances, $\sqrt{8.2cg}\,\frac{\pi}{2a}$ is the type that would rise.

Besides there is always a remarkable tendency to adjust one's stepping to the motion of a vibrating floor, and to tread heavier on the floor when it is sinking than when it is rising; which will of course produce a continually increasing disturbance.

In the case of the disturbance of which $\sqrt{24.2cg}\,\frac{\pi}{2a}$ is the type $\left(\text{neglecting terms multiplied by } \frac{s^2}{c^2}\right)$, the nodal points, or points of no vertical motion, are at $s = .15a$, $s = .62a$, and $s = a$.

11. To carry the approximation to the second and higher orders, we should have as before,

$$\frac{d^2v}{dt^2} = \frac{d}{ds}\left(T\frac{dv}{ds} + w\frac{dy}{ds} + w\frac{dv}{ds}\right)$$

$$\frac{d^2u}{dt^2} = \frac{d}{ds}\left(T\frac{du}{ds} + w\frac{dx}{ds} + w\frac{du}{ds}\right),$$

$$\frac{dv}{ds}\cdot\frac{dy}{ds} + \frac{du}{ds}\,\frac{dx}{ds} + \frac{1}{2}\left\{\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right\} = 0\,;$$

$$\int\frac{d^2v}{dt^2}\,\frac{dx}{ds}\,ds - \int\frac{d^2u}{dt^2}\,\frac{dy}{ds}\,ds = T\left(\frac{dv}{ds}\,\frac{dx}{ds} - \frac{du}{ds}\,\frac{dy}{ds}\right)$$

$$+ w\left(\frac{dv}{ds}\,\frac{dx}{ds} - \frac{du}{ds}\cdot\frac{dy}{ds}\right);$$

$$\frac{du}{ds} = -\frac{s}{c}\,\frac{d^3V}{ds^3} - \frac{1}{2}\left(1 + \frac{s^2}{c^2}\right)^{\frac{3}{2}}\cdot\left(\frac{d^3U'}{ds^3}\right)^2,$$

where $\frac{d^3V'}{ds^3}$ is $\frac{d^3V}{ds^3}$ found to a first approximation.

$$\int u\,ds = -\frac{s}{c}\,\frac{dV}{ds} + \frac{2}{c}(V - M_0) + M_1 s - \frac{1}{2}\iint\left(1 + \frac{s^2}{c^2}\right)^{\frac{3}{2}}\left(\frac{d^3V'}{ds^3}\right)^2.\,ds^2,$$

consequently we should have

$$\left.\begin{aligned}&\frac{d^2}{dt^2}\cdot\frac{dV}{ds} + \frac{d^2}{dt^2}(L_1 s + L_0)\left(1 - \frac{s^2}{c^2}\right) - 2\frac{s}{c^2}\,\frac{d^2}{dt^2}(V - M_0)\\ &- \frac{s^2}{c^2}\,\frac{d^2}{dt^2}(M_1) + \frac{1}{2}\,\frac{s}{c}\iint\left(1 + \frac{s^2}{c^2}\right)^{\frac{3}{2}}\frac{d^2}{dt^2}\left(\frac{d^3V'}{ds^3}\right)^2.\,ds^2\end{aligned}\right\},$$

$$\text{all} = cg\left(1 + \frac{s^2}{2c^2}\right)\frac{d^3 V}{ds^3} + u'\frac{d^3 V'}{ds^3}$$

$$= cg\left(1 + \frac{s^2}{2c^2}\right)\frac{d^3 V}{ds^3} - \frac{2}{c}\frac{d^2}{dt^2}(V' - M_0)\frac{d^3 V'}{ds^3}.$$

As before, we must approximate by first neglecting and then taking into account the terms multiplied by $\frac{s}{c}$ and its powers; the process would be simply laborious and inelegant, and as it is rather general laws than minute results, that we require, the labor of the calculation would be scarcely repaid. It may be observed that the most important term and the only one perhaps worth considering at all is

$$\frac{2}{c}\frac{d^2}{dt^2}M_0'\frac{d^3 V'}{ds^3}.$$

12. On the whole, then, we may state the following as the most important of the results we have arrived at.

(1) That for a slight disturbance of a chain of but small deflection the increase of tension is nearly uniform throughout, and consequently if we cause the increase of tension to be always small or zero at the points of suspension, we may be sure that it will not be very great at any other point.

(2) That the mere transit of loads at an ordinary pace across bridges of wide span is not likely to produce a sensible vibration.

(3) That stamping or tramping of any kind is dangerous in the extreme, as if it be periodic and coincide in period with any of the possible types of vibration we have already found, it will give rise to a continually increasing disturbance.

(4) That the rapid sweeping of gusts of wind along the platform is apt, under the conditions we have already found, to give rise to violent vibrations.

(5) That when the vibration arises from the action of a periodic force (coinciding in type with a type of vibration of the equation of disturbance,) continued for some time and then ceasing, the disturbance (approximately) will consist of terms containing but one type of vibration, and the chain will be separated by nodal points where there will be no vertical motion.

The most important points of difference between the motion of a suspension bridge and that of a simple chain are, first, that in a suspension bridge, the rigidity of the platform tends to check the vibrations of the chains, but probably not much in bridges of wide span: and secondly, that the links of the chain are finite in length, about thirty being the average number from pier to pier; this latter point of difference would not however greatly alter the results. A very common and practically one of the most important causes of danger I have not alluded to, I mean the improper manner in which sometimes the chains are bolted together, and which causes an enormous twisting force to be brought to bear upon the bolts. This is however a question entirely independent of the problem of a vibrating chain. If by means of buffers, or some kind of spring attachment, the increase of tension at the ends of the chain could be kept within moderate limits, it is clear that but little danger need be apprehended from the chain being thrown into a state of vibration; the very great amount of tension at the ends of chains of bridges of wide span seems a fatal obstacle to the employment of such modes of attachment; and probably it would be less costly to increase to a sufficient amount the thickness of the chains, than to adopt a complex mode of spring attachment, even if it were possible to construct one.

Mr. Homersham Cox has, I believe, shewn that by the use of oblique tension rods to support the platform, the same load could be sustained as by suspension chains of the same amount of material as the oblique rods; and for bridges of narrow span at least this would seem to be the preferable construction, as it would be less liable probably to vibration.

ADDENDA.

In the solution of the equation

$$\frac{d^2}{dt^2}\,\frac{dV}{ds}+\frac{2s}{c^2}\,\frac{d^2}{dt^2}M_0+\frac{d^2}{dt^2}L_1 s=cg\,\frac{d^3V}{ds^3},$$

we assumed in the first instance that $\frac{d^3V}{ds^3}$ could be expanded in a convergent series,

$$\Sigma\,.\,A_n\,\frac{2n+1}{2}\,\frac{\pi}{a}\cos\frac{2n+1}{2}\,\frac{\pi s}{a},$$

and shewed that in the majority of the kinds of disturbance arising from ordinary causes our assumption was true. It is curious, though practically useless, to examine some kinds of disturbance in which this is not the case.

Let us, for instance, suppose that $\frac{dV}{ds}$ could be expanded in a convergent series

$$\Sigma A_n\sin\frac{\pi s n}{a},\ \text{so that also}\ \Sigma\frac{n\pi}{a}A_n\cos\pi\frac{sn}{a}$$

was convergent, and let us suppose the chain free at its ends. The motion being supposed, as hitherto, to be always symmetrical about the vertical axis. Then, as before, we should have

$$\frac{dV}{ds}+\frac{2s}{c^2}M_0+L_1 s=h\sin p\sqrt{cg}\,t\,\sin(ps+\alpha),$$

$$\frac{d^2V}{ds^2}+L_1=0,\quad\therefore\ 0=-\frac{2}{c^2}M_0+hp\sin p\sqrt{cg}\,t\cos(pa+\alpha);$$

$$\frac{d^3V}{ds^3_0}=0,\quad\therefore\ 0=h\sin\alpha;$$

$$\because\ \alpha=0.$$

Hence, $0=-\frac{2}{c^2}M_0+hp\ \sin p\sqrt{cg}\,t\ \cos pa,$

$$0=-\frac{2a}{c^2}M_0+L_1a+h\sin p\sqrt{cg}\,t\sin pa,$$

which determines M_0 and L_1, when p is given.

In this kind of vibration the disturbed chain is always tangential to the equilibrium position of the chain at the points of suspension, but in those kinds we previously considered, the inclination of the disturbed chain to its equilibrium position was a maximum at the points of suspension.

If we examine the general equation,

$$\frac{d^2}{dt^2}\left(\frac{dV}{ds}+L_1 s+L_0+\frac{2s}{c^2}M_0-\frac{s^2}{c}M_1\right)=cg\frac{d^3V}{ds^3}\,.\quad .\quad .\quad .\quad .\quad .\quad .\quad (1),$$

and assume for $\frac{dV}{ds}$ the following form,

$$\Sigma\,.\,A\sin\sqrt{cg}\,(pt+\beta)\sin(ps+\alpha)+Ps^2+Rs+Q,$$

A being a constant and P, R and Q functions of t, we shall find for P, R and Q {substituting in (11) and equating} $P-\frac{M_1}{c}=0$,

$$R+L_1+\frac{2M_0}{c^2}=0,$$

$$\frac{d^2}{dt^2}(Q+L_0)=cg\times 2P=2gM_1,$$

whence P, Q and R are given in terms of L_0, L_1 M_0 and M_1.

We shall now consider the motion of the two portions of the chain reckoning from the right and the left of the origin separately, dashing the symbols which have reference to the left hand, or negative part of the chain. Hence

$$u_a=0=-\frac{a}{c}\frac{d^2V}{ds^2_a}+\frac{1}{c}\frac{dV}{ds_a}+M_1$$

$$u_a'=0=-\frac{a}{c}\frac{d^2V'}{ds^2_a{}'}+\frac{1}{c}\frac{dV}{ds_a}+M_1'$$

$$v_a=0=L_1+\frac{d^2V}{ds^2_a}$$

$$v_a'=0=L_1'+\frac{d^2V'}{ds^2_a{}'}\,;$$

$$\text{Also }u_0=-u_0'\ v_0=v_0',$$

$$\frac{d^3V}{ds^3_0}=-\frac{d^3V'}{ds^3_0}$$

$$\frac{2}{c}(V_0-M_0)=\frac{2}{c}(V_0'-M_0'),$$

which eight conditions are sufficient to determine L_0, L_1, M_0, M_1 and L_0', L_1', M_0', and M_1', whether the chain be symmetrical always about the vertical axis or not. Or we may consider the whole length of the chain between its two ends, if $a'=-a$, we shall then have

$$\left.\begin{aligned}u_a=u_{-a}=0\\ v_a=v_{-a}=0\end{aligned}\right\},$$

and employing only one set of equations, we shall thus by these four conditions, be enabled to determine L_1, M_0, L_0 and M_1, but I think the former would be found in practice the shorter method. When the disturbance arises from a travelling load, we should assume for

$$\frac{dV}{ds} \text{ a series } \Sigma\left(A_n \sin\frac{2n+1}{2}\frac{\pi s}{a}\right) + \Sigma C_n \cos\frac{\pi n s}{a},$$

$$\text{for } M_1\frac{s^2}{c} \text{ a series } \frac{M_1}{c}\left\{\frac{a^2}{3} + \frac{4a^2}{\pi^2}\left(-\cos\frac{\pi s}{a} + \frac{1}{4}\cos\frac{2\pi s}{a} - \&c.\right)\right\} 1,$$

and $f(s)$ would, as heretofore, be expanded in a series Σ. sines $\frac{2n+1}{2}\frac{\pi s}{a}$, and we should have to examine the two branches of the chain at the same time, the calculation would be lengthy, and the results of no more interest than those we have already obtained. Without going any way into the calculation; we can however see, that the types of vibration of the ΣA_n series would be but slightly modified, for since the introduction of $\frac{d^2}{dt^2} L_1 s$ scarcely changes the types of A_2, A_3, &c., when A_2, A_3, &c. are the leading terms from their values

$$\frac{3\pi}{2a}\sqrt{cg}\ \frac{5\pi}{2a}\sqrt{cg}, \text{ when } \frac{d^2}{dt^2}L_1 = 0,$$

and for all practical purposes produces no effect on the higher types of A_3, &c.; a slight variation in L_1, occasioned by the disturbance being not symmetrical, would make but little difference. Consequently we may state roughly, that whether the oscillations be symmetrical or unsymmetrical about the vertical axis, the time of a vibration arising from the causes we have considered, and which are expressible convergently in a $\Sigma A_n \sin\frac{2n+1}{2}\frac{\pi s}{a}$ series, will be nearly $= \frac{4a}{(2n+1)\sqrt{cg}}$ where n is a positive integer.

J. H. ROHRS.

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11. Approximation to second order.
12. Recapitulation of results obtained.

Addenda.

XVI. *On the Composition and Resolution of Streams of Polarized Light from different Sources. By* G. G. STOKES, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *Feb.* 16 and *March* 15, 1852.]

WHEN a stream of polarized light is decomposed into two streams which, after having been modified in a slightly different manner, are reunited, the mixture is found to have acquired properties which are quite distinct from those of the original stream, and give rise to a number of curious and apparently complicated phenomena. These phenomena have now, however, through the labours of Young and Fresnel, been completely reduced to law, and embraced in a theory, the wonderful simplicity of which is such as to bear with it the stamp of truth. But when two polarized streams from different sources mix together, the mixture possesses properties intermediate between those of the original streams, and none of the curious phenomena depending upon the interference of polarized light are manifested. The properties of such mixtures form but an uninviting subject of investigation; and accordingly, though to a certain extent they are obvious, and must have forced themselves upon the attention of all who have paid any special attention to the physical theory of light, they do not seem hitherto to have been studied in detail.

Were the only object of such a study to enable us to calculate with greater facility the results obtained by means of certain complicated combinations, the subject might deservedly be deemed of small importance. For the object of the philosopher is not to complicate, but to simplify and analyze, so as to reduce phenomena to laws, which in their turn may be made the stepping-stones for ascending to a general theory which shall embrace them all; and when such a theory has been arrived at, and thoroughly verified, the task of deducing from it the results which ought to be observed under a combination of circumstances which has nothing to recommend it for consideration but its complexity, may well be abandoned for new and more fertile fields of research. But in the present case certain difficulties seem to have arisen respecting the connexion between common and elliptically polarized light which it needed only a more detailed study of the laws of combination of polarized light to overcome; and accordingly the subject may be deemed not wholly devoid of importance.

The early part of the following paper is devoted to a demonstration of various properties of elliptically polarized light, and of oppositely polarized streams. When two streams of light are called oppositely polarized, it is meant that, so far as relates to its state of polarization, one stream is what the other becomes when it is turned in azimuth through 90°, and has its nature reversed as regards right-handed and left-handed. Most, if not all, of these properties have doubtless already occurred to persons studying the subject, but I am not aware of any formal demonstrations of them which have been published; and indeed some artifices were required in order to avoid being encumbered in the demonstrations with long analytical expressions. The combination of several independent polarized streams is next considered, and with respect to this subject a proposition is proved which may be regarded as the capital theorem of the paper. It is as follows.

When any number of independent polarized streams, of given refrangibility, are mixed together, the nature of the mixture is completely determined by the values of four constants, which are certain functions of the intensities of the streams, and of the azimuths and eccentricities of the ellipses by which they are respectively characterized; so that any two groups of polarized streams which furnish the same values for each of these four constants are optically equivalent.

It is a simple consequence of this theorem, that any group of polarized streams is equivalent to a stream of common light combined with a stream of elliptically polarized light from a different source. The intensities of these two streams, as well as the azimuth and eccentricity of the ellipse which characterizes the latter, are determined by certain formulæ, which will be found in their place.

The general principles established in this paper bear on two questions of physical interest. Strong reasons are adduced in favour of the universality of the law, that the two polarized pencils which a doubly refracting medium of any nature is capable of propagating independently in a given direction are polarized oppositely. In strictness, we ought to speak of two series of waves rather than two pencils; for it is the fronts of the waves, not the rays, which are supposed to have a common direction. The other point alluded to relates to the distinction between common, and elliptically polarized light. It is shewn that the changes which are continually taking place in the mode of vibration may be of any nature, and that there is no occasion, in the case of common light, to suppose the transition from a series of vibrations of one kind to a series of another kind to be abrupt.

At the end of the paper the general formulæ are applied to the case of some actual experiments, but these applications are not of sufficient importance to deserve separate mention.

1. Consider a stream of light polarized in the most general way, that is, elliptically polarized, and propagated through the free ether. Let the medium be referred to the rectangular axes of x, y, z, the axis of z being measured in the direction of propagation. Let α and $\alpha + 90°$ be the azimuths of the principal planes, that is, the planes of maximum and minimum polarization, azimuths being measured about the axis of z from x towards y. Let the rectangular components of the displacements of the ether be represented by lines drawn in the planes of polarization of the plane-polarized streams which these components, taken separately, would constitute. I make this assumption to avoid entering into the question whether the vibrations of plane-polarized light are parallel or perpendicular to the plane of polarization. If we adopt the former theory, the actual lines in the figures which we are to suppose drawn will represent in magnitude and direction the ethereal displacements; if we adopt the latter, the same will still be the case if we first suppose all our figures turned round the axis of z, in a given direction, through 90°.

Let the co-ordinates x', y' be measured in the principal planes whose azimuths are α, $\alpha + 90°$; let β be the angle whose tangent is equal to the ratio of the axis of the ellipse described, the numerical value of β being supposed not to lie beyond the limits 0 and 90°; let v be the velocity of propagation, t the time, λ the length of a wave, and put for shortness,

$$\frac{2\pi}{\lambda}(vt - x) = \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Then about the time t, and at no very great distance from a given point, suppose the origin, we may represent the diplacements belonging to the given stream of elliptically polarized light by

$$x' = c\cos\beta\sin(\phi + \epsilon), \quad y' = c\sin\beta\cos(\phi + \epsilon). \quad . \quad . \quad . \quad (2).$$

If the light be convergent or divergent, c will depend upon ϕ, but for our present object any variation of c arising from this cause will not enter into account. The value of a, which determines the direction in which x' is measured, as well as that of β, is given by the nature of the polarization. The polarization is right-handed or left-handed according to the sign of β. As to c and ϵ, the phenomena of optics oblige us to suppose that they are constant, or sensibly constant, for a great number of consecutive undulations, but that they change in an irregular manner a great number of times in the course of one second. The known rapidity of the luminous vibrations allows abundant scope for such a supposition, since c and ϵ may be constant for millions of consecutive undulations, and yet change millions of times in a second. This series of changes, rapid with respect to the duration of impressions on the retina, but slow compared with the periodic changes in the motion of the ethereal particles, is exactly what we might have expected beforehand from a consideration of the circumstances under which light is produced, so far at least as its sources are accessible to us; and thus in this point, as in so many others, the theory of undulations commends itself for its simplicity.

If c were constant c^2 would be a measure of the intensity, so long as we were only comparing different streams having the same refrangibility. But since c is liable to the changes just mentioned, if we wish to express ourselves exactly, avoiding conventional abbreviations, we must say that the intensity is measured, not by c^2, but by the mean value of c^2, which may conveniently be represented by $\mathfrak{M}(c^2)$.

2. Let us examine now whether it be always possible to resolve the given disturbance into two which, taken separately, would correspond to two elliptically polarized streams of given nature. For the sake of clear ideas, it may be supposed that the azimuths and eccentricities of the ellipses belonging to these two streams are given and invariable, while the azimuth and eccentricity of the ellipse belonging to the first stream are given for that stream, but vary from one to another of a set of streams which we wish to consider in succession.

Let x_1, c_1, &c. be for the first, and x_2, c_2, &c. be for the second stream of the pair, what x', c, &c. were for the original stream; and resolve all the displacements along the principal axes of the latter stream. Then, in order that the original disturbance may be equivalent to the pair, we must have, independently of ϕ,

$$\left.\begin{aligned} x_1\cos(a_1 - a) - y_1\sin(a_1 - a) + x_2\cos(a_2 - a) - y_2\sin(a_2 - a) &= x'; \\ x_1\sin(a_1 - a) + y_1\cos(a_1 - a) + x_2\sin(a_2 - a) + y_2\cos(a_2 - a) &= y'. \end{aligned}\right\} \quad (3)$$

Conceive x_1, y_1, x_2, y_2, x', and y' expressed in terms of ϕ by the formulæ (2) and the similar formulæ whereby x_1, y_1, &c. are expressed, and then let the sines and cosines of $\phi+\epsilon$, $\phi+\epsilon_1$, and $\phi+\epsilon_2$ be developed. In order that equations (3) may be satisfied independently of ϕ, the coefficients of $\sin\phi$ and $\cos\phi$ must separately be equal to zero, so that each of these equations will split into two. We shall thus have four equations to determine the four unknown quantities c_1, c_2, ϵ_1, and ϵ_2. For the sake of shortness, let

$$c\cos\epsilon = g, \qquad c\sin\epsilon = h,$$

whether the letters be or be not affected with suffixes; and further put

$$a_1 - a = \gamma_1, \qquad a_2 - a = \gamma_2;$$

then our four equations become

$$\left.\begin{aligned}
\cos\beta_1\cos\gamma_1 . g_1 + \sin\beta_1\sin\gamma_1 . h_1 + \cos\beta_2\cos\gamma_2 . g_2 + \sin\beta_2\sin\gamma_2 . h_2 &= \cos\beta . g,\\
\cos\beta_1\cos\gamma_1 . h_1 - \sin\beta_1\sin\gamma_1 . g_1 + \cos\beta_2\cos\gamma_2 . h_2 - \sin\beta_2\sin\gamma_2 . g_2 &= \cos\beta . h,\\
\cos\beta_1\sin\gamma_1 . g_1 - \sin\beta_1\cos\gamma_1 . h_1 + \cos\beta_2\sin\gamma_2 . g_2 - \sin\beta_2\cos\gamma_2 . h_2 &= -\sin\beta . h,\\
\cos\beta_1\sin\gamma_1 . h_1 + \sin\beta_1\cos\gamma_1 . g_1 + \cos\beta_2\sin\gamma_2 . h_2 + \sin\beta_2\cos\gamma_2 . g_2 &= \sin\beta . g.
\end{aligned}\right\} \quad (4)$$

Multiplying the first and second of these equations by 1, $\sqrt{-1}$, and adding, then multiplying the third and fourth by $-\sqrt{-1}$, 1, and adding, and putting generally

$$g + \sqrt{-1}\,h = G, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

we have

$$\left.\begin{aligned}
(\cos\beta_1\cos\gamma_1 - \sqrt{-1}\sin\beta_1\sin\gamma_1)\,G_1 + (\cos\beta_2\cos\gamma_2 - \sqrt{-1}\sin\beta_2\sin\gamma_2)\,G_2 &= \cos\beta . G,\\
(\sin\beta_1\cos\gamma_1 - \sqrt{-1}\cos\beta_1\sin\gamma_1)\,G_1 + (\sin\beta_2\cos\gamma_2 - \sqrt{-1}\cos\beta_2\sin\gamma_2)\,G_2 &= \sin\beta . G;
\end{aligned}\right\} \quad (6)$$

which two equations are equivalent to the four (4).

Putting for shortness p_1, p_2, q_1, q_2 for the coefficients in the left-hand members of equations (6), we have

$$\frac{G_1}{q_2\cos\beta - p_2\sin\beta} = \frac{G_2}{p_1\sin\beta - q_1\cos\beta} = \frac{G}{p_1 q_2 - p_2 q_1}. \quad (7)$$

On substituting for p_1, q_2, &c. their values, we find

$$p_1 q_2 - p_2 q_1 = \cos(\gamma_1 - \gamma_2)\sin(\beta_2 - \beta_1) + \sqrt{-1}\sin(\gamma_1 - \gamma_2)\cos(\beta_1 + \beta_2).$$

Now the equations (6) cannot be incompatible or identical unless the above quantity vanish. But this can only take place when

$$\sin(\beta_2 - \beta_1) = 0 \text{ and } \sin(\gamma_1 - \gamma_2) = 0,$$

or else

$$\cos(\beta_1 + \beta_2) = 0 \text{ and } \cos(\gamma_1 - \gamma_2) = 0,$$

or lastly

$$\sin(\beta_2 - \beta_1) = 0 \text{ and } \cos(\beta_1 + \beta_2) = 0.$$

The first case gives $\beta_2 = \beta_1$, $\gamma_1 - \gamma_2 = a_1 - a_2 = 0$, or $\pm 180^\circ$, so that the two streams into which it was proposed to resolve the first are of the same nature. The second case gives $\beta_2 = 90^\circ - \beta_1$, $\gamma_1 - \gamma_2 = a_1 - a_2 = \pm 90^\circ$, which shews that the two streams are identical in their nature, only the first and second principal planes of the first of these streams are accounted respectively the second and first of the second stream. The third case gives $\beta_2 = \beta_1 = \pm 45^\circ$, so that the two streams are circularly polarized and of the same kind, which is a particular instance of the first case.

Hence, universally, a stream of elliptically polarized light may be resolved into two streams of elliptically polarized light in which the polarizations are of any kind that we please, but different from one another.

Substituting for q_2, &c. their values in (7), and replacing γ_1, γ_2 by $\alpha_1 - \alpha$, $\alpha_2 - \alpha$, for which they had been temporarily written, we find

$$\left.\begin{aligned} &\{\sin(\beta_2 - \beta)\cos(\alpha_2 - \alpha) - \sqrt{-1}\cos(\beta_2 + \beta)\sin(\alpha_2 - \alpha)\}^{-1} G_1 \\ =&\{\sin(\beta - \beta_1)\cos(\alpha_1 - \alpha) - \sqrt{-1}\cos(\beta + \beta_1)\sin(\alpha_1 - \alpha)\}^{-1} G_2 \\ =&\{\sin(\beta_2 - \beta_1)\cos(\alpha_2 - \alpha_1) - \sqrt{-1}\cos(\beta_2 + \beta_1)\sin(\alpha_2 - \alpha_1)\}^{-1} G \end{aligned}\right\} . \quad (8)$$

3. Among these various modes of resolution there is one which possesses several peculiar properties, any one of which might serve to define it. Let us in the first place examine under what circumstances the intensity of the stream made up of the two components is independent of any retardation which the phase of vibration of one component may have undergone relatively to the phase of vibration of the other previously to the recomposition.

For this purpose there is evidently no occasion to consider the manner in which a_1, b_1, ϵ_1, a_2, b_2, ϵ_2 are made up of a, b, ϵ, but we may start with the components. Let ρ_1, ρ_2 be the retardations of phase which take place before recomposition, and resolve the disturbances along the axes of x, y. We shall have for the resolved parts

$$x = S\{a_1 \cos\alpha_1 \sin(\phi + \epsilon_1 - \rho_1) - b_1 \sin\alpha_1 \cos(\phi + \epsilon_1 - \rho_1)\}$$
$$y = S\{a_1 \sin\alpha_1 \sin(\phi + \epsilon_1 - \rho_1) + b_1 \cos\alpha_1 \cos(\phi + \epsilon_1 - \rho_1)\},$$

where S denotes the sum of the expression written down and that formed from it by replacing the suffix 1 by 2. To form the expression for the intensity, or rather what would be the intensity if the quantities c and ϵ were absolutely constant, not merely, constant for a great number of successive undulations, we must develope the expressions for x and y so as to contain the sine and cosine of $\phi + \kappa$, and take the sum of the squares of the coefficients. κ is here a constant quantity which may be chosen at pleasure, and which it will be convenient to take equal to $\epsilon_1 - \rho_1$. If I be the intensity, in the sense above explained, or as it may be called the *temporary intensity*, we find, putting δ for $\epsilon_2 - \rho_2 - \epsilon_1 + \rho_1$,

$$\begin{aligned} I =& \{a_1\cos\alpha_1 + a_2\cos\alpha_2\cos\delta + b_2\sin\alpha_2\sin\delta\}^2 + \{-b_1\sin\alpha_1 + a_2\cos\alpha_2\sin\delta - b_2\sin\alpha_2\cos\delta\}^2 \\ &+ \{a_1\sin\alpha_1 + a_2\sin\alpha_2\cos\delta - b_2\cos\alpha_2\sin\delta\}^2 + \{b_1\cos\alpha_1 + a_2\sin\alpha_2\sin\delta + b_2\cos\alpha_2\cos\delta\}^2 \\ =& a_1^2 + b_1^2 + a_2^2 + b_2^2 + 2(a_1a_2 + b_1b_2)\cos(\alpha_2 - \alpha_1)\cos\delta + 2(a_1b_2 + a_2b_1)\sin(\alpha_2 - \alpha_1)\sin\delta. \end{aligned}$$

On putting for a, b their values $c\cos\beta$, $c\sin\beta$, this expression becomes

$$I = c_1^2 + c_2^2 + 2c_1c_2\{\cos(\alpha_1 - \alpha_2)\cos(\beta_2 - \beta_1)\cos\delta + \sin(\alpha_2 - \alpha_1)\sin(\beta_2 + \beta_1)\sin\delta\} \quad . \quad (9).$$

In order that I may be independent of the difference of phase $\rho_2 - \rho_1$, and therefore of δ, we must have either

$$\cos(\alpha_2 - \alpha_1) = 0, \qquad \sin(\beta_2 + \beta_1) = 0, \quad . \quad . \quad . \quad . \quad (10),$$
$$\text{or } \sin(\alpha_2 - \alpha_1) = 0, \qquad \cos(\beta_2 - \beta_1) = 0, \quad . \quad . \quad . \quad . \quad (11).$$

The equations (10) give $\alpha_2 - \alpha_1 = \pm 90°$, $\beta_2 = -\beta_1$, so that the ellipses described in the case of the two streams are similar, their major axes perpendicular to each other, and the direction of revolution in the one stream contrary to that in the other. It will be easily seen that the equations (11) differ from (10) only in this, that what are regarded as the first and second principal planes of the second stream when equations (10) are satisfied, are accounted respectively the second and first when (11) are satisfied.

Two streams thus related may be said to be *oppositely polarized.* Two streams of plane-polarized light in which the planes of polarization are at right angles to each other, and two streams of circularly polarized light, one right-handed and the other left-handed, are particular cases of streams oppositely polarized.

In the reasoning of this article, nothing depends upon the precise relation between the two polarized streams and the original stream. All that it is necessary to suppose is, that the two polarized streams came originally from the same polarized source, so that the changes in epoch and intensity, that is, the changes in the quantities ϵ, c, are the same for the two streams. Nothing depends upon the precise nature of these changes, which may be either abrupt or continuous, but must be sufficiently infrequent if abrupt, or sufficiently gradual if continuous, to allow of our regarding c and ϵ as constant for a great number of successive undulations. Our results will apply just as well to the disturbance produced by the union of two neighbouring streams coming originally from the same polarized source, but having had their polarizations modified, as to that produced by the union, after recomposition, of the components of a single polarized stream. Since the resulting intensity is independent of δ, it follows that two oppositely polarized streams coming originally from the same polarized source are incapable of interfering, but two streams polarized otherwise than oppositely necessarily interfere, to a greater or less degree, when the difference in their retardation of phase is sufficiently small. Of course the interference here spoken of means only that which is exhibited without analyzation.

4. Two interfering streams may be said to interfere perfectly when the fluctuations of intensity are the greatest that the difference in the intensities of the interfering streams admits of, so that in case of equality the minima are absolutely equal to zero. Referring to (9), we see that in order that this may be the case the maximum value of the coefficient of $2c_1c_2$ must be equal to 1. Now the maximum value of $A\cos\delta + \beta\sin\delta$ is $\sqrt{(A^2 + B^2)}$, and therefore we must have

$$\cos^2(\alpha_2 - \alpha_1)\cos^2(\beta_2 - \beta_1) + \sin^2(\alpha_2 - \alpha_1)\sin^2(\beta_2 + \beta_1) = 1 = \cos^2(\alpha_2 - \alpha_1) + \sin^2(\alpha_2 - \alpha_1),$$

whence,

$$\cos^2(\alpha_2 - \alpha_1)\sin^2(\beta_2 - \beta_1) + \sin^2(\alpha_2 - \alpha_1)\cos^2(\beta_2 + \beta_1) = 0,$$

which lead to the very same conditions that have been already discussed in Art. 2. Hence two polarized streams coming from the same polarized source are capable of interfering perfectly if the polarizations are the same, not at all if the polarizations are opposite, and in intermediate cases of course in intermediate degrees.

5. When a stream of polarized light is resolved into two oppositely polarized streams, which are again compounded after their phases have been differently altered, we have from (9), taking account of (10) or (11),

$$I = c_1^2 + c_2^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

so that the intensity of the resultant is equal to the sum of the intensities of the components, and is therefore constant, that is, independent of $\rho_1 - \rho_2$, and is accordingly equal to what it was at first, when ρ_1 and ρ_2 were each equal to zero, that is, equal to the intensity of the original stream.

It may be readily proved from the formulæ (8) that it is only in the case in which the polarizations of the two components of the original polarized stream are opposite that the intensity of the original stream, whatever be the nature of its polarization, is equal to the sum of the intensities of the component streams. For, changing the sign of $\sqrt{-1}$ in these formulæ, multiplying the resulting equations, member for member, by the equations (8), and observing that if G' be what G becomes, $GG' = g^2 + h^2 = c^2$, we find

$$\left.\begin{aligned}&\{\sin^2(\beta_2 - \beta)\cos^2(\alpha_2 - \alpha) + \cos^2(\beta_2 + \beta)\sin^2(\alpha_2 - \alpha)\}^{-1}c_1^2\\ =\ &\{\sin^2(\beta - \beta_1)\cos^2(\alpha_1 - \alpha) + \cos^2(\beta + \beta_1)\sin^2(\alpha_1 - \alpha)\}^{-1}c_2^2\\ =\ &\{\sin^2(\beta_2 - \beta_1)\cos^2(\alpha_2 - \alpha_1) + \cos^2(\beta_2 + \beta_1)\sin^2(\alpha_2 - \alpha_1)\}^{-1}c^2.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

In order that $\mathfrak{m}(c^2)$ may be equal to $\mathfrak{m}(c_1^2) + \mathfrak{m}(c_2^2)$, it is necessary that c^2 be equal to $c_1^2 + c_2^2$, because, whatever fluctuations c_1 and c_2 may undergo in a moderate time, such as the tenth part of a second, c_1 and c_2 are always proportional to c. Hence the sum of the quantities whose reciprocals are the coefficients of c_1^2 and c_2^2 must be equal to that whose reciprocal is the coefficient of c^2. Since this has to be true independently of β, let the quantities $\sin^2(\beta_2 - \beta)$, &c. be replaced by sines and cosines of multiple arcs, and let our equation be put under the form

$$A + B\cos 2\beta + C\sin 2\beta = 0.$$

Then A, B, C must be separately equal to zero, or

$$\left.\begin{aligned}\sin^2(\beta_2 - \beta_1)\cos^2(\alpha_2 - \alpha_1) + \cos^2(\beta_2 + \beta_1)\sin^2(\alpha_2 - \alpha_1) = 1\,;\\ \cos 2\beta_2\cos 2(\alpha_2 - \alpha) + \cos 2\beta_1\cos 2(\alpha_1 - \alpha) = 0\,;\\ \sin 2\beta_2 + \sin 2\beta_1 = 0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

Replacing unity in the right-hand member of the first of these equations by

$$\cos^2(\alpha_2 - \alpha_1) + \sin^2(\alpha_2 - \alpha_1),$$

we find

$$\cos^2(\beta_2 - \beta_1)\cos^2(\alpha_2 - \alpha_1) + \sin^2(\beta_2 + \beta_1)\sin^2(\alpha_2 - \alpha_1) = 0\,;$$

whence $\beta_2 = -\beta_1$, $\alpha_2 = \alpha_1 + 90^\circ$, or else β_2 and β_1 differ by 90°, and $\alpha_2 = \alpha_1$, except in the particular case in which $\beta_1 = \pm 45^\circ$, when $\beta_2 = \mp 45^\circ$ satisfies the equation independently of α_2. Hence the streams must be polarized oppositely, a condition which may always be expressed by

$$\beta_2 = -\beta_1,\ \alpha_2 = \alpha_1 + 90^\circ,$$

which equations satisfy the second and third of equations (14) independently of α, as it might have been foreseen that they would, since it has been already shewn that the condition (12) is satisfied in the case of oppositely polarized streams. It now appears that it is only in the case of such streams that this is satisfied.

6. The properties of oppositely polarized pencils which have been proved, render it in a high degree probable that it is a general law that in a doubly refracting medium the two polarized pencils transmitted in a given direction are oppositely polarized. Were this not

the case, the two pencils, polarized otherwise than oppositely, into which a polarized pencil is resolved on entering into the medium, would at emergence compound a pencil of which the intensity would depend upon the retardations of phase of one pencil relatively to the other, so that such a medium, when examined with polarized light, ought to exhibit rings or colours without the employment of an analyzer. It is here supposed that the light enters the medium at the first surface, and leaves it at the second, in a direction normal to the surface, or very nearly so, and likewise that the refracting power of the medium for the two pencils is not very different, so that the effect of reflexion at the two surfaces may be disregarded, the only sensible effect being to diminish the intensity of each of the two pencils in the same proportion, without affecting its state of polarization. These views would lead us to scrutinize very carefully any experimental evidence brought forward which would lead to the conclusion, that the two polarized pencils which a doubly refracting medium was capable of propagating in a given direction were polarized otherwise than oppositely.

7. In ordinary doubly refracting crystals, whether uniaxal or biaxal, and in doubly refracting liquids, such as syrop of sugar, it is generally admitted that the two pencils transmitted in a given direction are oppositely polarized. In the former case the two pencils are polarized in rectangular planes, in the latter they are circularly polarized, one being right-handed and the other left-handed. The same is the case with quartz for pencils transmitted in the direction of the axis; but in following out the researches in which he so successfully connected the phenomena, identical with those of a doubly refracting liquid, which quartz exhibits in the direction of the axis, with the phenomena which it exhibits as a uniaxal crystal, Mr Airy met with an experimental result which seemed to shew that while the ellipses which characterize the two streams transmitted in a given direction, oblique to the axis, have their major axes situated, one in a principal plane, and the other perpendicular to the principal plane, and the directions of revolution are opposite, the eccentricities of the ellipses, though nearly, are not quite equal*. This conclusion depended upon the result of certain experiments made by means of a Fresnel's rhomb. The nature of the experiments seemed to eliminate the effect of an error in the rhomb, that is, a deviation from 90° in the retardation of phase which it produced in a pencil polarized in the plane of reflexion relatively to a pencil polarized in a plane perpendicular to the former. The effect of a possible index error in the plane of reflexion seemed also to be eliminated; and the quantities on which the results depended, though small, seemed to be beyond mere errors of pointing. Being impressed however with a strong conviction that the result depended in some way on the mode of observation, I was led to scrutinize the different steps of the process, and it occurred to me that the apparent inequality of eccentricities was probably due to defects of annealing in the rhomb. It is next to impossible to procure a piece of glass of such a size free from defects of that nature, for which reason I believe that even a good Fresnel's rhomb is not to be trusted for minute quantities, except in the case of merely differential observations.

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* *Cambridge Philosophical Transactions*, Vol. IV. p. 204.

The same views lead to the conclusion that the two pencils transmitted through magnetized glass in a direction oblique to the lines of magnetic force are oppositely polarized. Some theoretical investigations in which I was engaged some time ago led me to the result that these polarized streams are circularly polarized, as well as those transmitted along the line of magnetic force, and that the difference between the wave-velocities varies as the cosine of the inclination of the wave-normal to the line of magnetic force. I hope at some future time to bring these researches before the notice of this Society.

8. After these preliminary investigations respecting the nature of opposite polarization, which indeed contain, it is probable, but little that has not already occurred to persons who have studied the subject, it is time to come to the more immediate object of this paper, which relates to the combination of independent streams. But first it will be convenient to state explicitly a principle which is generally recognized.

When any number of polarized streams from different sources mix together, after having been variously modified by reflexion, refraction, transmission through doubly refracting media, tourmalines, &c., the intensity of the mixture is equal to the sum of the intensities due to the separate streams.

The reason of this law may be easily seen. The components whereby the disturbance due to any one stream is originally expressed have to be resolved, their components resolved again, and so on; and of these partial disturbances the phases of vibration have to be altered by quantities independent of the time, and the coefficients in some cases diminished in given ratios, and in some cases suppressed altogether. Each stream has to be treated in a similar way. The final disturbance being resolved in any two rectangular directions, each component must be put under the form $U \cos \phi + V \sin \phi$, and the sum of the squares of U and V must be taken to form the expression for the temporary intensity. All the quantities such as U and V will evidently be linear functions of $c \cos \epsilon$, $c \sin \epsilon$, $c' \cos \epsilon'$, $c' \sin \epsilon'$, &c., where c, $c' \ldots$ and ϵ, $\epsilon' \ldots$ refer to the different streams, so that U for instance will be of the form

$$Ac \cos \epsilon + Bc \sin \epsilon + A'c' \cos \epsilon' + B'c' \sin \epsilon' + \ldots$$

where A, B, A', $B' \ldots$ are independent of the time. The temporary intensity will involve U^2, but the actual intensity will involve $\mathfrak{m}(U^2)$, or

$$\mathfrak{m}\Sigma\,(Ac \cos \epsilon + Bc \sin \epsilon)^2 + 2\,\mathfrak{m}\Sigma\,\{(Ac \cos \epsilon + Bc \sin \epsilon)\,(A'c' \cos \epsilon' + B'c' \sin \epsilon')\}.$$

Now the products such as $\cos \epsilon \cos \epsilon'$, $\cos \epsilon \sin \epsilon'$, &c. will have a mean value zero, since the changes in ϵ and those in ϵ' have no relation to each other, and therefore the expression for $\mathfrak{m}\,(U^2)$ becomes

$$\mathfrak{m}\Sigma(Ac \cos \epsilon + Bc \sin \epsilon)^2, \text{ or } \Sigma\mathfrak{m}\,(Ac \cos \epsilon + Bc \sin \epsilon)^2,$$

that is, the sum of the quantities by which it would be expressed were the different streams taken separately.

Two streams which come from different sources, or which, though in strictness they come from the same source, are such that the changes of epoch and intensity in the one have no relation to the changes of epoch and intensity in the other, may be called *independent*.

9. Suppose that there are any number of independent polarized streams mixing together; let the mixture be resolved in any manner into two oppositely polarized streams, and let us examine the intensity of each.

Let us take one stream first. The intensities of its components are given by the formulæ (13), which become somewhat simpler in the case of opposite polarizations, since $\beta_2 = -\beta_1$, and $\alpha_2 = 90_0 + \alpha_1$. Hence

$$c_1^2 = \left\{\sin^2(\beta_1 + \beta)\sin^2(\alpha_1 - \alpha) + \cos^2(\beta_1 - \beta)\cos^2(\alpha_1 - \alpha)\right\}c^2;$$

whence we find

$$\mathfrak{m}(c_1^2) = \tfrac{1}{2}\left\{1 + \sin 2\beta_1 \sin 2\beta + \cos 2\alpha_1 \cos 2\alpha \cos 2\beta_1 \cos 2\beta + \sin 2\alpha_1 \sin 2\alpha \cos 2\beta_1 \cos 2\beta\right\}\mathfrak{m}(c^2). \quad (15)$$

There will be no occasion to write down the value of $\mathfrak{m}(c_2^2)$, since c_1 may be taken to refer to either component.

Let us pass now to the consideration of a group consisting of any number of independent polarized streams, let

$$\left.\begin{array}{lll} \Sigma\mathfrak{m}(c^2) = A, & \Sigma \sin 2\beta\mathfrak{m}(c^2) = B, & \Sigma \cos 2\alpha \cos 2\beta\mathfrak{m}(c^2) = C, \\ \Sigma \sin 2\alpha \cos 2\beta\mathfrak{m}(c^2) = D, & & \end{array}\right\} \quad (16)$$

and let c_1 now refer to one of the components of the whole group; then

$$2\mathfrak{m}(c_1^2) = A + B\sin 2\beta_1 + C\cos 2\alpha_1 \cos 2\beta_1 + D\sin 2\alpha_1 \cos 2\beta_1. \quad (17)$$

It follows that if there are two groups of independent polarized streams which are such as to give the same values to each of the four quantities A, B, C, D defined by (16), if the groups be resolved in any manner whatsoever, which is the same for both, into two oppositely polarized streams, the intensities of the components of the one group will be respectively equal to the intensities of the components of the other group. Conversely, if two groups of oppositely polarized streams are such that when they are resolved in any manner, the same for both, into two oppositely polarized streams, the intensities of the components of the one group are respectively equal to the intensities of the components of the other group, the quantities A, B, C, D must be the same for the two groups. For, if we take accented letters to refer to the second group, the second member of equation (17), and the expression thence derived by accenting A, B, C, D, must be equal independently of α_1 and β_1, which requires that A', B', C', D' be respectively equal to A, B, C, D.

DEFINITION. Two such groups will be said to be *equivalent*.

10. The theoretical definition of equivalence which has just been given agrees completely with the experimental tests of equivalence. One of the most ready as well as delicate modes of detecting minute traces of polarization, and at the same time determining qualitatively the nature of the polarization, consists in viewing the light to be examined through a plate of calcareous spar, or other crystal, cut for shewing rings, followed by a Nicol's prism. The plane-polarized pencils respectively stopped by and transmitted through the Nicol's prism consisted, on entering the crystal of pencils elliptically polarized in opposite ways; and the nature of this elliptic polarization changes in every possible manner from one point to another of the field of view. If these two streams of light be equivalent according to the definition

given in the preceding article, they will present exactly the same appearance on being viewed through a crystal followed by a Nicol's prism or other analyzer.

11. THEOREM. Let a polarized stream be resolved into two oppositely polarized streams; let the phase of vibration of one of the streams be altered by a given quantity relatively to that of the other, and let the streams be then compounded. If the polarization of the original stream be now changed to its opposite, the polarization of the final stream will also be changed to its opposite.

The straightforward mode of demonstrating this theorem, by making use of the general expressions, would lead to laborious analytical processes, which are wholly unnecessary. For the formulæ which determine the components of a given stream are expressed by simple equations, so that the results are unique, and accordingly whenever we can foresee what the result will be, it is sufficient to shew that the formulæ themselves, or the geometrical conditions of which the formulæ are merely the expressions, are satisfied.

For shortness' sake call the original stream X, and its components O, E. Let ρ be the given quantity, positive or negative, by which the phase of vibration of O is retarded relatively to that of E. Let o, e denote the streams O, E after the changes of phase, and Y the stream resulting from their reunion. Conceive now all the vibrations with which we are concerned to be turned in azimuth through 90°. This will not affect the geometrical relations connecting components and resultants. Let X', O', E', o', e', Y' be the streams which X, O, E, o, e, Y thus become. The streams O', E' are evidently polarized in the same manner respectively as E, O, except that *right-handed* is changed into *left-handed*, and *vice versa*; and in passing from O' to o' the phase is retarded by ρ. Now conceive the direction of motion of a given particle reversed, the motions of all other particles being derived from that of the first according to the general law of wave propagation. The relations between components and resultants will evidently not be violated; and if X'', O'', E'', o'', e'', Y'' denote the streams into which X', O', E', o', e', Y' are thus changed, it is evident that the polarizations of X'', O'', E'', o'', e'', Y'' are respectively opposite to those of X, O, E, o, e, Y. But on account of the reversion in direction of motion it is plain that there is reversion as regards acceleration and retardation of phase, so that in passing from the pair O'', E'' to the pair o'', e'' the phase of O'' is *accelerated* by ρ relatively to the phase of E''.

Hence the stream X'', polarized oppositely to X, is resolved into two E'', O'', polarized in the same manner respectively as O, E, which are recompounded after the phase of the one polarized in the same manner as O has been retarded by ρ relatively to the phase of the other, and the result is Y'', a stream polarized in a manner opposite to Y, which proves the theorem.

12. This theorem may be applied to the case of light transmitted through a slice of a doubly refracting crystal, and shews that two streams going in oppositely polarized come out oppositely polarized. Also, since nothing in the demonstration depends upon the order in which the decompositions and recompositions take place, it is immaterial whether X, X'' denote a pair of oppositely polarized incident streams, which give rise to the emergent streams

Y, Y'', or X, X'' denote a pair of oppositely polarized emergent streams, which came from the incident streams Y, Y''. A particular case of this theorem was assumed in the preceding article, when it was stated that the pencils, polarized in perpendicular planes, which on coming out of a crystal of calcareous spar are respectively stopped by and transmitted through a Nicol's prism, went into the crystal oppositely polarized.

The theorem will evidently be true of a train consisting of any number of crystalline plates, each possessing the property of resolving the incident light into two oppositely polarized streams, which are propagated within the medium with different velocities. For two oppositely polarized streams incident on the first plate give rise to two emergent streams which fall oppositely polarized on the second plate, and so on. Since the number of plates may be supposed to increase and their thickness to decrease indefinitely, while at the same time the steps by which the doubly refracting nature of the plates alters from one to another become separately insensible, the theorem will be true if the whole train, or part of it, consist of a substance of which the doubly refracting nature alters continuously, as for example a piece of strained or unannealed glass. If, however, the train contain a member which performs a partial analysis of the light, as for example a plate of smoky quartz, or a plate of glass inclined to the incident light at a considerable angle, it will no longer be true that two pencils going in oppositely polarized will come out oppositely polarized.

13. THEOREM. If two equivalent groups of polarized streams be resolved in any manner, which is the same for both, into two oppositely polarized groups, and these be recombined after the phase of one of the components has been retarded by a given quantity relatively to that of the other, the two groups of resultant streams will be equivalent.

Let the groups of resultants be each resolved in any manner into two oppositely polarized streams, and call these O, E. By Art. 11, if O', E' be the streams which furnish O, E respectively, O', E' are oppositely polarized. Now by Art. 3, the intensities of O, E are the same respectively as those of O', E'; but these are the same for the one group as for the other, by the definition of equivalence. Therefore the intensities of O, E are the same for the one group as for the other; but O, E are any two oppositely polarized components of the resultant groups; therefore these groups are equivalent.

Hence, if two equivalent groups be transmitted through a crystalline plate, the emergent groups will be equivalent; and by the same reasoning as in Art. 12 the theorem may be extended to an optical train consisting of any number of crystalline plates, pieces of unannealed glass, &c.

14. THEOREM. If two equivalent groups be resolved in any manner, the same for both, into two polarized streams, the intensities of the components of the one group will be respectively equal to the intensities of the components of the other group.

The proof of this theorem is very easy. It is sufficient to treat the general expressions (13) exactly as in Art. 9, only that no relations are to be introduced between α_2, β_2, and α_1, β_1. Since the coefficient of c^2 in (13) is constant, if we consider as variable such quantities only as may change in passing from the one group to the other, it will be easily seen that the intensity of either component depends on the same four constants A, B, C, D as before; and since these have the same values for equivalent groups, it follows that in the most general mode of resolution the components of any two such groups have the same intensities respectively.

15. THEOREM. If two equivalent groups be each resolved in the same manner into two oppositely polarized streams, and these be recombined after their vibrations have been diminished in two different given ratios, the resultant groups will be equivalent.

Let each of the resultant groups be resolved into two oppositely polarized streams O, E, and let O', E' denote the streams which furnished O, E respectively. It is easily seen that the streams O', E' are both polarized, though not in general oppositely. Were there any occasion to determine the nature of the polarizations, it might easily be done by following a process the reverse of that by which the modified are deduced from the original groups. Thus, if it were required to determine the polarization of O', we should resolve O into streams oppositely polarized in the manner originally given, augment the vibrations in ratios the inverse of those in which they are actually diminished, and recombine the streams so obtained. For our present purpose, however, it is sufficient to observe that the polarizations of O', E' depend only on the mode of resolution, and not on the nature of the original group, and that therefore they are the same for the one group as for the other. The intensity of O' is not the same as that of O, as in the case considered in Art. 13, but bears to it a ratio depending only on the mode of resolution, and therefore the same for each of the two original groups. The same is true of the intensity of E' compared with that of E. Now by Art. 14 the intensities of O', E' are the same for the two original groups, and the intensities of O, E bear to those of O', E' respectively, ratios the same for the two groups: therefore the intensities of O, E are the same for the two final groups; but O, E are any two oppositely polarized components of these groups; therefore these groups are equivalent.

16. It follows from this theorem that no partial analysis of light, such, for example, as would be produced by reflexion from the surface of glass or metal, or by transmission through a doubly absorbing medium, can from equivalent groups produce groups which are not equivalent to each other; and we have seen already that this cannot be done by means of the alteration of phase accompanying double refraction. It follows, therefore, that equivalent groups are optically undistinguishable.

In proving this property of equivalent groups, it has been supposed that the polarizations of the two streams into which any group was resolved were opposite, such being the case in nature. But did a medium exist such that the two streams of light which it transmitted independently were polarized otherwise than oppositely, it would still not enable us to distinguish between equivalent groups.

17. The experimental definition of common light is, light which is incapable of exhibiting rings of any kind when examined by a crystal of Iceland spar and an analyzer, or by some equivalent combination. Consequently, a group of independent polarized streams will together be equivalent to common light when, on being resolved in any manner into two

oppositely polarized pencils, the intensities of the two are the same, and of course equal to half that of the original group. Accordingly, in order that the group should be equivalent to common light, it is necessary and sufficient that the constants B, C, D should vanish.

18. Let us now see under what circumstances two independent streams of polarized light can together be equivalent to common light.

Let α, β refer to the first, and α', β' to the second stream, and let the intensities of the two streams be as 1 to n; then we get from the formulæ (16)

$$\sin 2\beta + n \sin 2\beta' = 0;$$
$$\cos 2\alpha \cos 2\beta + n \cos 2\alpha' \cos 2\beta' = 0;$$
$$\sin 2\alpha \cos 2\beta + n \sin 2\alpha' \cos 2\beta' = 0.$$

Transposing, squaring, and adding, we find $n^2 = 1$, and therefore $n = 1$, since n is essentially positive. Since β and β' are supposed not to lie beyond the limits -90° and $+90^\circ$, we get from the first equation $\beta' = -\beta$, or $\beta' = \pm 90^\circ - \beta$, + or − according as β is positive or negative. Now it is plain that any one solution must be expressed analytically in two ways, in which the values of β' are complementary, and the values of α' differ by 90°, since either principal axis of the ellipse belonging to the second stream may be that whose azimuth is α'. Accordingly, we may reject the second solution as being nothing more than the first expressed in a different way, and may therefore suppose $\beta' = -\beta$. The second and third equations then give $\cos 2\alpha' = -\cos 2\alpha$, $\sin 2\alpha' = -\sin 2\alpha$, and therefore α and α' differ by 90°. The equations are indeed satisfied by $\beta = -\beta' = \pm 45^\circ$, but this solution is only a particular case of the former.

It follows therefore that common light is equivalent to any two independent oppositely polarized streams of half the intensity; and no two independent polarized streams can together be equivalent to common light, unless they be polarized oppositely, and have their intensities equal.

19. We have seen that the nature of the mixture of a given group of independent polarized streams is determined by the values of the four constants A, B, C, D. Consider now the mixture of a stream of common light having an intensity J, and a stream, independent of the former, consisting of elliptically polarized light having an intensity J', and having α' for the azimuth of its plane of maximum polarization, and $\tan \beta'$ for the ratio of the axes of the ellipse which characterizes it.

By the preceding article, the stream of common light is equivalent to two independent streams, plane-polarized in azimuths 0° and 90°, having each an intensity equal to $\frac{1}{2}J$. Hence, applying the formulæ (16) and (17) to the mixture, we have

$$2\mathfrak{M}(c_1)^2 = J + J' + J' \sin 2\beta' \sin 2\beta_1 + J' \cos 2\alpha' \cos 2\beta' \cos 2\alpha_1 \cos 2\beta_1$$
$$+ J' \sin 2\alpha' \cos 2\beta' \sin 2\alpha_1 \cos 2\beta_1;$$

and this mixture will be equivalent to the original group of polarized streams, provided

$$\left.\begin{array}{ll} J + J' = A; & J' \sin 2\beta' = B; \\ J' \cos 2\alpha' \cos 2\beta' = C; & J' \sin 2\alpha' \cos 2\beta' = D. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

These equations give

$$\left.\begin{array}{ll} J' = \sqrt{(B^2 + C^2 + D^2)}\,; & J = A - \sqrt{(B^2 + C^2 + D^2)}\,; \\ \sin 2\beta' = \dfrac{B}{\sqrt{(B^2 + C^2 + D^2)}}\,; & \tan 2\alpha' = \dfrac{D}{C}\,. \end{array}\right\} \,. \quad \ldots\ldots \quad (19)$$

These formulæ can always be satisfied, and therefore it is always possible to represent the given group by a stream of common light combined with a stream of elliptically polarized light independent of the former. Moreover, there is only one way in which the group can be so represented. For, though the third of equations (19) gives two values for β' complementary to each other, these values, as before explained, lead only to two ways of expressing the same result. If we choose that value of β which is numerically the smaller, then among the different values of α', differing by 90°, which satisfy the fourth equation, we must choose one which gives to $\cos 2\alpha'$ the same sign as C.

20. Let us now apply the principles and formulæ which have just been established to a few examples. And first let us take one of the fundamental experiments by which MM. Arago and Fresnel established the laws of interference of polarized light, or rather an analogous experiment mentioned by Sir John Herschel. The experiment selected is the following.

Two neighbouring pencils of common light from the same source are made to form fringes of interference. A tourmaline, carefully worked to a uniform thickness, is cut in two, and its halves interposed in the way of the two streams respectively. It is found that when the planes of polarization of the two tourmalines are parallel the fringes are formed perfectly; but as one of the tourmalines is turned round in azimuth the fringes become fainter, and at last, when the planes of polarization become perpendicular to each other, the fringes disappear.

Let the planes of polarization of the tourmalines be inclined at an angle α, and let it be required to investigate an expression for the intensity of the fringes. Since common light is equivalent to two independent streams, of equal intensity, polarized in opposite ways, let the original light be represented by two independent streams, having each an intensity equal to unity, polarized in planes respectively parallel and perpendicular to the plane of polarization of the first tourmaline. If c, c' be the coefficients of vibration in the two streams respectively, $c \cos \alpha$, $c' \sin \alpha$ will be the coefficients of the resolved parts in the direction of the vibrations transmitted by the second tourmaline. Hence we shall have, mixing together, two independent streams, in one of which the temporary intensity fluctuates between the limits $(c \pm c \cos^2 \alpha)^2 + (c \cos \alpha \sin \alpha)^2$ as the interval of retardation changes in passing from one point to another in the field of view. The temporary intensity of the other stream being $(c' \sin \alpha)^2$, the intensity at different points will fluctuate between the limits

$$\mathfrak{m}\,(c^2 \pm 2c^2 \cos^2 \alpha + c^2 \cos^2 \alpha) + \mathfrak{m}\,(c' \sin \alpha)^2.$$

It is needless to take account of the absorption which takes place even on the pencils which the tourmalines do transmit, because it affects both pencils equally. Since $\mathfrak{m}\,(c^2) = \mathfrak{m}\,(c'^2) = 1$, we have for the limits of fluctuation of the intensity $2 \pm 2 \cos^2 \alpha$. When $\alpha = 0$ these limits become 4 and 0, and the interference is perfect. When $\alpha = 90°$ the limits coalesce, becoming each equal to 2, and there are no fringes. As α increases from 0 to 90°, the superior limit continually decreases, and the inferior increases, and consequently the fringes become fainter and fainter.

21. It is a well-known law of interference that if two rays of common light from the same source be polarized in rectangular planes, and afterwards be brought to the same plane of polarization, or in other words, analyzed so as to retain only light polarized in a particular plane, they will not interfere, but if the light be primitively polarized in one plane they will interfere. This law seems to have presented a difficulty to some, because, it would be argued, the most general kind of vibrations are elliptic, so that we must suppose the vibrations of the ether in the case of common light to be of this kind; and yet the phenomena of interference are exhibited perfectly well if the light be at first elliptically polarized instead of plane-polarized. For my own part, I never could see the difficulty, but on the other hand it seems to me that it would be an immense difficulty were the law anything else than what it is. For, if we consider the rectangular components of the vibrations which make up common light, these components being measured along any two rectangular axes perpendicular to the ray, we must suppose them to be independent of each other, or at least to have no fixed relation to each other so far as regards the changes in the mode of vibration, which we must suppose to be taking place continually, though slowly, it may be, in comparison with the time of a luminous vibration. To suppose otherwise would be contrary to the idea of common light, in which it is implied that on the average whatever we can say of one plane passing through the ray, we can say of another: whatever we can say of the direction one way round we can say of the other way round.

At the end of his excellent Tract on the Undulatory Theory, Mr Airy has shewn how the simple supposition of the existence, in common light, of successive series of undulations, in which the vibrations of one series have no relation to those of another, would account at the same time for the interference of common light and the non-interference of the pencils, polarized in rectangular planes, into which common light may be conceived to be decomposed. But he has, I think, introduced a gratuitous difficulty into the subject, by asserting that it is necessary to suppose the transition from one series into another to be abrupt, and that a gradual change in the nature of the vibrations is inadmissible. This assertion, which seems to have led others to conceive that there was here a difficulty with which the undulatory theory had to contend, seems to have resulted from an investigation from which it appeared that common light could not be represented by an indefinite series of elliptic vibrations, in which the major axis of the ellipse was supposed to revolve uniformly, rapidly, with regard to the duration of impressions on the retina, though slowly with regard to the time of a luminous vibration. I have elsewhere pointed out on what grounds I conceive that the instance of the revolving ellipse is not a case in point, namely, that it is not a fair representation of common light, because it gives a preponderance on the average to one direction of revolution over the contrary, which is contrary to the idea of common light*. Let us now apply the general formulæ (16) and (17) to this case.

Let $c \cos\beta$, $c \sin\beta$ be the semi-axes of the ellipse, a the azimuth of the first axis at a given time. So far as relates to the stream for which the azimuth of the first axis lies between α and $\alpha + d\alpha$, we have $\mathfrak{m}(c^2) = (2\pi)^{-1}c^2 d\alpha$; and in the application of the formulæ (16) the summation, to which Σ refers, will of course pass into an integration. We have therefore

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* Report of the meeting of the British Association at Swansea in 1848. Transactions of the Sections, p. 5.

$$A = \frac{c^2}{2\pi}\int_0^{2\pi} d\alpha = c^2; \quad B = \frac{c^2}{2\pi}\sin 2\beta \int_0^{2\pi} d\alpha = c^2 \sin 2\beta; \quad C = 0; \quad D = 0;$$

whence we get from the formulæ (19), supposing β positive,

$$J' = c^2 \sin 2\beta; \quad J = c^2(1 - \sin 2\beta); \quad \beta' = 45°;$$

while α' remains indeterminate. Hence the mixture is equivalent, not to common light alone, but to a stream of common light having an intensity equal to $c^2(1 - \sin 2\beta)$, combined with a stream of circularly polarized light, independent of the former, having an intensity equal to $c^2 \sin 2\beta$, and being of the same character as regards right-handed or left-handed as the original stream would be were the ellipse stationary. The result of supposing β negative is here assumed as obvious.

22. Suppose that a polarizing prism and a mica plate, which produce elliptic polarization, are made to revolve together with great rapidity. The stream of light thus produced will be equivalent to the former. The only difference is that in the former case c was supposed constant, whereas in the case of actual experiment it will be subject to the fluctuations mentioned at the beginning of this paper; but the mean values represented by $\mathfrak{m}$ will not be affected when these fluctuations are taken into account, and therefore the same formulæ will continue to apply. Hence, if the polarization be circular the rotation will make no difference; if it be plane, the light will appear completely depolarized; in intermediate cases the result will be intermediate, and the light will be equivalent to a mixture of common light and circularly polarized light. The reader may compare these conclusions of theory with some experiments by Professor Dove*.

23. As a last example, let light polarized by transmission through a Nicol's prism be transmitted through a second Nicol's prism, which is made to revolve uniformly and rapidly while the first remains fixed.

Let α be the azimuth of the plane of polarization of the second Nicol's prism, measured from that of the first, c the coefficient of vibration in the stream transmitted through the first prism. The stream passing through the second prism is made up of an infinite number of independent streams such as that whose intensity is $(2\pi)^{-1}\cos^2\alpha d\alpha$ multiplied by the mean value of c^2. Hence we have from the formulæ (16)

$$A = \tfrac{1}{2}\mathfrak{m}(c^2); \quad B = 0; \quad C = \tfrac{1}{4}\mathfrak{m}(c^2); \quad D = 0;$$

whence, taking the intensity of the original stream as unity, we have

$$J' = \tfrac{1}{4}; \quad J = \tfrac{1}{4}; \quad \beta' = 0; \quad \alpha' = 0;$$

or the light is equivalent to a mixture of common light having an intensity $\frac{1}{4}$, and light independent of the former, of the same intensity, polarized at an azimuth zero. This result may be compared with one of Professor Dove's experiments.

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* See *Philosophical Magazine*, Vol. XXX. (1847) p. 465.

If a rotating crystalline plate, cut from a doubly refracting crystal in a direction not nearly coinciding with the optic axis or one of the optic axes, and not sufficiently thin to exhibit colours in polarized light, be substituted for the rotating Nicol's prism, since the plate is too thick to allow of the exhibition of any phenomena depending on the interference of the oppositely polarized pencils, the effect will be just the same as in the case of the Nicol's prism, only the intensity of each steam will be doubled.

In applying the formulæ of this paper to experiments in which one part of an optical train is made to revolve rapidly, it must be understood that the other parts of the train are at rest, or at least do not revolve with a velocity nearly equal to the former. Otherwise, particular phenomena will be exhibited depending on the simultaneous movements of two or more parts of the train, as appeared in Professor Dove's experiments; and in the calculation of these phenomena it will not be allowable to substitute for the stream of light emerging from the polarizer, or first revolving piece whatever it be, the streams of common and elliptically polarized light to which, for general purposes, it is equivalent.

G. G. STOKES.

XVII. *On some recent Improvements in Clock-Escapements.* By Edmund Beckett Denison, Esq., M.A., *of Trinity College.*

[Read *Feb.* 7, 1853.]

For many years past the great problem of clock-making has been the invention of a simple *gravity escapement*, which will give a constant impulse to the pendulum, and without any friction capable of sensibly affecting the arc of vibration. For if that can be done, it is evident that all mechanical causes of disturbance of the time of vibration will also be removed; and if the escapement can also be made so that its effect on the pendulum will still remain the same, notwithstanding any small variation of arc which may arise from changes in the density of the air, or any accidental causes, then the pendulum (so far as it depends on the escapement) must remain isochronous as long as it is kept going at all.

Although Graham's well-known *dead escapement*, together with that modification of it called the *pinwheel escapement*, has now held its place against all rivals for more than a century, as the best for astronomical and turret-clocks, it is very far from satisfying the above conditions. Its success is due to a very different cause; viz. to the fact that the several errors, which arise both from the inconstancy of the impulse and the variations of its large amount of friction, have, by a fortunate coincidence, a tendency to counteract and compensate each other, and sometimes exactly do so. Mr Airy shewed, in a paper on the Disturbances of Pendulums, in Vol. III. of these Transactions, that from this cause the difference between the time of a free pendulum, and of the same pendulum under the influence of this escapement, is less than the difference produced by any other escapement. It does not follow indeed, as he thence inferred, that the *variation* of this difference from day to day (which is the '*rate*' of the clock) must be less with this escapement than any other; and in fact (as I shewed in my paper read here in 1848, Vol. VIII. p. 633, and in the Rudimentary Treatise on Clocks afterwards published,) this variation may be made to vanish altogether in what is called a *gravity* or *remontoire escapement*, by a construction which makes the difference of time from that of a free pendulum a *maximum*. But I also shewed that this compensation of errors in the dead escapement may be traced still farther, into the expression for the variation of the difference from a free pendulum; and that the good going of these clocks depends in a great measure upon two different errors, which arise from the changes of friction in different parts of the clock, also tending to counteract each other[*].

But it must be remembered that all these self-acting corrections are at the best only approximations to accuracy; especially as the different errors have no constant or definite relation to each other, but depend on the accidental state of the friction in different parts of the clock, and upon the construction of each individual escapement. And it is by no means an unimportant circumstance, that such accuracy as can be obtained in a clock of this kind depends very much upon the amount of skill and care which have been employed in its construction. In the best clocks the pallets are jewelled, the pivots turned as small as possible, the wheels and pinions made with very high numbers of teeth, and various other refinements are introduced, because the clockmakers have long ago found out that nothing like accuracy of performance can be obtained, except by reducing all the friction to the lowest possible amount. The consequence is that such clocks are very expensive, costing from £30 to £80, according to the degree of finish, though they contain very little machinery, and that of an extremely simple kind. For though it may be theoretically true that the friction on the dead part of the pallets (called for shortness the 'dead friction') does not directly affect the time of vibration, yet if you try the experiment of artificially thickening the oil there by putting a little resin to it (you may clean it off again with spirits of turpentine), you will find that the friction does in practice affect the time of vibration very sensibly by altering the arc, on several functions of which the time depends.

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[*] I take this opportunity of saying that there are several typographical errors in my former paper. The most material one is the omission, in the last page but one, of a^2 in the denominator of the final expression for $86400\,d\Delta$, or the — daily rate of a gravity escapement.

One of those functions is the *circular error* (as far as it is uncorrected by the pendulum-spring, and how far that is is very uncertain), and this tends to make the clock gain if the arc is diminished. But, on the other hand, there appears in the expression for the daily rate another term involving da, or the variation of the arc, and of the opposite sign to the circular error. And I have found, both from previous experience, and from some experiments expressly made for the purpose of testing the value of an invention which professed to make a pendulum with a dead escapement always isochronous, that sometimes one of these terms may preponderate, and sometimes the other. There is, besides, another term in the expression, depending on the variation of force of the impulse, independently of the variation of the arc. And it appears that when a decrease of arc arises chiefly from an increase of friction on the pallets, the clock will generally lose in spite of the circular error; but where it arises chiefly from an increase of friction in the train the clock will gain. But at any rate the amount of the increase or decrease of time which may be expected to accompany any given increase or decrease of arc, arising in the natural state of things, is quite uncertain. And therefore (if not for other reasons) all such things as cycloidal checks, springs for accelerating the pendulum as the arc increases, and any other 'isochronous' contrivances, are practically useless, and in many cases are even more likely to aggravate the actual error of the clock than to correct it, however well they may appear to answer in arbitrary experiments for producing some enormous developement of one of the errors without taking any notice of another, which in the natural state of things may be much larger in an opposite direction.

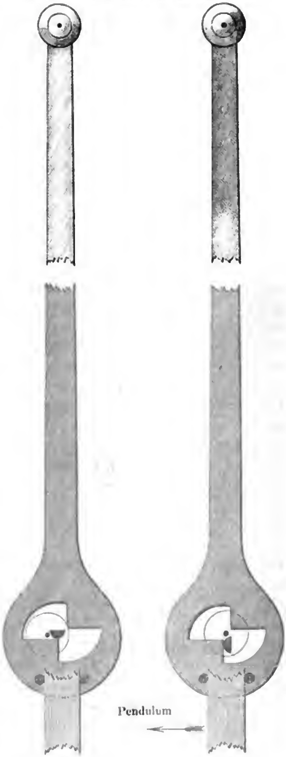
The only exception to this remark is that, in turret-clocks, in which the variations of friction are much greater than in regulators, it seems to be found beneficial to give a very slight recoil to the dead part of the pallets. But I suspect that the advantage is not so much in the accelerative effect thereby produced, as in the tendency of such a recoil to check the variations of the arc, by the increased friction which it opposes to any increased swing of the pendulum. The most valuable addition however to a turret-clock for the purpose of equalizing the force of the train is a remontoire in the train just below the scape-wheel, which is then driven by a spiral spring or a small weight wound up at short intervals by the train and the great clock-weight. The spring remontoire described in the supplement to my former paper has been ever since used by Mr Dent in all his largest clocks. There are two of his clocks in the same town similar in all respects, except that one has this remontoire and the other has not; and the clockmaker there who takes care of them both, and who has access to a dipleidoscope for correcting them, reports that he finds it impossible to keep the one without the remontoire to so good a rate as the other. Nevertheless a train remontoire, however effectual in doing what it professes to do, cannot remove the friction on the pallets or keep it constant; and it is subject to continual variation according to the freshness of the oil and the temperature. In fact all that a train remontoire does for a large clock is to put it on a level with an astronomical clock, of which the train is so highly finished that the force on the scape-wheel has no sensible variation; except that the long and heavy pendulum of the large clock then gives it an advantage over the small one, and consequently turret-clocks with cast-iron wheels can be made, by means of this remontoire, to go better than the best astronomical clocks, provided the pallets are kept properly clean and well oiled.

The result of all this evidently is that it is hopeless to expect any farther material improvement in clock-escapements in any direction, except that of materially reducing, if not completely removing, the friction which affects the pendulum. And I shall now proceed to give an account of the latest attempts that have been made in that direction—so far at least as they appear to me to be worth notice; for such attempts are being made every day, many of them in ignorance of what is really wanted, and none of them with the slightest chance of coming into general use.

Until lately I thought that Mr Bloxam's escapement, which is mentioned in my former paper and fully described in the Rudimentary Treatise on Clocks, was capable of satisfying all the requisite conditions of a gravity escapement. But when it became necessary for me to consider whether it ought to be introduced into the great Westminster clock, which Mr Dent is making under the direction of the Astronomer Royal and myself, I found that it could not be constructed so as to satisfy the proper mathematical conditions without exposing it to such mechanical objections as would prevent its acting at all. In the only specimen of it which (as far as I know) has ever been made, the scape-wheel lifts the gravity arms, or pallets, through a very small angle, and therefore lifts them slowly, and without giving them such a velocity as to carry them out so far that the tooth of the scape-wheel can slip by instead of being caught and stopped: which is called *tripping*. But if the lift is increased to anything near the amount which renders the escapement independent of any moderate change of arc, it will be quite as liable to trip as any other gravity escapement, and probably more, on account of the arms being lifted with scarcely any friction. And therefore, although I believe that, even as it was made by Mr Bloxam, it would go better than any dead escapement, because there is nothing to cause any variation of the arc, still it cannot be regarded as a perfect gravity escapement; and it was also represented to me by those who are better judges of that point than I am, that it would be too delicate and expensive to construct to be at all suitable for general use. And so I found that we were really as far as ever from a practicable gravity escapement, capable of satisfying all the requisite conditions.

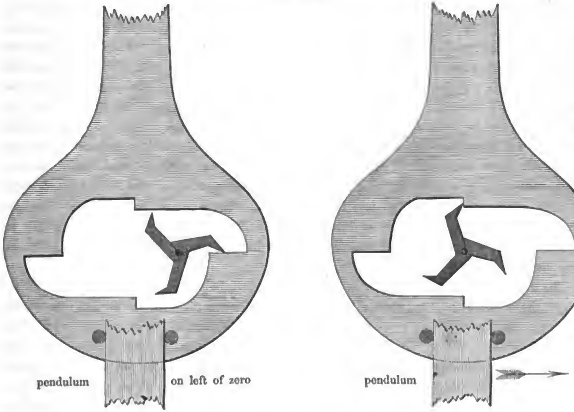
There were, as might be expected, several new gravity escapements in the Exhibition of 1851, but none of them by any means perfect, or at all likely to come into use. But there was also a new form of dead escapement, which had just before been patented by Mr C. Macdowall, a clockmaker, and which certainly possessed considerable advantages in some respects over Graham's, or any escapement of that class. It was called the *single-pin escapement,* because the scape-wheel is only a small disc with a single (ruby) pin in it, parallel and very near to the arbor (or axis) of the disc. The disc turns half round at every beat of the pendulum, and the pin gives the impulse on the vertical faces of the pallets; and the dead friction takes place on the horizontal faces, as in the other dead escapements. This drawing will sufficiently explain its action. The principal advantage of it is that the greatest part of the impulse is nearly direct, and is therefore given with hardly any friction; and the consequence is, that the pressure on the dead part of the pallets is also much less than usual, so that scarcely any oil is required. A further proof of the small friction in this escapement is, that I know a watch of this kind has been going very well for above a year, and was exposed on some cold nights last winter which would have stopped a watch with a common lever escapement. It is however an objection to it, that two additional wheels are required in the train, and the last pivot has to turn with such a velocity, that the force on the scape-disc would be liable to vary a good deal from variations in the state of the oil.

C. Macdowall's Single-pin Escapement.



But it afterwards occurred to me that this last objection might be removed, and also the direct or middle portion of the impulse retained, omitting altogether the oblique part, or that which is given in about the first 60^0 and the last 60^0 of the 180^0 through which the scape-disc moves at every beat. This is done by using a scape-wheel with three pins or teeth instead of one; and the wheel and pallets then assume this form, which pretty obviously suggests the name of the *three-legged dead escapement.* The impulse is given on the vertical faces of the pallets, as the tooth in action passes through its highest or lowest arc of 60^0, with scarcely any friction. And the consequence of this great saving of friction in the impulse is a diminution of the pressure and friction on the dead pallets far beyond what I had expected, although I had already learnt from Macdowall's escapement that there was more force wasted in friction in the common dead escapement than anybody appears to have been aware of before. The pendulum of the Westminster clock, in order to get its length adjusted, had been previously set going with a common church-clock movement and pinwheel escapement, and then it required a weight of 80 lbs. to make it swing the usual arc; but when this three-legged escapement was put to it, they were obliged to take off all the weight except 15 lbs. So that at least four-fifths of the proper *duty* of the scape-wheel is wasted in first producing and then overcoming the friction of the common dead escapement. I say 'at least' four-fifths, because even in this new escapement there is some dead friction, and moreover, I found that it had been made with an unnecessary amount of clearance, so that nearly a quarter of the impulse was lost in the drop of the teeth on to the pallets. The dead friction however is so small,

Three-legged dead Escapement: full-size for a Turret-clock; (the pallet arbor to be 12 inches above the scape-wheel.)



that the application of oil made very little difference in the arc; and small as it is, it might be made still less by using a double scape-wheel (as suggested in my Rudimentary Treatise, for the common dead escapement), with long teeth for the locking and short ones for the impulse; in fact, one has since been made in this way in Paris by a person to whom a drawing of the escapement was given.

It is worth remarking too, that I found the clock always gained with an increase of arc, even though it was produced by increasing the clock-weight: thus affording another proof of the uselessness of any such accelerating contrivances as I before mentioned. And as a small recoil had been given to the dead pallets, it would perhaps be possible so to adjust the amount of this recoil, that the pendulum should neither gain nor lose for any probable amount of variation of the arc. And at any rate, as the force of the scape-wheel may be kept almost constant by a remontoire in the train, and any variation of the dead friction may be made quite insensible by the 'duplex' arrangement just now referred to, there can be little doubt that this escapement might be made more constant in its action than any other in which the pendulum receives its impulse from a scape-wheel. It requires however more delicacy in construction than the single-pin escapement, on account of the small depth of the intersection of the circles described by the points of the scape-wheel teeth and the corners of the vertical pallets; and it probably could not be applied to a watch at all.

The *crutch*, or arm which carries the pallets, must be about 25 times as long as the legs of the scape-wheel, in order to let the escape take place at 1° (there will always be a little lost in the drop); and therefore the legs cannot well be more than $\frac{1}{4}$ inch long in an astronomical clock. In a turret-clock they may be two or three times this size. The scape-wheel which drove the pendulum of the Westminster clock (weighing above 6 cwt. and $14\frac{1}{2}$ ft. long), was only half an inch in radius, and $\frac{1}{7}$ of an ounce in weight; and though the (temporary) pallets were only of iron case-hardened, the teeth had not made a mark upon them after it had been going for four months, when it was taken off to be replaced by another escapement which I have yet to describe.

For after it had been going for some time, I was consulted about a clock for the Cathedral at Fredericton, where the cold is sometimes as great as -40°; and some apprehension was expressed as to the possibility of making any ordinary escapement to go at that temperature; I think not without reason, as even chronometers will stop at a less degree of cold than that, and a watch with a lever escapement (which is the common dead escapement without the dead friction) will stop at a temperature far above it. I have no doubt the three-legged escapement would keep a clock going through any cold which does not actually freeze the oil in the pivot-holes. But as even oil which will not freeze is rendered much less fluid by extreme cold, the force of the escapement would probably vary so much as to make the pendulum swing a good deal further in summer than in winter, and thereby affect the rate. I therefore resumed the attempt, which I had long abandoned, to invent a gravity or remontoire escapement, capable of satisfying all the requisite mechanical and mathematical conditions, and sufficiently easy to make. And I found that it could be done by turning this three-legged impulse escapement into a gravity escapement, in the manner which I will explain presently. But first let us consider what are the conditions which such an escapement ought to satisfy*.

The first of them is that it must be safe from tripping, or from any risk of the pallets by which the scape-wheel teeth are stopped, being thrown so far in lifting that the tooth misses the stop altogether; whereby at least two beats of the pendulum are missed, and perhaps more, and then the scape-wheel acquires such a velocity that several of the teeth may be bent by striking against the stops, and then of course it begins to trip worse than ever. A great many contrivances have been resorted to for preventing this primary difficulty of gravity escapements. Cumming, the first inventor of them, used two pairs of arms, one pair being only for locking and not lifted by the scape-wheel at all, but only by the pendulum in unlocking it. And this was to a certain extent effectual; and I have heard of one of those escapements going (while it did go) better than a very expensive dead escapement clock, by which it was superseded after it had taken to tripping, from the points of the teeth having gradually got bent by their continual impact against the stops without anything to moderate the velocity. Hardy's escapement was just the same, except that he put all the four arms upon springs, I suppose for the sake of avoiding the friction of so many pivots. But he thereby introduced a worse evil; for the springs being stiffer in cold than hot weather, and acting on the pendulum at the extremity of its arc, where it is most affected by any variation of force, made the clock gain in winter; and accordingly Hardy's escapement has gone the way of all other gravity escapements, and was even taken out of the transit clock at Greenwich, and replaced by a dead one. Captain Kater proposed to get rid of tripping by making the impulse arms drop on to an anchor, like the pallets of a dead escapement with the impulse faces cut off, and so unlock the scape-wheel by their weight alternately acting on the different sides of the anchor. Mr Gowland, who had a patent escapement in the Exhibition on the same principle as regards the unlocking, provided against tripping by putting paddles to the pallets, going down into a pot of oil which resisted their motion; not a very elegant contrivance certainly, though apparently effectual. There was also a French escapement by M. Gannery in the Exhibition, very like Gowland's, except that it dispensed with the pot of oil, by giving a large motion of the scape-wheel for a small motion of the arms, as Mr Bloxam's escapement did. But for various reasons not worth enlarging upon now, none of these things have ever come into use, or ever will; and I know of no other gravity escapements which are not open to still more serious objections.

But, secondly, it is necessary that the escapement should not only be secure against tripping, but against what may be called *approximate tripping*. I have seen escapements in which the depth of the locking was sufficient to keep them from actual tripping with any probable amount of force on the scape-wheel, but nevertheless the arc of the pendulum very sensibly increased when the clock-weight was increased. The reason is that the scape-wheel has then force enough to send the arms a little too far, though not far enough to carry the stop quite out of the way of the tooth, and then the pressure of the tooth on the stop keeps the arm there until the pendulum comes and carries it off; and consequently the difference between the angle at which the arm begins to ascend, and that to which it descends with the pendulum (which difference is the angle of impulse), becomes larger than it ought to be, and the swing of the pendulum is increased. This defect of most gravity escapements seems to have escaped the notice of clockmakers altogether.

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* For a description of the general principle of gravity escapements I must refer again to my book on Clocks, p. 70.

Thirdly, the escapement should have no friction sufficient to require oil on any of the parts affecting the pendulum. I do not mean that it is necessary that no oil should be used, but that the friction should be so small that no difference can be observed in the arc of the pendulum whether oil is applied or not. So far as any gravity escapement fails in this respect, it descends to the condition of a common impulse escapement. And on the other hand, any contrivance for diminishing the friction below the point at which the pendulum is indifferent to the absence or presence of oil, is a merely superfluous refinement, and is generally made at the expense of more important objects, as was the case in several of the escapements I mentioned just now.

The fourth condition which a gravity escapement ought to satisfy, or at least not to deviate far from, is the mathematical condition investigated in my former paper, which may be shortly stated thus, with respect to what I there shewed to be the best form of gravity escapement, viz. that in which the pendulum takes up one arm just as it leaves the other:— If γ is the arc of the pendulum from zero when this change of arms takes place, and a the extreme arc, then γ ought to $= \frac{a}{\sqrt{2}} = .71\,a$ nearly; for that is the proportion which makes the difference of the time from that of a free pendulum a maximum, and therefore makes the variation of that difference evanescent for any small variation of a. And now that we know how small a moving force is really required to keep up the vibration of the pendulum when it is delivered from the incumbrance of friction, it will be found by calculation that the error will still be insignificant if γ deviates from the above proportion so far as to $= \frac{a}{2}$ instead of $\frac{a}{\sqrt{2}}$. For if W is the clock-weight (or rather so much of it as arrives at the pendulum clear of friction) and h its daily fall, and M and l the weight and length of the pendulum, then the daily rate

$$= -\frac{Wh}{Ml\pi a^2}\cdot\frac{\frac{a^2}{\gamma^2}-2}{\sqrt{\frac{a^2}{\gamma^2}-1}}\,\frac{da}{a}.$$

In the Westminster clock Wh is certainly not above 10lbs. × 6ft., and Ml is 680lbs. × 13ft. Consequently, taking $\gamma = \frac{a}{2}$ or 1° in this last expression, and assuming a variation of arc of $5'$, the daily rate due to the escapement would not be more than $\frac{1}{10}$ sec.; and that is much less than the daily rate due to the circular error corresponding to such a variation of arc, which is about $\frac{1}{2}$ a second. Even if $\gamma = \frac{a}{3}$, the daily rate will be only $\frac{1}{7}$ sec.; below that angle it increases rapidly. (It may be observed that this error is of the same sign as the circular error, and therefore cannot be employed to correct it.)

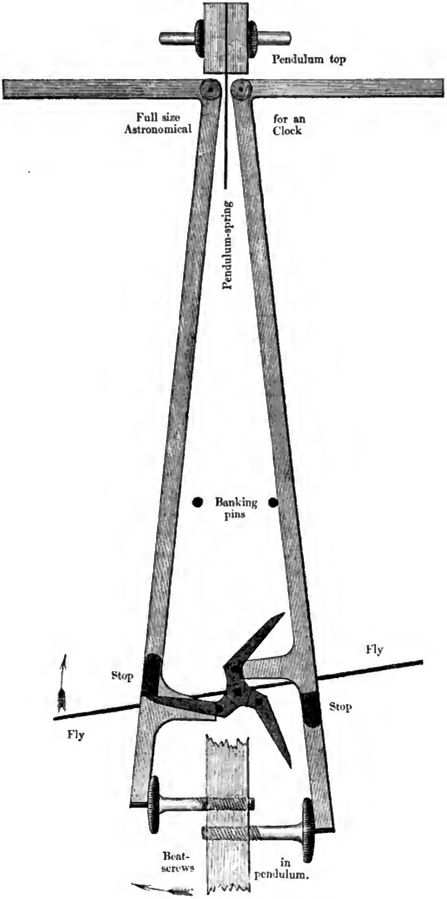
It appears therefore that, within moderate limits, there is no need to be particularly careful about the proportions of a and γ. And this is very convenient, because it happens that, in small clocks especially, the escapement is easier to make safely with γ less than $\frac{a}{\sqrt{2}}$; because when it is so large, the arms can hardly be made light enough for a pendulum of the usual weight, without making them so thin that they are liable to bend (so little impulse does a pendulum require when it is delivered from friction); and moreover the smaller the lift and the heavier the arms, the more they tend to protect themselves against tripping.

Lastly, it is essential to the success of any gravity escapement that it should be easy to make and cheap. If it does not render the pendulum so independent of the train that the 'movement' of a common house-clock will do for an astronomical clock, it may as well do nothing; and if the escapement itself is so complicated or delicate, and therefore expensive, as to bring up the total cost of the clock to much the same as the present price of a first-rate clock, it will have very little chance of coming into use, however good it may be.

All these conditions are satisfied by the escapement of which a drawing is given on the next page. The gravity arms are lifted by the three pins set in the three legs of the scape-wheel. In this drawing the upper pin is about to lift the right arm as soon as the pendulum, now going to the left, has lifted the left arm, and so carried the stop away from the long tooth which is stopped or locked by it. The pin which is lifting always moves from the vertical line through the center of the scape-wheel to 60° beyond it; not as in the three-legged dead escapement from -30° to $+30^\circ$; and consequently there is no occasion here for such delicate work as in that escapement. By the time the pin has got to 60° from the vertical, the tooth belonging to it will have reached the stop on the right arm, and so the scape-wheel will be stopped until the pendulum returns to that side and releases it; and then the pin at the bottom will lift the left arm. Each arm, you observe, (as in all gravity escapements) descends with the pendulum to a lower position than that at which the pendulum begins to lift it, and thus the impulse is given. The business of the scape-wheel is merely to replace the arm at the proper height for the pendulum to take it up and unlock the wheel; and this raising of the arms by the wheel must be done so slowly, that there may be no risk of the arms being sent out too far and allowing the tooth to slip past, or 'trip,' instead of being caught by the stop.

1. This first condition is satisfied in this escapement by the simple addition of a fan-fly set with a friction-spring on the scape-wheel arbor, just like the striking-fly of a common clock. Probably such a fly could not be used with advantage unless where the scape-wheel has very few teeth, and consequently moves through a large angle at each beat. The construction of the scape-wheel also requires the pallets or gravity arms to be much longer than they are in any other escapement, and consequently the depth of the locking, which must occupy something less than the angle $a - \gamma$, is greater than usual, and this also is an additional security against tripping.

Three-legged Gravity Escapement.



2. The approximate tripping is prevented by the length of the locking-teeth of the scape-wheel; for as their points are made three times as far from the center as the pins which lift the arms, their pressure on the stops is so small that the friction due to it is not enough to resist the weight of the arms; and so, if they do by accident get thrown a little too high by the scape-wheel, they immediately fall down again and rest against the pin which has lifted them until the pendulum arrives.

3. These long teeth (as in Mr Bloxam's escapement) also render the momentary friction at unlocking quite insensible, and they will act perfectly well without any oil, though there is no harm in putting a little oil on the stops, if it be only to keep them from rusting. And there is no other friction in the escapement capable of affecting the pendulum; for that of the pivots on which the arms move, and the small amount of sliding of the fork-pins on the beat screws in the pendulum-rod, are too small for any possible variation of such friction to be of any consequence. These two frictions only correspond to those of the pallet-arbor and the fork in a common dead escapement; for there is always some sliding of the fork on the rod, because no point of the pendulum moves in a circle, but in something like a cycloid, on account of its suspension by a spring; and moreover we saw that the pressure of the fork on the pendulum in a common escapement must be at least five times as great as in an escapement which has scarcely any friction.

4. The difficulty of satisfying the proper mathematical condition in most other gravity escapements is the risk of tripping, and the want of sufficient length of arms to make the locking safe within the angle $\alpha - \gamma$, if γ is made of the proper size. From what has been said before, it is evident that no such difficulty exists here in satisfying the mathematical condition of $\gamma = \frac{\alpha}{2}$; and perhaps $\gamma = \frac{\alpha}{3}$ is a more convenient size, and still more secure.

5. The best proof of the facility with which this escapement can be made, and its consequent cheapness, is the fact that two of them, one for a turret-clock and the other for a regulator, were made immediately from my drawings, without any mistake, by Mr Dent's ordinary workmen, in fact, chiefly by a boy; and they say it is the easiest escapement to make that there is. It is too soon yet to be able to give an exact estimate of the price for which these clocks can be made; but as we have now got an escapement independent of the friction of the train, and there is nothing either of difficulty or delicacy in making it, I see no reason to doubt that astronomical clocks may be made with this escapement for £20, to go better than those which have hitherto cost £80. It is true that a great deal of the work for which these large sums are paid, is not worth a farthing to any body, contributing nothing to the accurate performance of the clock, and being probably never seen by any body except the men who do it, and who put the clock together, though the only pretence for doing it is that the clock-makers think it looks well. But so long as a great part of the work *must* be highly finished, and the clock-trade is carried on as it is in England, you will be obliged to pay for the finishing in the same way of the rest of the work, though it would do just as well if it was not 'finished' at all: in fact it is by no means uncommon to see that greater attention has been bestowed on the parts which required none, than on those which could not be done too carefully. A gravity escapement which requires no great delicacy in construction will remove

all necessity and all excuse for spending money in polishing, and in other refinements, some necessary and others superfluous, which are now introduced in all the best astronomical clocks. And, as regards turret-clocks, it will remove what little excuse there ever was for the prejudice against the use of cast-iron wheels among the clockmakers—at least those of them who have had no experience of it, and refuse to listen to the experience of those who have.

It will moreover supersede the necessity for a train remontoire, except where it is desired to have a visible jump of the minute-hand at every half minute, as in the clock at the Great Northern Railway Station in London, and several others, and among them the great Westminster clock, according to Mr Airy's original requisitions. But if the first blow of the hour is made to take place the moment the going part lets off the striking, as it easily can be, this will in almost all cases be a sufficiently exact and convenient mode of fixing the precise time, and therefore a train remontoire will probably be very seldom used with this escapement. Another saving in large clocks may be effected in the pendulum, as a long and heavy compensated pendulum adds a good deal to the expense of a clock, and a short one will do as well when the escapement gives a constant impulse without friction. The reason why a large pendulum is advantageous to a clock with a common escapement is, that it mitigates the errors due to the escapement; but it does not remove them. There are several turret-clocks now making on this plan with $1\frac{1}{4}$ second pendulums (61 inches), instead of the longer ones which they would have had if the escapement had been of the usual kind.

There is another incidental advantage in an escapement of this kind, which I may as well notice here; viz. that it allows you to put the clock forward or backward any even number of beats of the pendulum without touching the pendulum, and without any risk of injuring the scape-wheel. For if you lift one of the arms, so as to make the wheel trip one tooth, you put it forward two beats; and if you stop the scape-wheel, of course you put the clock back: whereas with a common escapement you cannot put the clock forward at all except by the hand-adjusting work, which you cannot judge of to seconds; and it requires some care to stop the scape-wheel without letting the pallets catch the point of one of the teeth. The still smaller adjustments are to be made in the Westminster clock in what I believe to be a new way. The place where any additional weight produces the greatest acceleration of a pendulum, is half-way between the centers of suspension and oscillation; at this point therefore a collar is fixed, and on it lives a weight of the form called a 'shifter,' about $\frac{1}{100}$ the weight of the pendulum itself; and there is another weight like it, to be kept in the clock-room. Taking off the permanent one for about 10 minutes will retard the clock 1 second, and putting on the other will accelerate it. This method produces less disturbance of the pendulum than any other I know of, though of course it requires some care in putting on the weight (it is very easy to take it off); but every other method that I have seen requires much more care in its application, as well as more trouble in construction.

The permanent regulation of the pendulum is done in the same way, by small weights laid upon the collar half-way down the pendulum. One ounce just accelerates this pendulum of 6cwt. a second a day. This is far the best way of doing the pendulum regulation, as it avoids all screwing up and down of the bob and other disturbance of the pendulum, and requires much smaller weights than when they are laid on the top of the bob. Moreover in long church-clock pendulums the bob is often unapproachable, except by a ladder.

There was an astronomical clock from Denmark in the Exhibition, which struck a bell at every minute, in order that an observer might look through his telescope without interruption, taking the time from the clock entirely by ear. I never could get hold of any person to shew the inside of it, and therefore I do not know whether there was any provision to prevent the friction of raising the striking work from affecting the pendulum; and without such a provision it is impossible to make a striking-clock keep very good time. But with a gravity escapement tolerably independent of the force of the train, it may be done very easily; and I should suppose that the striking at every minute would be a convenient addition to an observatory clock.

It may save some trouble if I give a few practical directions as to the best mode of arranging the works of a clock with this escapement.

The scape-wheel should be near the bottom, on account of the length of the arms, so that the seconds' dial and the hour dial will be reversed from their usual places in astronomical clocks. The scape-wheel stands behind the back plate with its back pivot in a cock; and the front pivot had better be in a cock also for convenience of taking it out. The fly on the scape-wheel arbor is between the plates: a fly about an inch wide, and each arm $1\frac{1}{2}$ in. long, will be large enough. And in order to leave room for the fly, the arbor of the wheel next to the scape-wheel is made short, and one end set in a cock within the front plate of the clock, (with the screw-head outside), and the other (which carries the seconds' hand) goes through the cock which carries the front pivot of the scape-wheel arbor. The lower part of both the plates had better be cut away in the middle, in something like a semicircle, up to the cocks of the scape-wheel, so as to let the escapement be seen from the front. To make the angle γ (at which the pendulum changes the arms) $= 1^0$ the distance of the centers of the scape-wheel pins from the center of the wheel must be $\frac{1}{24}$th of the length of the arms down to the stops; or $\frac{1}{36}$th to make $\gamma = 40'$ or $\frac{a}{3}$.

The pendulum must be made to swing 2^0 by adjusting the weight of the arms, on which horizontal projecting pieces may be left at first, as shewn in the drawing of the escapement to be cut or filed down afterwards until the arc is reduced to 2^0.

The adjustment for beat is made, as shewn in the drawing of the escapement, by merely bending out the lower ends of the arms below the stops (making them long enough for the purpose) so as to embrace the pendulum. They will not however touch it, and the contact is made by inserting in the pendulum-rod on opposite sides two steel screws with broad heads, a little convex, and with the edges rough so as to turn by hand. The screw on one side must be a little above that on the other, that the screw-holes may be clear of each other; and the arms should touch the screws just at or a little above the middle of the head. In turret-clocks, where there is more room to get at the adjusting work, it may be done by screwing long eccentric pins into the bottom of the arms (which should come down just beyond the range of the fly) with nuts: then as you turn the pins the eccentricity does the adjustment for beat. In all cases there should be somewhere on the clock-frame a couple of fixed *banking-*

pins on which each arm should rest just when it rests also against the pin of the scape-wheel which is going to lift it.

In turret-clocks the fly is outside the back frame; then the scape-wheel with its pins pointing backwards (or towards the pendulum); then the arms with their stops pointing the other way; and then a large cock to carry the back pivot of the scape-wheel arbor and clear the fly, leaving the arms and fork-pins to come through below the cock. The fly should be about 4 or 4½ inches long and an inch broad, according to the size of the pendulum, i. e. according to the weight of the arms. The stops had better be made of soft steel, screwed on where it is seen they ought to come, and then either they or the pallets (as the case may be) filed away till you have got the proper depth and inclination for locking safely. The depth of locking may be about $\frac{1}{10}$ inch for arms 9 inches long, and for others in the same proportion; and the inclination had better be such as to give what they call a little 'recoil the wrong way,' so as slightly to resist the unlocking, and so make the locking more safe: the resistance is far too little to affect the pendulum, in consequence of the small pressure due to the length of the teeth. When the stops are adjusted, they must be taken off and hardened, and their faces polished. The scape-wheel should be of steel about $\frac{1}{10}$ in. thick for a turret-clock, and about half that thickness for a regulator. Such a scape-wheel, for a turret-clock, will weigh only a quarter of an ounce. The lifting-pins may be of brass wire rivetted in. The lifting-faces of the pallets should be hard and polished; and the whole of the arms had better be of steel, in order that they might be as light as possible with the requisite stiffness, as the action is steadier if some considerable portion of the weight to be lifted is in the horizontal arms.

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THE

THEORY OF THE LONG INEQUALITY

OF

URANUS AND NEPTUNE,

DEPENDING ON THE NEAR COMMENSURABILITY OF THEIR MEAN MOTIONS:

AN ESSAY,

WHICH OBTAINED THE ADAMS PRIZE FOR THE YEAR 1850, IN THE
UNIVERSITY OF CAMBRIDGE.

By R. PEIRSON, M.A.
FELLOW OF ST JOHN'S COLLEGE.

Immenses pendules de l'éternité, qui battent les siècles comme les nôtres battent les secondes.

CAMBRIDGE:
PRINTED AT THE PITT PRESS,
BY JOHN W. PARKER, PRINTER TO THE UNIVERSITY.

M.DCCC.LII.

ADVERTISEMENT.

THE Subject of the Prize was announced in the following terms:—

The University having accepted a Fund raised by several Members of St John's College, for the purpose of founding a Prize to be called the ADAMS PRIZE, for the best [illegible] or other branch of

ERRATA.

	ERROR.	CORRECTION.
Page 3, line 14,	π'	Π'
... 7, ... 6,	Π	Π'
... 46, ... 7,	m^2	m_2
... 64, ... 27,	mean motion and epoch	mean motion

At the end of Article (76) it should have been mentioned that the inequality of epoch of P is caused by the elliptical part of the normal force, and may be investigated in the same way as in Art. (72).

[illegible] exposition, apart from the symbolic reasoning, of the mode in which the forces produce the results arrived at. It is desired that the Essay may be prefaced by a statement of the course of the investigation, and of the principal heads under which the subject is treated, with a summary of the results.

H. W. COOKSON, *Vice-Chancellor.*
J. F. W. HERSCHEL.
J. CHALLIS.
M. O'BRIEN.

March 7, 1849.

THE THEORY

OF THE

LONG INEQUALITY OF URANUS AND NEPTUNE.

1. If the mean motions n and n' of two planets be nearly in the ratio of two small integers k' and k, then $kn \sim k'n'$ is small compared with n and n', and therefore those terms of the disturbing function which involve $(kn - k'n')t$ in their arguments will be much increased by integration; and since $k \sim k'$ is small, the principal terms of that form are of a low order. The variations of the elements, therefore, depending on such terms, are of considerable magnitude, and occupy a long period of time in going through all their changes.

Now the mean motion of Uranus is nearly equal to twice that of Neptune. Hence there is a long inequality in the elements of these planets, the principal part of which depends upon terms of the first order in the eccentricities involving $(n - 2n')t$ in their arguments. There are also terms of the third order in eccentricities and inclinations depending on the same angle, and terms of the second and third orders involving $2(n - 2n')t$ and $3(n - 2n')t$ respectively, all of which are sufficiently important to be retained. The object of the following essay is to calculate the variations of the elements depending on these terms; then, to calculate the principal terms depending on the square of the disturbing force; and, finally, to give some explanation of the mode in which the disturbing forces act in producing the long inequality.

SECTION I.

VARIATIONS OF THE ELEMENTS DEPENDING ON THE FIRST POWER OF THE DISTURBING FORCE.

2. Let x, y, z, be the co-ordinates of Uranus, x', y', z', those of Neptune, x, y being the plane of the ecliptic, m and m' their masses, and let

$$R = \frac{m'(xx' + yy' + zz')}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{m'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}},$$

also let r = radius vector of Uranus,

θ = longitude measured from mean equinox upon the ecliptic as far as the node, and thence on the plane of the orbit, a, e, ϖ and ϵ the semi-major axis, eccentricity, longitude of perihelion, and epoch, n the mean motion in one mean solar day, i and Ω the inclination of the orbit to the ecliptic, and the longitude of the ascending node: and let similar letters accented refer to Neptune. Then

$$x = r\,\{\cos\Omega\,\cos(\theta - \Omega) - \sin\Omega\,\sin(\theta - \Omega)\cos i\},$$
$$y = r\,\{\sin\Omega\,\cos(\theta - \Omega) + \cos\Omega\,\sin(\theta - \Omega)\cos i\},$$
$$z = r\sin(\theta - \Omega)\sin i,$$

and similar equations hold for Neptune.

Substituting these values of x, y, z, &c. in R, we have

$$R = \frac{m'r}{r'^2}\{\cos(\theta - \theta') + I\} - \frac{m'}{\sqrt{r^2 - 2rr'\,\{\cos\theta - \theta') + I\} + r'^2}},$$

$$\begin{aligned} \text{where } I = &- 2\sin(\theta - \Omega)\sin(\theta' - \Omega)\sin^2\tfrac{1}{2}i \\ &- 2\sin(\theta' - \Omega')\sin(\theta - \Omega')\sin^2\tfrac{1}{2}i' \\ &+ \sin(\theta - \Omega)\sin(\theta' - \Omega')\sin i\sin i' \\ &+ 4\cos(\Omega - \Omega')\sin(\theta - \Omega)\sin(\theta' - \Omega')\sin^2\tfrac{1}{2}i\sin^2\tfrac{1}{2}i', \end{aligned}$$

and the equations for the variations of the elements of Uranus disturbed by Neptune, are, putting μ = mass of Sun + m,

$$\frac{da}{dt} = -\frac{2na^2}{\mu}\,\frac{dR}{d\epsilon},$$

$$\frac{de}{dt} = \frac{na\sqrt{1 - e^2}}{\mu e}\left\{\frac{dR}{d\varpi} + (1 - \sqrt{1 - e^2})\frac{dR}{d\epsilon}\right\},$$

$$\frac{d\varpi}{dt} = -\frac{na\sqrt{1 - e^2}}{\mu e}\,\frac{dR}{de} - \frac{2na\sin^2\frac{1}{2}i}{\mu\sin i\sqrt{1 - e^2}}\,\frac{dR}{di},$$

$$\frac{d\,(nt+\epsilon)}{dt} = n + \frac{2na^2}{\mu}\frac{dR}{da} + \frac{na\sqrt{1-e^2}}{\mu e}\left(\sqrt{1-e^2}-1\right)\frac{dR}{de}$$
$$-\frac{2na\sin^2\tfrac{1}{2}i}{\mu\sin i\sqrt{1-e^2}}\frac{dR}{di},$$
$$\frac{di}{dt} = \frac{na}{\mu\sin i\sqrt{1-e^2}}\left\{\frac{dR}{d\Omega} + 2\sin^2\tfrac{1}{2}i\left(\frac{dR}{d\epsilon}+\frac{dR}{d\varpi}\right)\right\},$$
$$\frac{d\Omega}{dt} = -\frac{na}{\mu\sin i\sqrt{1-e^2}}\frac{dR}{di}.$$

3. In order to diminish the number of terms depending on the inclinations, we shall, in calculating a, e, ϖ, and $nt+\epsilon$, take the plane of the orbit of m at instant of epoch as plane of reference, and then the first four equations become, putting $\mu = 1$, and omitting terms higher than the third order,

$$\frac{da}{dt} = -2na^2\frac{dR}{d\epsilon},$$
$$\frac{de}{dt} = \frac{na\sqrt{1-e^2}}{e}\frac{dR}{d\varpi} + \tfrac{1}{2}nae\frac{dR}{d\epsilon},$$
$$\frac{d\varpi}{dt} = -\frac{na\sqrt{1-e^2}}{e}\frac{dR}{de},$$
$$\frac{d\,(nt+\epsilon)}{dt} = n + 2na^2\frac{dR}{da} - \tfrac{1}{2}nae\left(1-\tfrac{1}{4}e^2\right)\frac{dR}{de};$$

and, putting $i' = \gamma'$ and $\Omega' = \Pi'$,

$$I = -2\sin(\theta'-\Pi')\sin(\theta-\pi')\sin^2\tfrac{1}{2}\gamma'$$
$$= \left\{\cos(\theta+\theta'-2\Pi') - \cos(\theta-\theta')\right\}\sin^2\tfrac{1}{2}\gamma'.$$

In calculating the inclination and longitude of node, we shall take the plane of the orbit of m' at epoch for plane of reference; therefore, putting γ and Π for the inclination and longitude of the ascending node on this plane,

$$\frac{d\gamma}{dt} = \frac{na}{\sqrt{1-e^2}\sin\gamma}\frac{dR}{d\Pi} + na\tan\tfrac{1}{2}\gamma\left(\frac{dR}{d\epsilon}+\frac{dR}{d\varpi}\right),$$
$$\frac{d\Pi}{dt} = -\frac{na}{\sqrt{1-e^2}\sin\gamma}\frac{dR}{d\gamma},$$
$$\text{and } I = \left\{\cos(\theta+\theta'-2\Pi) - \cos(\theta-\theta')\right\}\sin^2\tfrac{1}{2}\gamma.$$

4. In changing the plane of reference from the ecliptic to the orbit of m, it becomes necessary to apply a small correction to the values of ϵ' and ϖ' as referred to the ecliptic, but not to those of ϵ and ϖ. Let *PQR* (fig. 1) be the spherical triangle formed by the planes of the orbits and the ecliptic, *PR* being in the plane of the orbit of m, *QR* in that of m', and *PQ* the ecliptic. Now θ, ϵ, and ϖ, are supposed to be measured from ♈ upon the ecliptic to

P, and thence upon the orbit of m. If, therefore, we take $PN = P♈$, and take N as the origin from which the angles are measured, and NPR for the plane of reference, the angles θ, ϵ, and ϖ will be the same as before. But θ', ϵ', ϖ', when referred to the ecliptic, were measured from ♈ to Q, and thence upon the orbit of m'; and when they are referred to the plane NPR, they are measured from N to R, and thence upon the plane of the orbit of m'; consequently, the values of these angles are less now than they were before by the quantity $PQ + QR - PR$. Also $\Pi' = ☊' + PR - PQ$.

Now upon again changing the plane of reference from the orbit of m to that of m', no further alteration needs to be made in the angles. For we may take $RN' = RN$ and take N' as the origin of angles, and recollecting that the node opposite to R is the ascending node of m upon the orbit of m' the angles θ, ϵ, ϖ will be altered by 360°, and the angles θ', ϵ', ϖ' are not altered at all. Also $\gamma = \gamma'$, and Π differs from Π' by 180°, and therefore 2Π differs from $2\Pi'$ by 360°; consequently the same value may be used for Π as for Π'. It appears, therefore, that when ϵ' and Π' have received the above correction, no further alteration is required, whichever orbit is taken for plane of reference.

5. The values of ϵ, ϖ and ☊ at Greenwich mean noon, Jan. 1, 1801, given in Baily's *Astron. Tables and Formulæ*, are

$$\epsilon = 177^\circ\ 48'\ 23''\cdot 0$$
$$\varpi = 167\ \ 31\ \ 16\ \cdot 1$$
$$☊ = \ \ 72\ \ 59\ \ 35\ \cdot 3$$

The apparent annual increase of $\varpi = 52''\cdot 50$,

that of $☊ = 14''\cdot 16$.

The mean sidereal period = 30686·8208296 mean solar days. Hence the value of n is $42''\cdot 23311392$.

The number of days between mean noon, Jan. 1, 1801, and mean noon, Jan. 1, 1847, is $46 \times 365 + 11$. Therefore the mean motion of Uranus during that interval is $197^\circ\ 5'\ 58''\cdot 55$, and using the value of precession given by Bessel $50''\cdot 2294$, the precession in 46 years $= 38'\ 30''\cdot 55$. Therefore adding these quantities to the above value of ϵ, we have for the epoch of Greenwich mean noon, Jan. 1, 1847,

$$\epsilon = 15^\circ\ 32'\ 52''\cdot 10.$$

Again, the increase of ϖ in 46 years $= 40'\ 15''\cdot 0$,

and that of ☊ in the same period $= 10'\ 51''\cdot 36$.

Therefore adding these quantities to the values of ϖ and ☊ at the former epoch, respectively, we have at the new epoch

$$\varpi = 168^\circ\ 11'\ 31''\cdot 1,$$
$$☊ = \ \ 73\ \ 10\ \ 26\ \cdot 66.$$

The values of the inclination and eccentricity at the former epoch are (Baily's *Astr. Tables and Form.*)

$$i = 0^\circ\, 46'\, 28''\cdot44,$$
$$e = 0.04667938,$$

which values we shall employ for the latter epoch.[*]

The value of the mass according to Mr Adams, (*Monthly Notices of R.A.S.*, Vol. IX. p. 159,) is

$$m = \frac{1}{21000}.$$

The value of the $\frac{1}{2}$ major axis deduced by the formula $a^{\frac{3}{2}} = \dfrac{30686\cdot8208296}{365\cdot2563612}\sqrt{\dfrac{1+m}{1+\mathrm{E}}}$, where E = mass of the earth $= \dfrac{1}{389551}$, (Encke, *Astronomische Nachrichten*, No. 443), is

$$a = 19\cdot18267909.$$

The elements of Uranus, therefore, which will be used, are,

Epoch Greenwich mean noon January 1, 1847.

$$\epsilon = 15^\circ\, 32'\, 52''\cdot10,$$
$$\varpi = 168\ \ 11\ \ 31\cdot1,$$
$$e = 0\cdot04667938,$$
$$n = 42''\cdot23311392,$$
$$a = 19\cdot18267909,$$
$$\left.\begin{array}{l}\gamma = \gamma' \\ \Pi = \Pi'\end{array}\right\}, \text{ see next Article,}$$
$$m = \frac{1}{21000}, \text{ mass of Sun being } 1.$$

6. The elements of Neptune for the same epoch given by Mr Sears C. Walker, of the United States of America, (*Monthly Notices of R.A.S.* Vol. VIII. p. 202), are

$$\epsilon' = 328^\circ\, 32'\, 44''\cdot20,$$
$$\varpi' = \ \ 47\ \ 12\ \ \ 6\cdot50,$$
$$\Omega' = 130\ \ \ 4\ \ 20\cdot81,$$

$$\text{Periodic time} = 164\cdot6181 \text{ tropical years,}$$
$$= 60125\cdot4838 \text{ mean solar days.}$$

From this value of the periodic time, we shall find

$$n' = 21''\cdot55492011.$$

The mass, deduced by Professor Peirce, from Bond's and Lassel's observations on the satellite combined, is

$$m' = \frac{1}{18780}.$$

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[*] The secular variation of e during the 46 years between 1801 and 1847, as well as that part of the secular variation of ϖ which is due to the action of Neptune during the same period, is taken account of afterwards in Art. (57).

The value of a' deduced in the same way as a in the last Article, is

$$a' = 30\cdot0363992,$$

also, $\left.\begin{array}{l} i' = 1^\circ\,46'\,58''\cdot97 \\ e' = 0\cdot00871946 \end{array}\right\}$. By Mr Sears C. Walker.

Solving the triangle PQR (fig. 1), we shall find

$$\begin{aligned} PQ &= 56^\circ\,53'\,54''\cdot15, \\ QR &= 25\ \ 30\ \ 30\cdot34, \\ PR &= 82\ \ 23\ \ 48\cdot14. \end{aligned}$$

Hence the quantity to be subtracted from ϵ' and ϖ' or $PQ + QR - PR = 36''\cdot35$, and the quantity to be added to ☊$'$ in order to give Π', or $PR - PQ$, $= 25^\circ\,29'\,53''\cdot99$.

The values of the elements of Neptune, then, which will be employed, are,

Epoch Greenwich mean noon, January 1, 1847.

$$\begin{aligned} \epsilon' &= 328^\circ\,32'\ \ 7''\cdot85, \\ \varpi' &= \ \ 47\ \ 11\ \ 30\cdot15, \\ e' &= 0\cdot00871946, \\ n' &= 21''\cdot55492011, \\ a' &= 30\cdot0363992, \\ \gamma' &= 1^\circ\,30'\,24''\cdot64, \\ \Pi' &= 155^\circ\,34'\,14''\cdot80, \\ m' &= \frac{1}{18780}. \end{aligned}$$

7. Expanding the value of R given in (Art. 2), according to ascending powers of I, we have

$$\begin{aligned} R = \frac{m'r}{r'^2}\cos(\theta-\theta') &- \frac{m'}{\sqrt{r^2 + r'^2 - 2rr'\cos(\theta-\theta')}} \\ + \frac{m'r}{r'^2}\,I &- \frac{m'rr'I}{\{r^2 + r'^2 - 2rr'\cos(\theta-\theta')\}^{\frac{3}{2}}}, \text{ \&c.} \end{aligned}$$

The developement of the first two terms was verified by comparison with the expansion given by Pontécoulant (*Théorie Analyt. du Syst. du Monde*, Liv. VI). The developement of the last two terms is, putting

$$\{a^2 + a'^2 - 2aa'\cos(\theta-\theta')\}^{-\frac{3}{2}} - \frac{1}{a'^3} = \tfrac{1}{2}B_0 + \dots + B_k\cos k(\theta-\theta'),$$

$$\text{and}\quad \xi = nt + \epsilon, \qquad \xi = n't + \epsilon',$$

$$\beta = nt + \epsilon - \varpi, \qquad \beta' = n't + \epsilon' - \varpi',$$

$$\begin{aligned} R = &-\frac{m'}{2}\,aa'B_k\sin^2\tfrac{1}{2}\gamma'\cos\{k(\xi-\xi') - (\xi+\xi'-2\Pi')\} \\ &+ \frac{m'}{2}\,aa'B_2\sin^2\tfrac{1}{2}\gamma'\cos\{(k-1)(\xi-\xi')\} \end{aligned}$$

$$+\frac{m'}{4}aa'\left\{(2k-1)B_k+a\frac{dB_k}{da}\right\}e\sin^2\tfrac{1}{2}\gamma'\cos\{k(\xi-\xi')-\beta-(\xi+\xi'-2\Pi')\}$$

$$+\frac{m'}{4}aa'\left\{(2k+3)B_k+a\frac{dB_k}{da}\right\}e\sin^2\tfrac{1}{2}\gamma'\cos\{k(\xi-\xi')-\beta+(\xi+\xi'-2\Pi')\}$$

$$-\frac{m'}{4}aa'\left\{(2k-1)B_k+a\frac{dB_k}{da}\right\}e\sin^2\tfrac{1}{2}\gamma'\cos\{(k-1)(\xi-\xi')-\beta\}$$

$$-\frac{m'}{4}aa'\left\{(2k+3)B_k+a\frac{dB_k}{da}\right\}e\sin^2\tfrac{1}{2}\gamma'\cos\{(k+1)(\xi-\xi')-\beta\}$$

$$-\frac{m'}{4}aa'\left\{(2k+1)B_k-a'\frac{dB_k}{da'}\right\}e'\sin^2\tfrac{1}{2}\gamma'\cos\{k(\xi-\xi')-\beta'-(\xi+\xi'-2\Pi')\}$$

$$-\frac{m'}{4}aa'\left\{(2k-3)B_k-a'\frac{dB_k}{da'}\right\}e'\sin^2\tfrac{1}{2}\gamma'\cos\{k(\xi-\xi')-\beta'+(\xi+\xi'-2\Pi)\}$$

$$+\frac{m'}{4}aa'\left\{(2k-3)B_k-a'\frac{dB_k}{da'}\right\}e'\sin^2\tfrac{1}{2}\gamma'\cos\{(k-1)(\xi-\xi')-\beta'\}$$

$$+\frac{m'}{4}aa'\left\{(2k+1)B_k-a'\frac{dB_k}{da'}\right\}e'\sin^2\tfrac{1}{2}\gamma'\cos\{(k+1)(\xi-\xi')-\beta'\},$$

in which positive and negative values of k are to be taken, including zero.

8. We shall find in the developement of R the following terms of the first order in eccentricities which involve $(n-2n')t$, putting $\lambda = nt+\epsilon-2(n't+\epsilon')$,

$$\begin{aligned}R_1 &= m'M_1 e\cos(\lambda+\varpi)\\ &\quad + m'M_2 e'\cos(\lambda+\varpi'),\end{aligned}$$

where
$$aM_1 = \frac{\alpha}{2}\left(4b_2^{(\frac{1}{2})}+\alpha\frac{db_2^{(\frac{1}{2})}}{d\alpha}\right),$$

$$aM_2 = -\frac{\alpha}{2}\left(3b_1^{(\frac{1}{2})}-4\alpha+\alpha\frac{db_1^{(\frac{1}{2})}}{d\alpha}\right),$$

α being $=\frac{a}{a'}$, and $b_k^{(\frac{1}{2})}$ the coefficient of $\cos k(\theta-\theta')$ in the expansion of

$$\{1-2\alpha\cos(\theta-\theta')+\alpha^2\}^{-\frac{1}{2}}.$$

Terms of the third order involving λ, are

$$\begin{aligned}R &= m'(M_3e^3+M_4ee'^2+M_5e\sin^2\tfrac{1}{2}\gamma')\cos(\lambda+\varpi)\\ &\quad + m'(M_6e'^3+M_7e^2e'+M_8e'\sin^2\tfrac{1}{2}\gamma')\cos(\lambda+\varpi')\\ &\quad + m'M_9\,e^2e'\cos(\lambda+2\varpi-\varpi')\\ &\quad + m'M_{10}\,ee'^2\cos(\lambda+2\varpi'-\varpi)\\ &\quad + m'M_{11}\,e\sin^2\tfrac{1}{2}\gamma'\cos(\lambda+2\Pi'-\varpi)\\ &\quad + m'M_{12}\,e'\sin^2\tfrac{1}{2}\gamma'\cos(\lambda+2\Pi'-\varpi'),\end{aligned}$$

where

$$aM_3 = -\frac{a}{16}\left\{28b_2^{(\frac{1}{2})} + 5a\frac{db_2^{(\frac{1}{2})}}{da} - 6a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2} - a^3\frac{d^3b_2^{(\frac{1}{2})}}{da^3}\right\},$$

$$aM_4 = -\frac{a}{8}\left\{64b_2^{(\frac{1}{2})} + 6a\frac{db_2^{(\frac{1}{2})}}{da} - 8a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2} - a^3\frac{d^3b_2^{(\frac{1}{2})}}{da^2}\right\},$$

$$aM_5 = -\frac{a^2}{4}\left\{5\left(b_3^{(\frac{3}{2})} + b_1^{(\frac{3}{2})}\right) + a\frac{db_3^{(\frac{3}{2})}}{da} + a\frac{db_1^{(\frac{3}{2})}}{da}\right\},$$

$$aM_6 = \frac{a}{16}\left\{28b_1^{(\frac{1}{2})} - 24a - 4a\frac{db_1^{(\frac{1}{2})}}{da} - 9a^2\frac{d^2b_1^{(\frac{1}{2})}}{da^2} - a^3\frac{d^3b_1^{(\frac{1}{2})}}{da^3}\right\},$$

$$aM_7 = \frac{a}{8}\left\{12b_1^{(\frac{1}{2})} - 8a - 4a\frac{db_1^{(\frac{1}{2})}}{da} - 7a^2\frac{d^2b_1^{(\frac{1}{2})}}{da^2} - a^3\frac{d^3b_1^{(\frac{1}{2})}}{da^3}\right\},$$

$$aM_8 = \frac{a^2}{4}\left\{4\left(b_0^{(\frac{3}{2})} + b_2^{(\frac{3}{2})}\right) - 8 + a\frac{db_0^{(\frac{3}{2})}}{da} + a\frac{db_2^{(\frac{3}{2})}}{da}\right\}.$$

$$aM_9 = \frac{a}{16}\left\{105b_3^{(\frac{1}{2})} + 19a\frac{db_3^{(\frac{1}{2})}}{da} - 7a^2\frac{d^2b_3^{(\frac{1}{2})}}{da^2} - a^3\frac{d^3b_3^{(\frac{1}{2})}}{da^3}\right\},$$

$$aM_{10} = \frac{a}{16}\left\{10a\frac{db_0^{(\frac{1}{2})}}{da} + 8a^2\frac{d^2b_0^{(\frac{1}{2})}}{da^2} + a^3\frac{d^3b_0^{(\frac{1}{2})}}{da^3}\right\},$$

$$aM_{11} = \frac{a^2}{4}\left\{b_1^{(\frac{3}{2})} + a\frac{db_1^{\frac{3}{2}}}{da}\right\},$$

$$aM_{12} = \frac{a^2}{4}\left\{4b_3^{(\frac{3}{2})} - a\frac{db_3^{\frac{3}{2}}}{da}\right\},$$

where $b_k^{(\frac{3}{2})}$ is the coefficient of $\cos k(\theta - \theta')$ in the expansion of

$$\left\{1 - 2a\cos(\theta - \theta') + a^2\right\}^{-\frac{3}{2}}.$$

There are also terms of the second order involving 2λ. For these terms,

$$\begin{aligned} R_2 = {} & m'N_1e^2\cos(2\lambda + 2\varpi) \\ & + m'N_2e'^2\cos(2\lambda + 2\varpi') \\ & + m'N_3ee'\cos(2\lambda + \varpi + \varpi') \\ & + m'N_4\sin^2\tfrac{1}{2}\gamma'\cos(2\lambda + 2\Pi'), \end{aligned}$$

where

$$aN_1 = -\frac{a}{8}\left\{44b_4^{(\frac{1}{2})} + 14a\frac{db_4^{(\frac{1}{2})}}{da} + a^2\frac{d^2b_4^{(\frac{1}{2})}}{da^2}\right\},$$

$$aN_2 = -\frac{a}{8}\left\{38b_2^{(\frac{1}{2})} + 14a\frac{db_2^{(\frac{1}{2})}}{da} + a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2}\right\},$$

$$aN_3 = \frac{a}{4}\left\{42b_3^{(\frac{1}{2})} + 14a\frac{db_3^{(\frac{1}{2})}}{da} + a^2\frac{d^2b_3^{(\frac{1}{2})}}{da^2}\right\},$$

$$aN_4 = -\frac{a^2}{2}b_3^{(\frac{3}{2})}.$$

The terms of the third order involving 3λ are

$$\begin{aligned} R_3 = {} & m'K_1 e^3 \cos(3\lambda + 3\varpi) \\ & + m'K_2 e'^3 \cos(3\lambda + 3\varpi') \\ & + m'K_3 e^2 e' \cos(3\lambda + 2\varpi + \varpi') \\ & + m'K_4 e e'^2 \cos(3\lambda + \varpi + 2\varpi') \\ & + m'K_5 e \sin^2 \tfrac{1}{2}\gamma' \cos(3\lambda + 2\Pi' + \varpi) \\ & + m'K_6 e' \sin^2 \tfrac{1}{2}\gamma' \cos(3\lambda + 2\Pi' + \varpi'), \end{aligned}$$

where

$$aK_1 = \frac{\alpha}{48}\left\{804 b_6^{(\frac{1}{2})} + 279\alpha \frac{db_6^{(\frac{1}{2})}}{d\alpha} + 30\alpha^2 \frac{d^2 b_6^{(\frac{1}{2})}}{d\alpha^2} + \alpha^3 \frac{d^3 b_6^{(\frac{1}{2})}}{d\alpha^3}\right\},$$

$$aK_2 = -\frac{\alpha}{48}\left\{816 b_3^{(\frac{1}{2})} + 312\alpha \frac{db_3^{(\frac{1}{2})}}{d\alpha} + 33\alpha^2 \frac{d^2 b_3^{(\frac{1}{2})}}{d\alpha^2} + \alpha^3 \frac{d^3 b_3^{(\frac{1}{2})}}{d\alpha^3}\right\},$$

$$aK_3 = -\frac{\alpha}{16}\left\{825 b_5^{(\frac{1}{2})} + 291\alpha \frac{db_5^{(\frac{1}{2})}}{d\alpha} + 31\alpha^2 \frac{d^2 b_5^{(\frac{1}{2})}}{d\alpha^2} + \alpha^3 \frac{d^3 b_5^{(\frac{1}{2})}}{d\alpha^3}\right\},$$

$$aK_4 = \frac{\alpha}{16}\left\{832 b_4^{(\frac{1}{2})} + 302\alpha \frac{db_4^{(\frac{1}{2})}}{d\alpha} + 32\alpha^2 \frac{d^2 b_4^{(\frac{1}{2})}}{d\alpha^2} + \alpha^3 \frac{d^3 b_4^{(\frac{1}{2})}}{d\alpha^3}\right\},$$

$$aK_5 = \frac{\alpha^2}{4}\left\{9 b_5^{(\frac{3}{2})} + \alpha \frac{db_5^{(\frac{3}{2})}}{d\alpha}\right\},$$

$$aK_6 = -\frac{\alpha^2}{4}\left\{12 b_4^{(\frac{3}{2})} + \alpha \frac{db_4^{(\frac{3}{2})}}{d\alpha}\right\}.$$

9. The relation existing between the above coefficients and the corresponding coefficients for Neptune is, for those terms which arise from the symmetrical parts of R and R', very simple. If $m'M$ be the coefficient of any term in the symmetrical part of R, and mM' the corresponding coefficient in R', then $M = M'$. And there also exists a very simple relation between the corresponding coefficients in the unsymmetrical parts of R and R', which may be proved in the following manner. Considering only the unsymmetrical part,

$$R = m'\frac{xx' + yy' + zz'}{r'^3} = -\frac{m'}{1 + m'}\left(x\frac{d^2x'}{dt^2} + y\frac{d^2y'}{dt^2} + z\frac{d^2z'}{dt^2}\right),$$

since $\dfrac{x'}{r'^3} = -\dfrac{1}{1 + m'}\,\dfrac{d^2x'}{dt^2}$ &c., the differential coefficients $\dfrac{d^2x'}{dt^2}$, &c. being taken as if the elements were constant.

Similarly,

$$R' = -\frac{m}{1 + m}\left(x'\frac{d^2x}{dt^2} + y'\frac{d^2y}{dt^2} + z'\frac{d^2z}{dt^2}\right).$$

Now any term in this part of R involving $(kn \pm k'n')\,t$ in its argument can arise only from terms in x, y, and z involving knt combined with terms in $\dfrac{d^2x'}{dt^2}$, $\dfrac{d^2y'}{dt^2}$, and $\dfrac{d^2z'}{dt^2}$ respectively, involving $k'n't$ in their arguments; and similarly, with respect to the terms of that form in R'.

Let, then, $p\cos(knt + q)$ be a term in x having in its coefficient some particular power of the eccentricity or some particular combination of eccentricity and inclination. And let $p'\cos(k'n't+q')$ be a particular term in x': then $-p'k'^2n'^2\cos(k'n't+q')$ will be the corresponding term in $\frac{d^2x'}{dt^2}$,

$$\therefore \text{ the coefficient of their product in } x\frac{d^2x'}{dt^2} = -pp'k'^2n'^2,$$

$$\text{and the coefficient of the product in } x'\frac{d^2x}{dt^2} = -pp'k^2n^2,$$

$$\text{the ratio of these coefficients} = \frac{k'^2n'^2}{k^2n^2},$$

and the same ratio holds for the similar combination in $y\frac{d^2y'}{dt^2}$ and $y'\frac{d^2y}{dt^2}$, and also for any combination in $z\frac{d^2z'}{dt^2}$ and $z'\frac{d^2z}{dt^2}$.

Hence the coefficient of the product $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(knt+q)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(k'n't+q')$ in R bears to the corresponding coefficient in R' the ratio

$$\frac{m'}{m}\,\frac{1+m}{1+m'}\,\frac{k'^2n'^2}{k^2n^2} \text{ which equals } \frac{m'k'^2a^3}{mk^2a'^3}$$

accurately: and this is the ratio of the coefficients of the terms involving $(kn \pm k'n')t$ to which the combination $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(knt+q)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(k'n't+q')$ gives rise.

Hence, if $m'M$ and mM' be the corresponding coefficients in the unsymmetrical parts of R and R',

$$\frac{M'}{M} = \frac{k^2}{k'^2a^3},$$

or in the case of the long inequality, in which $\frac{k}{k'} = \frac{1}{2}$,

$$\frac{M'}{M} = \frac{1}{4a^3}.$$

(Since this ratio is nearly $=1$, it follows that the usual approximate relation between the principal parts of the long inequality in mean longitude, $\frac{\zeta}{\zeta'} = -\frac{m'a'^{\frac{1}{2}}}{ma^{\frac{1}{2}}}$ or, more accurately, $-\frac{m'k}{mk'a^2}$ holds true when some of the terms arise from the unsymmetrical part of R, but the ratio is not sufficiently exact to be of much use except in small terms depending on the square of the disturbing force.)

10. Making use of the relations given in the last article, the coefficients for Neptune may be calculated from those of Uranus by the following formulæ,

$$a'M_1' = \frac{aM_1}{a} \quad a'M_2' = \frac{aM_2}{a} - 2\left(a - \frac{1}{4a^2}\right),$$

$$a'M_3' = \frac{aM_3}{a} \quad a'M_4' = \frac{aM_4}{a} \quad a'M_5' = \frac{aM_5}{a},$$

$$a'M_6' = \frac{aM_6}{\alpha} + \frac{3}{2}\left(\alpha - \frac{1}{4\alpha^2}\right),$$

$$a'M_7' = \frac{aM_7}{\alpha} + \left(\alpha - \frac{1}{4\alpha^2}\right),$$

$$a'M_8' = \frac{aM_8}{\alpha} + 2\left(\alpha - \frac{1}{4\alpha^2}\right),$$

$$a'M_9' = \frac{aM_9}{\alpha} \quad a'M_{10}' = \frac{aM_{10}}{\alpha} \quad a'M_{11}' = \frac{aM_{11}}{\alpha},$$

$$a'M_{12}' = \frac{aM_{12}}{\alpha}.$$

Also, generally, $a'N' = \dfrac{aN}{\alpha}$, and $a'K' = \dfrac{aK}{\alpha}$.

11. The values of the fundamental quantities $b_0^{(\frac{1}{2})}$, $b_1^{(\frac{1}{2})}$, &c. were calculated in the following manner. First $b_0^{(\frac{1}{2})}$ and $b_1^{(\frac{1}{2})}$ were determined by the formulæ,

$$b_0^{(\frac{1}{2})} = \frac{4}{\pi} F(\alpha), \quad b_1^{(\frac{1}{2})} = \frac{4}{\alpha\pi}\left\{F(\alpha) - E(\alpha)\right\},$$

(see Pontécoulant, *Théorie Analyt.* Liv. VI. chap. ii.), and the others by the formula

$$b_{k+1}^{(\frac{1}{2})} = \frac{2k}{2k+1}\,\frac{1+\alpha^2}{\alpha}\,b_k^{(\frac{1}{2})} - \frac{2k-1}{2k+1}\,b_{k-1}^{(\frac{1}{2})}.$$

The quantities $\alpha\dfrac{db_0^{(\frac{1}{2})}}{d\alpha}$, $\alpha\dfrac{db_1^{(\frac{1}{2})}}{d\alpha}$, were found from the formulæ,

$$\alpha\frac{db_0^{(\frac{1}{2})}}{d\alpha} = \frac{\alpha^2 b_0^{(\frac{1}{2})} - \alpha b_1^{(\frac{1}{2})}}{1-\alpha^2}, \quad \alpha\frac{db_1^{(\frac{1}{2})}}{d\alpha} = \frac{db_0^{(\frac{1}{2})}}{d\alpha},$$

and the higher differential coefficients of $b_0^{\frac{1}{2}}$, $b_1^{\frac{1}{2}}$ were found from these by differentiating them.

The others were calculated by the formula

$$\alpha\frac{db_{k-1}^{(\frac{1}{2})}}{d\alpha} - \alpha\frac{db_{k+1}^{(\frac{1}{2})}}{d\alpha} = (k-1)\,b_{k-1}^{(\frac{1}{2})} + (k+1)\,b_{k+1}^{(\frac{1}{2})} - 2k\alpha b_k^{(\frac{1}{2})},$$

and the higher differential coefficients by successively differentiating this formula.

The values of $b_0^{(\frac{3}{2})}$, &c. were calculated by the formulæ

$$b_k^{(\frac{3}{2})} + b_{k+1}^{(\frac{3}{2})} = \frac{2k+1}{(1-\alpha)^2}\left(b_k^{(\frac{1}{2})} - b_{k+1}^{(\frac{1}{2})}\right),$$

$$b_k^{(\frac{3}{2})} - b_{k+1}^{(\frac{3}{2})} = \frac{2k+1}{(1+\alpha)^2}\left(b_k^{(\frac{1}{2})} + b_{k+1}^{(\frac{1}{2})}\right);$$

and $\alpha\dfrac{db_0^{(\frac{3}{2})}}{d\alpha}$, &c. by the formulæ,

$$\alpha\frac{db_k^{(\frac{3}{2})}}{d\alpha} + \alpha\frac{db_{k+1}^{(\frac{3}{2})}}{d\alpha} = \frac{2k+1}{(1-\alpha)^2}\left(\alpha\frac{db_k^{(\frac{1}{2})}}{d\alpha} - \alpha\frac{db_{k+1}^{(\frac{1}{2})}}{d\alpha}\right) + \frac{2\alpha}{1-\alpha}\left(b_k^{(\frac{3}{2})} + b_{k+1}^{(\frac{3}{2})}\right),$$

$$\alpha\frac{db_k^{(\frac{3}{2})}}{d\alpha} - \alpha\frac{db_{k+1}^{(\frac{3}{2})}}{d\alpha} = \frac{2k+1}{(1+\alpha)^2}\left(\alpha\frac{db_k^{(\frac{1}{2})}}{d\alpha} + \alpha\frac{db_{k+1}^{(\frac{1}{2})}}{d\alpha}\right) - \frac{2\alpha}{1+\alpha}\left(b_k^{(\frac{3}{2})} - b_{k+1}^{(\frac{3}{2})}\right);$$

and $a^2 \dfrac{d^2 b_0^{(\frac{3}{2})}}{da^2}$, &c., by the formulæ,

$$a^2 \frac{d^2 b_k^{(\frac{3}{2})}}{da^2} + a^2 \frac{d^2 b_{k+1}^{(\frac{3}{2})}}{da^2} = \frac{2k+1}{(1-a)^2}\left(a^2 \frac{d^2 b_k^{(\frac{1}{2})}}{da^2} - a^2 \frac{d^2 b_{k+1}^{(\frac{1}{2})}}{da^2}\right)$$

$$+ \frac{4a}{1-a}\left(a \frac{d b_k^{(\frac{3}{2})}}{da} + a \frac{d b_{k+1}^{(\frac{3}{2})}}{da}\right)$$

$$- \frac{2a^2}{(1-a)^2}\left(b_k^{(\frac{3}{2})} + b_{k+1}^{(\frac{3}{2})}\right),$$

$$a^2 \frac{d^2 b_k^{(\frac{3}{2})}}{da^2} - a^2 \frac{d^2 b_{k+1}^{(\frac{3}{2})}}{da^2} = \frac{2k+1}{(1+a)^2}\left(a^2 \frac{d^2 b_k^{(\frac{1}{2})}}{da^2} + a^2 \frac{d^2 b_{k+1}^{(\frac{1}{2})}}{da^2}\right)$$

$$- \frac{4a}{1+a}\left(a \frac{d b_k^{(\frac{3}{2})}}{da} - a \frac{d b_{k+1}^{(\frac{3}{2})}}{da}\right)$$

$$- \frac{2a^2}{(1+a)^2}\left(b_k^{(\frac{3}{2})} - b_{k+1}^{(\frac{3}{2})}\right),$$

all which are deduced by differentiating the formulæ for $b_k^{(\frac{3}{2})}$ and $b_{k+1}^{(\frac{3}{2})}$.

The values of $b_6^{(\frac{1}{2})}$, $a \dfrac{d b_6^{(\frac{1}{2})}}{da}$, $a^2 \dfrac{d^2 b_6^{(\frac{1}{2})}}{da^2}$, $a^3 \dfrac{d^3 b_6^{(\frac{1}{2})}}{da^3}$ and $a^4 \dfrac{d^4 b_6^{(\frac{1}{2})}}{da^4}$ were also verified by means of the series

$$b_6^{(\frac{1}{2})} = \frac{3 \,.\, 5 \,.\, 7 \,.\, 9 \,.\, 11}{4 \,.\, 6 \,.\, 8 \,.\, 10 \,.\, 12} a^6 \left\{1 + \frac{1}{2} \,.\, \frac{13}{14} a^2 + \frac{1 \,.\, 3}{2 \,.\, 4} \,.\, \frac{13 \,.\, 15}{14 \,.\, 16} a^4 + \&c.\right\},$$

and its differential coefficients.

12. The values of the fundamental quantities are

$$a = 0{\cdot}638647761 = \sin(39^\circ{\cdot}69106023)$$

$$\log a = \bar{1}{\cdot}805261394,$$

$b_0^{(\frac{1}{2})} = 2{\cdot}27020865$, $b_1^{(\frac{1}{2})} = 0{\cdot}77224885$,

$b_2^{(\frac{1}{2})} = 0{\cdot}37818958$, $b_3^{(\frac{1}{2})} = 0{\cdot}20361255$,

$b_4^{(\frac{1}{2})} = 0{\cdot}11459739$, $b_5^{(\frac{1}{2})} = 0{\cdot}06619017$,

$b_6^{(\frac{1}{2})} = 0{\cdot}03888697$.

$b_0^{(\frac{3}{2})} = 6{\cdot}3025242$, $b_1^{(\frac{3}{2})} = 5{\cdot}1694631$,

$b_2^{(\frac{3}{2})} = 3{\cdot}8841368$, $b_3^{(\frac{3}{2})} = 2{\cdot}8007736$,

$b_4^{(\frac{3}{2})} = 1{\cdot}9712274$, $b_5^{(\frac{3}{2})} = 1{\cdot}3652731$.

$a \dfrac{d b_0^{(\frac{1}{2})}}{da} = 0{\cdot}73084947$, $a \dfrac{d b_1^{(\frac{1}{2})}}{da} = 1{\cdot}14437020$,

$a \dfrac{d b_2^{(\frac{1}{2})}}{da} = 0{\cdot}96086031$, $a \dfrac{d b_3^{(\frac{1}{2})}}{da} = 0{\cdot}72740340$,

$a \dfrac{d b_4^{(\frac{1}{2})}}{da} = 0{\cdot}52631179$, $a \dfrac{d b_5^{(\frac{1}{2})}}{da} = 0{\cdot}37111385$,

$a \dfrac{d b_6^{(\frac{1}{2})}}{da} = 0{\cdot}25732242$.

$$a^2\frac{d^2b_0^{(\frac{1}{2})}}{da^2} = 1{\cdot}83976715, \qquad a^2\frac{d^2b_1^{(\frac{1}{2})}}{da^2} = 1{\cdot}73635257,$$

$$a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2} = 2{\cdot}13612463, \qquad a^2\frac{d^2b_3^{(\frac{1}{2})}}{da^2} = 2{\cdot}24746380,$$

$$a^2\frac{d^2b_4^{(\frac{1}{2})}}{da^2} = 2{\cdot}11125291, \qquad a^2\frac{d^2b_5^{(\frac{1}{2})}}{da^2} = 1{\cdot}84049561,$$

$$a^2\frac{d^2b_6^{(\frac{1}{2})}}{da^2} = 1{\cdot}52389289.$$

$$a^3\frac{d^3b_0^{(\frac{1}{2})}}{da^3} = 6{\cdot}4772865, \qquad a^3\frac{d^3b_1^{(\frac{1}{2})}}{da^3} = 6{\cdot}6694842,$$

$$a^3\frac{d^3b_2^{(\frac{1}{2})}}{da^3} = 6{\cdot}7535555, \qquad a^3\frac{d^3b_3^{(\frac{1}{2})}}{da^3} = 7{\cdot}5346529,$$

$$a^3\frac{d^3b_4^{(\frac{1}{2})}}{da^3} = 8{\cdot}2727191, \qquad a^3\frac{d^3b_5^{(\frac{1}{2})}}{da^3} = 8{\cdot}5685410,$$

$$a^3\frac{d^3b_6^{(\frac{1}{2})}}{da^3} = 8{\cdot}353575.$$

$$a^4\frac{d^4b_0^{(\frac{1}{2})}}{da^4} = 35{\cdot}4438, \qquad a^4\frac{d^4b_1^{(\frac{1}{2})}}{da^4} = 35{\cdot}4897,$$

$$a^4\frac{d^4b_2^{(\frac{1}{2})}}{da^4} = 36{\cdot}2803, \qquad a^4\frac{d^4b_3^{(\frac{1}{2})}}{da^4} = 37{\cdot}2441,$$

$$a^4\frac{d^4b_4^{(\frac{1}{2})}}{da^4} = 39{\cdot}8328, \qquad a^4\frac{d^4b_5^{(\frac{1}{2})}}{da^4} = 43{\cdot}3230,$$

$$a^4\frac{d^4b_6^{(\frac{1}{2})}}{da^4} = 46{\cdot}3636.$$

$$a\frac{db_0^{(\frac{3}{2})}}{da} = 18{\cdot}59952015, \qquad a\frac{db_1^{(\frac{3}{2})}}{da} = 18{\cdot}78435767,$$

$$a\frac{db_2^{(\frac{3}{2})}}{da} = 17{\cdot}43417869, \qquad a\frac{db_3^{(\frac{3}{2})}}{da} = 15{\cdot}13495487,$$

$$a\frac{db_4^{(\frac{3}{2})}}{da} = 12{\cdot}513242, \qquad a\frac{db_5^{(\frac{3}{2})}}{da} = 9{\cdot}977628.$$

$$a^2\frac{d^2b_0^{(\frac{3}{2})}}{da^2} = 97{\cdot}3428, \qquad a^3\frac{d^2b_1^{(\frac{3}{2})}}{da^2} = 96{\cdot}0670,$$

$$a^2\frac{d^2b_2^{(\frac{3}{2})}}{da^2} = 94{\cdot}2358, \qquad a^2\frac{d^2b_3^{(\frac{3}{2})}}{da^2} = 89{\cdot}9867,$$

$$a^2\frac{d^2b_4^{(\frac{3}{2})}}{da^2} = 82{\cdot}9630, \qquad a^2\frac{d^2b_5^{(\frac{3}{2})}}{da^2} = 73{\cdot}8548.$$

13. Hence we shall find for the coefficients aM_1, &c., in (Art. 8,)

$\log aM_1 = \bar{1}\cdot8975641$, $\qquad \log aM_2 = \bar{1}\cdot4616115 -$
$\log aM_3 = \bar{1}\cdot2219740$, $\qquad \log aM_4 = \bar{1}\cdot6894009 -$,
$\log aM_5 = 0\cdot8763454 -$, $\qquad \log aM_6 = \bar{1}\cdot9145597 -$,
$\log aM_7 = 0\cdot1864584 -$, $\qquad \log aM_8 = 0\cdot8459271$,
$\log aM_9 = \bar{1}\cdot6778939$, $\qquad \log aM_{10} = 0\cdot0560459$,
$\log aM_{11} = 0\cdot3878377$, $\qquad \log aM_{12} = \bar{1}\cdot28667 -$.

$\log aN_1 = 0\cdot0641950 -$, $\qquad \log aN_2 = 0\cdot3787041 -$,
$\log aN_3 = 0\cdot5250656$, $\qquad \log aN_4 = \bar{1}\cdot7567708 -$.

$\log aK_1 = 0\cdot3202750$, $\qquad \log aK_2 = 0\cdot8005295 -$,
$\log aK_3 = 0\cdot9595044 -$, $\qquad \log aK_4 = 1\cdot1198185$,
$\log aK_5 = 0\cdot3560816$, $\qquad \log aK_6 = 0\cdot5667868 -$.

A negative sign written after the log of a quantity signifies that the quantity is negative.

14.
$$a^2\frac{dM_1}{da} = \frac{a}{2}\left(5a\frac{db_2^{(\frac{1}{2})}}{da} + a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2}\right),$$

$$a^2\frac{dM_2}{da} = -a\left(2a\frac{db_1^{(\frac{1}{2})}}{da} - 2a + \tfrac{1}{2}a^2\frac{d^2b_1^{(\frac{1}{2})}}{da^2}\right);$$

$$\therefore\ \log a^2\frac{dM_1}{da} = 0\cdot3456178,\quad \log a^2\frac{dM_2}{da} = 0\cdot0793316 -$$

$$a^2\frac{dM_3}{da} = -\frac{a}{16}\left(33a\frac{db_2^{(\frac{1}{2})}}{da} - 7a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2} - 9a^3\frac{d^3b_2^{(\frac{1}{2})}}{da^3} - a^4\frac{d^4b_2^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dM_4}{da} = -\frac{a}{8}\left(70a\frac{db_2^{(\frac{1}{2})}}{da} - 10a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2} - 11a^3\frac{d^3b_2^{(\frac{1}{2})}}{da^3} - a^4\frac{d^4b_2^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dM_5}{da} = -\frac{a^2}{4}\left\{5\left(b_3^{(\frac{3}{2})} + b_1^{(\frac{3}{2})}\right) + 7a\left(\frac{db_3^{(\frac{3}{2})}}{da} + \frac{db_1^{(\frac{3}{2})}}{da}\right) + a^2\left(\frac{d^2b_3^{(\frac{3}{2})}}{da^2} + \frac{d^2b_1^{(\frac{3}{2})}}{da^2}\right)\right\},$$

$$a^2\frac{dM_6}{da} = \frac{a}{16}\left(24a\frac{db_1^{(\frac{1}{2})}}{da} - 24a - 22a^2\frac{d^2b_1^{(\frac{1}{2})}}{da^2} - 12a^3\frac{d^3b_1^{(\frac{1}{2})}}{da^3} - a^4\frac{d^4b_1^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dM_7}{da} = \frac{a}{8}\left(8a\frac{db_1^{(\frac{1}{2})}}{da} - 8a - 18a^2\frac{d^2b_1^{(\frac{1}{2})}}{da^2} - 10a^3\frac{d^3b_1^{(\frac{1}{2})}}{da^3} - a^4\frac{d^4b_1^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dM_8}{da} = \frac{a^2}{4}\left\{4\left(b_0^{(\frac{3}{2})} + b_2^{(\frac{3}{2})}\right) - 8 + 6a\frac{d}{da}\left(b_0^{(\frac{3}{2})} + b_2^{(\frac{3}{2})}\right) + a^2\frac{d^2}{da^2}\left(b_0^{(\frac{3}{2})} + b_2^{(\frac{3}{2})}\right)\right\},$$

$$a^2\frac{dM_9}{da} = \frac{a}{16}\left(124a\frac{db_3^{(\frac{1}{2})}}{da} + 5a^2\frac{d^2b_3^{(\frac{1}{2})}}{da^2} - 10a^3\frac{d^3b_3^{(\frac{1}{2})}}{da^3} - a^4\frac{d^4b_3^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dM_{10}}{da} = \frac{a}{16}\left(10a\frac{db_0^{(\frac{1}{2})}}{da} + 26a^2\frac{d^2b_0^{(\frac{1}{2})}}{da^2} + 11a^3\frac{d^3b_0^{(\frac{1}{2})}}{da^3} + a^4\frac{d^4b_0^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dM_{11}}{da}=\frac{a^2}{4}\left(b_1^{(\frac{3}{2})}+3a\frac{db_1^{(\frac{3}{2})}}{da}+a^2\frac{d^2b_1^{(\frac{3}{2})}}{da^2}\right),$$

$$a^2\frac{dM_{12}}{da}=\frac{a^2}{4}\left(4b_2^{(\frac{3}{2})}+2a\frac{db_2^{(\frac{3}{2})}}{da}-a^2\frac{d^2b_2^{(\frac{3}{2})}}{da^2}\right).$$

Therefore,

$$\log a^2\frac{dM_3}{da}=0\cdot50589,\qquad \log a^2\frac{dM_4}{da}=0\cdot71288,$$

$$\log a^2\frac{dM_5}{da}=1\cdot67486\ -,\qquad \log a^2\frac{dM_6}{da}=0\cdot75214\ -,$$

$$\log a^2\frac{dM_7}{da}=1\cdot01408\ -,\qquad \log a^2\frac{dM_8}{da}=1\cdot65243,$$

$$\log a^2\frac{dM_9}{da}=\bar{1}\cdot64862\ -,\qquad \log a^2\frac{dM_{10}}{da}=0\cdot81021,$$

$$\log a^2\frac{dM_{11}}{da}=1\cdot20599,\qquad \log a^2\frac{dM_{12}}{da}=0\cdot65024\ -,$$

$$a^2\frac{dN_1}{da}=-\frac{a}{8}\left(58a\frac{db_4^{(\frac{1}{2})}}{da}+16a^2\frac{d^2b_4^{(\frac{1}{2})}}{da^2}+a^3\frac{d^3b_4^{(\frac{1}{2})}}{da^3}\right),$$

$$a^2\frac{dN_2}{da}=-\frac{a}{8}\left(52a\frac{db_2^{(\frac{1}{2})}}{da}+16a^2\frac{d^2b_2^{(\frac{1}{2})}}{da^2}+a^3\frac{d^3b_2^{(\frac{1}{2})}}{da^3}\right),$$

$$a^2\frac{dN_3}{da}=a\left(14a\frac{db_3^{(\frac{1}{2})}}{da}+4a^2\frac{d^2b_3^{(\frac{1}{2})}}{da^2}+\tfrac{1}{4}a^3\frac{d^3b_3^{(\frac{1}{2})}}{da^3}\right),$$

$$a^2\frac{dN_4}{da}=-\frac{a^2}{2}\left(b_3^{(\frac{3}{2})}+a\frac{db_3^{(\frac{3}{2})}}{da}\right).$$

Therefore,

$$\log a^2\frac{dN_1}{da}=0\cdot7629815\ -,\qquad \log a^2\frac{dN_2}{da}=0\cdot8607176\ -,$$

$$\log a^2\frac{dN_3}{da}=1\cdot1286613,\qquad \log a^2\frac{dN_4}{da}=0\cdot5632118\ -,$$

$$a^2\frac{dK_1}{da}=\frac{a}{48}\left(1083a\frac{db_6^{(\frac{1}{2})}}{da}+339a^2\frac{d^2b_6^{(\frac{1}{2})}}{da^2}+33a^3\frac{d^3b_6^{(\frac{1}{2})}}{da^3}+a^4\frac{d^4b_6^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dK_2}{da}=-\frac{a}{48}\left(1128a\frac{db_3^{(\frac{1}{2})}}{da}+378a^2\frac{d^2b_3^{(\frac{1}{2})}}{da^2}+36a^3\frac{d^3b_3^{(\frac{1}{2})}}{da^3}+a^4\frac{d^4b_3^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dK_3}{da}=-\frac{a}{16}\left(1116a\frac{db_5^{(\frac{1}{2})}}{da}+353a^2\frac{d^2b_5^{(\frac{1}{2})}}{da^2}+34a^3\frac{d^3b_5^{(\frac{1}{2})}}{da^3}+a^4\frac{d^4b_5^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2\frac{dK_4}{da}=\frac{a}{16}\left(1134a\frac{db_4^{(\frac{1}{2})}}{da}+366a^2\frac{d^2b_4^{(\frac{1}{2})}}{da^2}+35a^3\frac{d^3b_4^{(\frac{1}{2})}}{da^3}+a^4\frac{d^4b_4^{(\frac{1}{2})}}{da^4}\right),$$

$$a^2 \frac{dK_5}{da} = \frac{a^2}{4}\left(9b_5^{(\frac{3}{2})} + 11a\frac{db_5^{(\frac{3}{2})}}{da} + a^2\frac{d^2b_5^{(\frac{3}{2})}}{da^2}\right),$$

$$a^2 \frac{dK_6}{da} = -\frac{a^2}{4}\left(12b_4^{(\frac{3}{2})} + 14a\frac{db_4^{(\frac{3}{2})}}{da} + a^2\frac{d^2b_4^{(\frac{3}{2})}}{da^2}\right).$$

Therefore,

$\log a^2 \dfrac{dK_1}{da} = 1{\cdot}1721943,$ $\log a^2 \dfrac{dK_2}{da} = 1{\cdot}4203660\ -$

$\log a^2 \dfrac{dK_3}{da} = 1{\cdot}7467763\ -,$ $\log a^2 \dfrac{dK_4}{da} = 1{\cdot}8313179,$

$\log a^2 \dfrac{dK_5}{da} = 1{\cdot}3004887,$ $\log a^2 \dfrac{dK_6}{da} = 1{\cdot}4584086\ -.$

15. The following logarithms will be required.

$\log\sin\varpi = \bar{1}{\cdot}3109761,$ $\log\cos\varpi = \bar{1}{\cdot}9907711\ -,$

$\log\sin\varpi' = \bar{1}{\cdot}8654780,$ $\log\cos\varpi' = \bar{1}{\cdot}8322198,$

$\log\sin 2\varpi = \bar{1}{\cdot}6027172\ -,$ $\log\cos 2\varpi = \bar{1}{\cdot}9620142,$

$\log\sin 2\varpi' = \bar{1}{\cdot}9987278,$ $\log\cos 2\varpi' = \bar{2}{\cdot}8832664\ -,$

$\log\sin(\varpi + \varpi') = \bar{1}{\cdot}7627153\ -$ $\log\cos(\varpi + \varpi') = \bar{1}{\cdot}9113136\ -,$

$\log\sin 2\Pi' = \bar{1}{\cdot}8768448\ -$ $\log\cos 2\Pi' = \bar{1}{\cdot}8181742,$

$\log\sin(2\varpi - \varpi') = \bar{1}{\cdot}9751656\ -,$ $\log\cos(2\varpi - \varpi') = \bar{1}{\cdot}5168508,$

$\log\sin(2\varpi' - \varpi) = \bar{1}{\cdot}9824229\ -,$ $\log\cos(2\varpi' - \varpi) = \bar{1}{\cdot}4453671,$

$\log\sin(2\Pi' - \varpi) = \bar{1}{\cdot}7799697,$ $\log\cos(2\Pi' - \varpi) = \bar{1}{\cdot}9020604\ -,$

$\log\sin(2\Pi' - \varpi') = \bar{1}{\cdot}9975742\ -,$ $\log\cos(2\Pi' - \varpi') = \bar{1}{\cdot}0228363\ -,$

$\log\sin 3\varpi = \bar{1}{\cdot}7631462,$ $\log\cos 3\varpi = \bar{1}{\cdot}9110959\ -,$

$\log\sin 3\varpi' = \bar{1}{\cdot}7934327,$ $\log\cos 3\varpi' = \bar{1}{\cdot}8939967\ -,$

$\log\sin(2\varpi + \varpi') = \bar{1}{\cdot}6020161,$ $\log\cos(2\varpi + \varpi') = \bar{1}{\cdot}9621480,$

$\log\sin(2\varpi' + \varpi) = \bar{1}{\cdot}9963434\ -,$ $\log\cos(2\varpi' + \varpi) = \bar{1}{\cdot}1113349\ -,$

$\log\sin(2\Pi' + \varpi) = \bar{1}{\cdot}9404083,$ $\log\cos(2\Pi' + \varpi) = \bar{1}{\cdot}6901009\ -,$

$\log\sin(2\Pi' + \varpi') = \bar{2}{\cdot}4636830\ -,$ $\log\cos(2\Pi' + \varpi') = \bar{1}{\cdot}9998162,$

$\log e = \bar{2}{\cdot}6691250,$ $\log e' = \bar{3}{\cdot}9404896,$

$\log e^2 = \bar{3}{\cdot}3382500,$ $\log ee' = \bar{4}{\cdot}6096146,$ $\log e'^2 = \bar{5}{\cdot}8809792,$

$\log e^3 = \bar{4}{\cdot}0073750,$ $\log e^2e' = \bar{5}{\cdot}2787396,$ $\log ee'^2 = \bar{6}{\cdot}5501042,$

$\log e'^3 = \bar{7}{\cdot}8214688,$ $\log\sin\gamma = \bar{2}{\cdot}4198955,$ $\log\sin^2\frac{1}{2}\gamma = \bar{4}{\cdot}2378065$

$\log e\sin^2\frac{1}{2}\gamma = \bar{6}{\cdot}9069315,$ $\log e'\sin^2\frac{1}{2}\gamma = \bar{6}{\cdot}1782961,$

$$\log \frac{m'n^2}{\omega^2 \sin 1''} = 4{\cdot}4063077, \qquad \log \frac{mn'^2}{\omega^2 \sin 1''} = 3{\cdot}7735706,$$

$$\log \frac{m'n}{\omega \sin 1''} = 2{\cdot}7235186\ -, \qquad \log \frac{mn'}{\omega \sin 1''} = 2{\cdot}3828882\ -,$$

$$\log \left(a - \frac{1}{4a^2}\right) = \bar{2}{\cdot}41008, \qquad \log \left(a + \frac{1}{8a^2}\right) = \bar{1}{\cdot}97549,$$

16. We shall now proceed to calculate the variations of the elements depending on terms of the first order. By (Art. 8),

$$R_1 = m'(P \cos\lambda - P' \sin\lambda),$$

where $P = M_1 e \cos \varpi + M_2 e' \cos \varpi'$,

P' = same function with sines for cos;

and $R_1' = m\,(Q \cos\lambda - Q' \sin\lambda)$,

where $Q = M_1' e \cos \varpi + M_2' e' \cos \varpi'$,

Q' = same with sines for cos.

When the equations of (Art. 3) are integrated, we have for the variations of the elements of Uranus, putting $\omega = n - 2n'$,

$$\delta\,(nt + \epsilon) = \left(\frac{3m'n^2aP}{\omega^2 \sin 1''} + \frac{2m'na^2}{\omega \sin 1''}\,\frac{dP}{da} - \frac{m'nae}{2\omega \sin 1''}\,\frac{dP}{de} + \frac{m'nae^3}{8\omega \sin 1''}\,\frac{dP}{de}\right) \sin\lambda$$

$$+\ (\text{same with } P' \text{ instead of } P) \cos\lambda,$$

$$\delta e = \left(-\frac{m'na}{\omega \sin 1''}\,\frac{dP'}{de} + \frac{m'nae^2}{2\omega \sin 1''}\,\frac{dP'}{de} - \frac{m'naeP'}{2\omega \sin 1''}\right) \sin\lambda$$

$$-\ (\text{same with } P \text{ for } P') \cos\lambda,$$

$$e\delta\varpi = \left(-\frac{m'na}{\omega \sin 1''}\,\frac{dP}{de} + \frac{m'nae^2}{2\omega \sin 1''}\,\frac{dP}{de}\right) \sin\lambda$$

$$+\ (\text{same with } P' \text{ for } P) \cos\lambda,$$

$$\delta n = -\frac{3m'n^2aP'}{\omega} \sin\lambda + \frac{3m'n^2aP}{\omega} \cos\lambda,$$

$$\delta a = \frac{2m'na^2P'}{\omega} \sin\lambda - \frac{2m'na^2P}{\omega} \cos\lambda,$$

$$\delta\gamma = \left(-\frac{m'nae \tan\frac{1}{2}\gamma}{\omega \sin 1''}\,\frac{dP'}{de} - \frac{m'na \tan\frac{1}{2}\gamma}{\omega \sin 1''}\,P'\right) \sin\lambda$$

$$-\ (\text{same with } P \text{ for } P') \cos\lambda,$$

$$\delta\Pi = 0;$$

and the variations of the elements of Neptune are

$$\delta\,(n't + \epsilon') = \left(-\frac{6mn'^2a'Q}{\omega^2 \sin 1''} + \frac{2mn'a'^2}{\omega \sin 1''}\,\frac{dQ}{da'} - \frac{mn'a'e'}{2\omega \sin 1''}\,\frac{dQ}{de'} + \frac{mn'a'e'^3}{8\omega \sin 1''}\,\frac{dQ}{de'}\right) \sin\lambda$$

$$+\ (\text{same with } Q' \text{ for } Q) \cos\lambda,$$

$$\delta e' = \left(-\frac{mn'a'}{\omega \sin 1''} \cdot \frac{dQ'}{de'} + \frac{mn'a'e'^2}{2\,\omega \sin 1''}\,\frac{dQ'}{de'} + \frac{mn'a'e'Q'}{\omega \sin 1''}\right) \sin\lambda$$

$$-\ (\text{same with } Q \text{ for } Q') \cos\lambda,$$

$$e'\delta\varpi' = \left(-\frac{mn'a'}{\omega\sin 1''}\,\frac{dQ}{de'} + \frac{mn'a'e'^2}{2\omega\sin 1''}\,\frac{dQ}{de'}\right)\sin\lambda$$

$$+ \text{(same with } Q' \text{ for } Q)\cos\lambda,$$

$$\delta n' = +\frac{6mn'^2a'Q'}{\omega}\sin\lambda - \frac{6mn'^2a'Q}{\omega}\cos\lambda,$$

$$\delta a' = -\frac{4mn'a'^2Q'}{\omega}\sin\lambda + \frac{4mn'a'^2Q}{\omega}\cos\lambda,$$

$$\delta\gamma' = \left(\frac{2mn'a'\tan\frac{1}{2}\gamma'}{\omega\sin 1''}\,Q' - \frac{mn'a'e'\tan\frac{1}{2}\gamma'}{\omega\sin 1''}\,\frac{dQ'}{de'}\right)\sin\lambda$$

$$- \text{(same with } Q \text{ for } Q')\cos\lambda,$$

$$\delta\Pi' = 0.$$

17. By means of quantities given in (Arts. 13 and 15), we shall find

$$\log aP = \bar{2}\cdot 5775645\ -, \qquad \log aP' = \bar{3}\cdot 7553692\,;$$

and therefore, by (Art. 15),

$$\frac{3m'n^2aP}{\omega^2\sin 1''} = -\,2890''\cdot 6362, \qquad \frac{3m'n^2aP'}{\omega^2\sin 1''} = +\,435''\cdot 3096.$$

$$\text{Again,}\quad a'Q = \frac{aP}{a} - 2\left(a - \frac{1}{4a^2}\right)e'\cos\varpi',$$

$$a'Q' = \frac{aP'}{a} - 2\left(a - \frac{1}{4a^2}\right)e'\sin\varpi',$$

$$\therefore\ \log a'Q = \bar{2}\cdot 7745325\ -, \qquad \log a'Q' = \bar{3}\cdot 9337808,$$

$$\therefore\ \frac{6mn'^2a'Q}{\omega^2\sin 1''} = -\,2119''\cdot 6020, \qquad \frac{6mn'^2a'Q'}{\omega^2\sin 1''} = +\,305''\cdot 8459.$$

18. By (Arts 14 and 15),

$$\log a^2\frac{dP}{da} = \bar{1}\cdot 0349352\ -, \qquad \log a^2\frac{dP'}{da} = \bar{2}\cdot 1300441,$$

$$\therefore\ \frac{2m'na^2}{\omega\sin 1''}\,\frac{dP}{da} = +\,114''\cdot 6790, \qquad \frac{2m'na^2}{\omega\sin 1''}\,\frac{dP'}{da} = -\,14''\cdot 2755,$$

$$\text{and}\quad a'^2\frac{dQ}{da'} = -\frac{1}{a}\left(a^2\frac{dP}{da} + aP\right) + 4\left(a + \frac{1}{8a^2}\right)e'\cos\varpi',$$

$$a'^2\frac{dQ'}{da'} = -\frac{1}{a}\left(a^2\frac{dP'}{da} + aP'\right) + 4\left(a + \frac{1}{8a^2}\right)e'\sin\varpi',$$

$$\therefore\ \log a'^2\frac{dQ}{da'} = \bar{1}\cdot 4001832, \qquad \log a'^2\frac{dQ'}{da'} = \bar{3}\cdot 7675884\ -,$$

$$\therefore\ \frac{2mn'a'^2}{\omega\sin 1''}\,\frac{dQ}{da'} = -\,121''\cdot 3672, \qquad \frac{2mn'a'^2}{\omega\sin 1''}\,\frac{dQ'}{da'} = +\,2''\cdot 8282.$$

19.
$$ae\,\frac{dP}{de} = aM_1\,e\cos\varpi, \qquad ae\,\frac{dP'}{de} = aM_1\,e\sin\varpi,$$

$$\therefore\ \log ae\,\frac{dP}{de} = \bar{2}\cdot5574002\,-, \qquad \log ae\,\frac{dP'}{de} = \bar{3}\cdot8776652,$$

$$\therefore\ \frac{m'nae}{2\omega\sin 1''}\,\frac{dP}{de} = +\,9''\cdot5475, \qquad \frac{m'nae}{2\omega\sin 1''}\,\frac{dP'}{de} = -\,1''\cdot9960,$$

$$\frac{m'nae^3}{8\omega\sin 1''}\,\frac{dP}{de} = +\,0''\cdot0052, \qquad \frac{m'nae^3}{8\omega\sin 1''}\,\frac{dP'}{de} = -\,0''\cdot0011,$$

$$a'e'\,\frac{dQ}{de'} = a'M_2'e'\cos\varpi', \qquad a'e'\,\frac{dQ'}{de'} = a'M_2'e'\sin\varpi',$$

$$\therefore\ \log a'e'\,\frac{dQ}{de'} = \bar{3}\cdot4757260\,-, \qquad \log a'e'\,\frac{dQ'}{de'} = \bar{3}\cdot5089842\,-,$$

$$\therefore\ \frac{mn'a'e'}{2\omega\sin 1''}\,\frac{dQ}{de'} = +\,0''\cdot3611, \qquad \frac{mn'a'e'}{2\omega\sin 1''}\,\frac{dQ'}{de'} = +\,0''\cdot3898,$$

$$\frac{mn'a'e'^3}{8\omega\sin 1''}\,\frac{dQ}{de'} = +\,0''\cdot0000, \qquad \frac{mn'a'e'^3}{8\omega\sin 1''}\,\frac{dQ'}{de'} = +\,0''\cdot0000,$$

$$\therefore\ \delta(nt+\epsilon) = -\,2785''\cdot499\sin\lambda + 423''\cdot029\cos\lambda,$$
$$\delta(n't+\epsilon') = +\,1997''\cdot874\sin\lambda - 303''\cdot407\cos\lambda.$$

20. By the last Article,

$$\log a\,\frac{dP}{de} = \bar{1}\cdot8882752\,-, \qquad \log a\,\frac{dP'}{de} = \bar{1}\cdot2085402,$$

$$\therefore\ \frac{m'na}{\omega\sin 1''}\,\frac{dP}{de} = +\,409''\cdot0664, \qquad \frac{m'na}{\omega\sin 1''}\,\frac{dP'}{de} = -\,85''\cdot5183,$$

$$\frac{m'nae^2}{2\omega\sin 1''}\,\frac{dP}{de} = +\,0''\cdot4456, \qquad \frac{m'nae^2}{2\omega\sin 1''}\,\frac{dP'}{de} = -\,0''\cdot0931,$$

$$\log a'\,\frac{dQ}{de'} = \bar{1}\cdot5352364\,-, \qquad \log a'\,\frac{dQ'}{de'} = \bar{1}\cdot5684946\,-,$$

$$\therefore\ \frac{mn'a'}{\omega\sin 1''}\,\frac{dQ}{de'} = +\,82''\cdot8180, \qquad \frac{mn'a'}{\omega\sin 1''}\,\frac{dQ'}{de'} = +\,89''\cdot4093,$$

$$\frac{mn'a'e'^2}{2\omega\sin 1''}\,\frac{dQ}{de'} = +\ \ 0''\cdot0031, \qquad \frac{mn'a'e'^2}{2\omega\sin 1''}\,\frac{dQ'}{de'} = +\ \ 0''\cdot0034,$$

also by Art. 17,

$$\frac{m'naeP}{2\omega\sin 1''} = +\,0''\cdot4668, \qquad \frac{m'naeP'}{2\omega\sin 1''} = -\,0''\cdot0703,$$

$$\frac{mn'a'e'Q}{\omega\sin 1''} = +\,0''\cdot1253, \qquad \frac{mn'a'e'Q'}{\omega\sin 1''} = -\,0''\cdot0181,$$

Hence,

$$\delta e = +\ \ 85''\cdot495\sin\lambda + 409''\cdot088\cos\lambda,$$
$$e\delta\varpi = -\,408''\cdot621\sin\lambda +\ \ 85''\cdot425\cos\lambda,$$
$$\delta e' = -\ \ 89''\cdot424\sin\lambda +\ \ 82''\cdot690\cos\lambda,$$
$$e'\delta\varpi' = -\ \ 82''\cdot815\sin\lambda -\ \ 89''\cdot406\cos\lambda.$$

21. Since $\log a = 1\cdot2829093$, and $\log n = 1\cdot6256531$ and $\log \dfrac{m'n}{\omega} = \bar{3}\cdot4090935\ -$, we have,

$$\frac{2m'na^2P}{\omega} = +\,0\cdot003720466, \qquad \frac{2m'na^2P'}{\omega} = -\,0\cdot000560276,$$

$$\text{and}\quad \frac{3m'n^2aP}{\omega} = +\,0''\cdot012286620, \qquad \frac{3m'n^2aP'}{\omega} = -\,0''\cdot001850278\,;$$

and since $\log a' = 1\cdot4776479$, $\log n' = 1\cdot3335464$, and $\log \dfrac{mn'}{\omega} = \bar{3}\cdot0684631\ -$, we have

$$\frac{4mn'a'^2Q}{\omega} = +\,0\cdot008369576, \qquad \frac{4mn'a'^2Q'}{\omega} = -\,0\cdot001207680,$$

$$\frac{6mn'^2a'Q}{\omega} = +\,0''\cdot009009346, \qquad \frac{6mn'^2a'Q'}{\omega} = -\,0''\cdot001299995.$$

Therefore,

$$\begin{aligned}
\delta n &= +\,0''\cdot001850278 \sin\lambda + 0''\cdot\,012286620 \cos\lambda,\\
\delta a &= -\,0\ \ \cdot000560276 \sin\lambda - 0\ \ \cdot003720466 \cos\lambda,\\
\delta n' &= -\,0''\cdot001299995 \sin\lambda - 0''\cdot009009346 \cos\lambda,\\
\delta a' &= +\,0\ \ \cdot001207680 \sin\lambda + 0\ \ \cdot008369576 \cos\lambda.
\end{aligned}$$

22. Since $\log\tan\frac{1}{2}\gamma = \bar{2}\cdot11894$, we have

$$\frac{m'nae\tan\frac{1}{2}\gamma}{\omega\sin 1''}\,\frac{dP}{de} = +\,0''\cdot2511, \qquad \frac{m'nae\tan\frac{1}{2}\gamma}{\omega\sin 1''}\,\frac{dP'}{de} = -\,0''\cdot0525,$$

$$\frac{m'na\tan\frac{1}{2}\gamma}{\omega\sin 1''}\,P = +\,0''\cdot2630, \qquad \frac{m'na\tan\frac{1}{2}\gamma}{\omega\sin 1''}\,P' = -\,0''\cdot0396,$$

$$\text{and}\quad \frac{mn'a'e'\tan\frac{1}{2}\gamma'}{\omega\sin 1''}\,\frac{dQ}{de'} = +\,0''\cdot0095, \qquad \frac{mn'a'e'\tan\frac{1}{2}\gamma'}{\omega\sin 1''}\,\frac{dQ'}{de'} = +\,0''\cdot0103,$$

$$\frac{2mn'a'\tan\frac{1}{2}\gamma'}{\omega\sin 1''}\,Q = +\,0''\cdot3779, \qquad \frac{2mn'a'\tan\frac{1}{2}\gamma'}{\omega\sin 1''}\,Q' = -\,0''\cdot0545\,;$$

$$\begin{aligned}
\therefore\ \ \delta\gamma &= +\,0''\cdot092 \sin\lambda + 0''\cdot514 \cos\lambda,\\
\delta\Pi &= \quad 0,\\
\delta\gamma' &= -\,0''\cdot065 \sin\lambda - 0''\cdot368 \cos\lambda,\\
\delta\Pi' &= \quad 0.
\end{aligned}$$

23. The part of R of the third order involving λ is, (Art. 8)

$$R = m'\,(P_1\cos\lambda - P_1'\sin\lambda),$$

$$\begin{aligned}
\text{where}\ \ aP_1 = {} & a\,(M_3e^3 + M_4ee'^2 + M_5e\sin^2\tfrac{1}{2}\gamma')\cos\varpi,\\
& + a\,(M_6e'^3 + M_7e^2e' + M_8e'\sin^2\tfrac{1}{2}\gamma')\cos\varpi'.\\
& + aM_9e^2e'\cos(2\varpi - \varpi'),\\
& + aM_{10}ee'^2\cos(2\varpi' - \varpi),\\
& + aM_{11}e\sin^2\tfrac{1}{2}\gamma'\cos(2\Pi' - \varpi),\\
& + aM_{12}e'\sin^2\tfrac{1}{2}\gamma'\cos(2\Pi' - \varpi'),
\end{aligned}$$

and aP_1' = same with sines for cosines.

Also $R' = m\,(Q_1 \cos\lambda - Q_1' \sin\lambda)$,

where $a'Q_1 = \dfrac{aP_1}{\alpha} + \left(\alpha - \dfrac{1}{4\alpha^2}\right)\left(\dfrac{3}{2}e'^3 + e^2e' + 2e' \sin^2 \tfrac{1}{2}\gamma\right) \cos \varpi'$,

$a'Q_1'$ = same with sines for cosines.

The equations for Uranus will be

$$\delta\,(nt + \epsilon) = \left(\frac{3m'n^2aP_1}{\omega^2 \sin 1''} + \frac{2m'na^2}{\omega \sin 1''}\frac{dP_1}{da} - \frac{m'nae}{2\omega \sin 1''}\frac{dP_1}{de}\right) \sin\lambda,$$

$$+ \text{(same with } P_1' \text{ for } P_1) \cos\lambda,$$

$$\delta e = \frac{m'na}{e\omega \sin 1''}\frac{dP_1}{d\varpi} \sin\lambda + \frac{m'na}{e\omega \sin 1''}\frac{dP_1'}{d\varpi} \cos\lambda,$$

$$e\delta\varpi = -\frac{m'na}{\omega \sin 1''}\frac{dP_1}{de} \sin\lambda - \frac{m'na}{\omega \sin 1''}\frac{dP_1'}{de} \cos\lambda,$$

$$\delta n = -\frac{3m'n^2aP_1'}{\omega} \sin\lambda + \frac{3m'n^2aP_1}{\omega} \cos\lambda,$$

$$\delta a = \frac{2m'na^2P_1'}{\omega} \sin\lambda - \frac{2m'na^2P_1}{\omega} \cos\lambda,$$

$$\delta\gamma = \frac{m'n}{\omega \sin 1''}\frac{a}{\sin\gamma}\frac{dP_1}{d\Pi} \sin\lambda + \frac{m'n}{\omega \sin 1''}\frac{a}{\sin\gamma}\frac{dP_1'}{d\Pi} \cos\lambda,$$

$$\delta\Pi = -\frac{m'n}{\omega \sin 1''}\frac{a}{\sin\gamma}\frac{dP_1}{d\gamma} \sin\lambda - \frac{m'n}{\omega \sin 1''}\frac{a}{\sin\gamma}\frac{dP_1'}{d\gamma} \cos\lambda,$$

and those for Neptune are,

$$\delta\,(n't + \epsilon') = \left(-\frac{6mn'^2a'Q_1}{\omega^2 \sin 1''} + \frac{2mn'a'^2}{\omega \sin 1''}\frac{dQ_1}{da'} - \frac{mn'a'e'}{2\omega \sin 1''}\frac{dQ_1}{de'}\right) \sin\lambda,$$

$$+ \text{(same with } Q_1' \text{ for } Q_1) \cos\lambda,$$

$$\delta e' = \frac{mn'a'}{e'\omega \sin 1''}\frac{dQ_1}{d\varpi'} \sin\lambda + \frac{mn'a'}{e'\omega \sin 1''}\frac{dQ_1'}{d\varpi'} \cos\lambda,$$

$$e'\delta\varpi' = -\frac{mn'a'}{\omega \sin 1''}\frac{dQ_1}{de'} \sin\lambda - \frac{mn'a'}{\omega \sin 1''}\frac{dQ_1'}{de'} \cos\lambda,$$

$$\delta n' = \frac{6mn'^2a'Q_1'}{\omega} \sin\lambda - \frac{6mn'^2a'Q_1}{\omega} \cos\lambda,$$

$$\delta a' = -\frac{4mn'a'^2Q_1'}{\omega} \sin\lambda + \frac{4mn'a'^2Q_1}{\omega} \cos\lambda,$$

$$\delta\gamma' = \frac{mn'}{\omega \sin 1''}\frac{a'}{\sin\gamma'}\frac{dQ_1}{d\Pi'} \sin\lambda + \frac{mn'}{\omega \sin 1''}\frac{a'}{\sin\gamma'}\frac{dQ_1'}{d\Pi'} \cos\lambda,$$

$$\delta\Pi' = -\frac{mn'}{\omega \sin 1''}\frac{a'}{\sin\gamma'}\frac{dQ_1}{d\gamma'} \sin\lambda - \frac{mn'}{\omega \sin 1''}\frac{a'}{\sin\gamma'}\frac{dQ_1'}{d\gamma'} \cos\lambda.$$

24. $\log aP_1 = \bar{5}\cdot29907, \qquad \log aP_1' = \bar{5}\cdot37328 -;$

$$\therefore \frac{3m'n^2aP_1}{\omega^2 \sin 1''} = +1''\cdot5223, \qquad \frac{3m'n^2aP_1'}{\omega^2 \sin 1''} = -1''\cdot8060.$$

$$\log a'Q_1 = \bar{5}\cdot49927, \qquad \log a'Q_1' = \bar{5}\cdot56277 -;$$

$$\therefore \frac{6mn'^2a'Q_1}{\omega^2 \sin 1''} = +1''\cdot1246, \qquad \frac{6mn'^2a'Q_1'}{\omega^2 \sin 1''} = -1''\cdot3016.$$

$$\log a^2 \frac{dP_1}{da} = \bar{4}\cdot18441 -, \qquad \log a^2 \frac{dP_1'}{da} = \bar{5}\cdot52892 -;$$

$$\therefore \frac{2m'na^2}{\omega \sin 1''} \frac{dP_1}{da} = +0''\cdot1618, \qquad \frac{2m'na^2}{\omega \sin 1''} \frac{dP_1'}{da} = +0''\cdot0358.$$

$$\begin{aligned} ae\frac{dP_1}{de} = {} & (3aM_3e^3 + aM_4ee'^2 + aM_5e \sin^2 \tfrac{1}{2}\gamma') \cos \varpi, \\ & + 2aM_7e^2e' \cos \varpi' + 2aM_9e^2e' \cos (2\varpi - \varpi'), \\ & + aM_{10}ee'^2 \cos (2\varpi' - \varpi) + aM_{11}e \sin^2 \tfrac{1}{2}\gamma' \cos (2\Pi' - \varpi), \end{aligned}$$

$ae\dfrac{dP_1'}{de}$ = same with sines for cosines;

$$\therefore \log ae\frac{dP_1}{de} = \bar{5}\cdot5681 -, \qquad \log ae\frac{dP_1'}{de} = \bar{5}\cdot7347 -;$$

$$\therefore \frac{m'nae}{2\omega \sin 1''} \frac{dP_1}{de} = +0''\cdot0098, \qquad \frac{m'nae}{2\omega \sin 1''} \frac{dP_1'}{de} = +0'',0144,$$

$$\begin{aligned} a'^2\frac{dQ_1}{da'} = {} & -\frac{1}{\alpha}\left(a^2\frac{dP_1}{da} + aP_1\right) \\ & - \left(\alpha + \frac{1}{8\alpha^2}\right)(3e'^3 + 2e^2e' + 4e' \sin^2 \tfrac{1}{2}\gamma) \cos \varpi', \end{aligned}$$

$a'^2\dfrac{dQ_1'}{da'}$ = same with sines for cosines.

$$\therefore \log a'^2\frac{dQ_1}{da'} = \bar{4}\cdot25108, \qquad \log a'^2\frac{dQ_1'}{da'} = \bar{5}\cdot7634;$$

$$\therefore \frac{2mn'a'^2}{\omega \sin 1''} \frac{dQ_1}{da'} = -0''\cdot0861, \qquad \frac{2mn'a'^2}{\omega \sin 1''} \frac{dQ_1'}{da'} = -0''\cdot0280.$$

$$a'e'\frac{dQ_1}{de'} = \frac{ae'}{\alpha} \frac{dP_1}{de'} + \left(\alpha - \frac{1}{4\alpha^2}\right)\left(\frac{9}{2}e'^3 + e^2e' + 2e' \sin^2 \tfrac{1}{2}\gamma\right) \cos \varpi',$$

$a'e'\dfrac{dQ_1'}{de'}$ = same with sines for cosines,

$$\begin{aligned} ae'\frac{dP_1}{de'} = {} & 2aM_4ee'^2 \cos \varpi + (3aM_6e'^3 + aM_7e^2e' + aM_8e' \sin^2 \tfrac{1}{2}\gamma) \cos \varpi' \\ & + aM_9e^2e' \cos (2\varpi - \varpi') + 2aM_{10}ee'^2 \cos (2\varpi' - \varpi) \\ & + aM_{12}e' \sin^2 \tfrac{1}{2}\gamma \cos (2\Pi - \varpi'); \end{aligned}$$

$$\therefore \log a'e'\frac{dQ_1}{de'} = \bar{6}\cdot8785 -, \qquad \log a'e'\frac{dQ_1'}{de'} = \bar{5}\cdot6900 -,$$

$$\therefore \frac{mn'a'e'}{2\omega \sin 1''} \frac{dQ_1}{de'} = +0''\cdot0009, \qquad \frac{mn'a'e'}{2\omega \sin 1''} \frac{dQ_1'}{de'} = +0''\cdot0059,$$

Hence,

$$\delta(nt+\epsilon) = +1''{\cdot}674 \sin\lambda - 1''{\cdot}785 \cos\lambda,$$
$$\delta(n't+\epsilon') = -1''{\cdot}212 \sin\lambda + 1''{\cdot}268 \cos\lambda.$$

25. $$a\frac{dP_1}{d\varpi} = -a\left(M_3e^3 + M_4ee'^2 + M_5e\sin^2\tfrac{1}{2}\gamma'\right)\sin\varpi,$$
$$-2aM_9e^2e'\sin(2\varpi-\varpi')$$
$$+aM_{10}ee'^2\sin(2\varpi'-\varpi)$$
$$+aM_{11}e\sin^2\tfrac{1}{2}\gamma'\sin(2\Pi'-\varpi),$$

$a\dfrac{dP_1'}{d\varpi} = -$ the same with cosines for sines.

$$\therefore\ \log a\frac{dP_1}{d\varpi} = \bar{5}{\cdot}53656, \qquad \log a\frac{dP_1'}{d\varpi} = \bar{5}{\cdot}81345,$$

$$\therefore\ \frac{m'na}{e\omega\sin 1''}\frac{dP_1}{d\varpi} = -0''{\cdot}3899, \qquad \frac{m'na}{e\omega\sin 1''}\frac{dP_1'}{d\varpi} = -0''{\cdot}7376,$$

and by last Art.

$$\frac{m'na}{\omega\sin 1''}\frac{dP_1}{de} = +0''{\cdot}4193, \qquad \frac{m'na}{\omega\sin 1''}\frac{dP_1'}{de} = +0''{\cdot}6153.$$

Again,

$$a'\frac{dQ_1}{d\varpi'} = \frac{a}{a}\frac{dP_1}{d\varpi'} - \left(a - \frac{1}{4a^2}\right)\left(\frac{3}{2}e'^3 + e^2e' + 2e'\sin^2\tfrac{1}{2}\gamma\right)\sin\varpi',$$

$a'\dfrac{dQ_1'}{d\varpi'} = -$ same with cosines for sines,

$$a\frac{dP_1}{d\varpi'} = -a\left(M_6e'^3 + M_7e^2e' + M_8e'\sin^2\tfrac{1}{2}\gamma'\right)\sin\varpi'$$
$$+aM_9e^2e'\sin(2\varpi-\varpi') - 2aM_{10}ee'^2\sin(2\varpi'-\varpi)$$
$$+aM_{12}e'\sin^2\tfrac{1}{2}\gamma'\sin(2\Pi'-\varpi').$$

Hence $\log a'\dfrac{dQ_1}{d\varpi'} = \bar{5}{\cdot}31765,$ $\qquad \log a'\dfrac{dQ_1'}{d\varpi'} = \bar{5}{\cdot}32552\ -,$

$$\therefore\ \frac{mn'a'}{e'\omega\sin 1''}\frac{dQ_1}{d\varpi'} = -0''{\cdot}5755, \qquad \frac{mn'a'}{e'\omega\sin 1''}\frac{dQ_1'}{d\varpi'} = +0''{\cdot}5860,$$

and by last Article,

$$\frac{mn'a'}{\omega\sin 1''}\frac{dQ_1}{de'} = +0''{\cdot}2094, \qquad \frac{mn'a'}{\omega\sin 1''}\frac{dQ_1'}{de'} = +1''{\cdot}3565.$$

Therefore,

$$\delta e = -0''{\cdot}390\sin\lambda - 0''{\cdot}738\cos\lambda,$$
$$e\delta\varpi = -0''{\cdot}419\sin\lambda - 0''{\cdot}615\cos\lambda,$$
$$\delta e' = -0''{\cdot}575\sin\lambda + 0''{\cdot}586\cos\lambda,$$
$$e'\delta\varpi' = -0''{\cdot}209\sin\lambda - 1''{\cdot}356\cos\lambda.$$

26. By means of the values of $\log a$, $\log a'$, $\log n$, $\log n'$. $\log \frac{m'n}{\omega}$, and $\log \frac{mn'}{\omega}$, given in (Art. 21), we shall find

$$\delta n = -\,0''{\cdot}000007676 \sin\lambda - 0''{\cdot}000006470 \cos\lambda,$$
$$\delta a = +\,0{\cdot}000002324 \sin\lambda + 0{\cdot}000001959 \cos\lambda,$$
$$\delta n' = +\,0''{\cdot}000005533 \sin\lambda + 0''{\cdot}000004780 \cos\lambda,$$
$$\delta a' = -\,0{\cdot}000005140 \sin\lambda - 0{\cdot}000004441 \cos\lambda.$$

27.
$$a\frac{dP_1}{d\Pi} = -\,2aM_{11}e\sin^2\tfrac{1}{2}\gamma\sin(2\Pi - \varpi)$$
$$-\,2aM_{12}e'\sin^2\tfrac{1}{2}\gamma\sin(2\Pi - \varpi'),$$
$$a\frac{dP_1'}{d\Pi} = -\text{ same with cosines for sines.}$$
$$\therefore\ \log a\frac{dP_1}{d\Pi} = \bar{5}{\cdot}38632\,-, \qquad \log a\frac{dP_1'}{d\Pi} = \bar{5}{\cdot}49693\,-,$$
$$\therefore\ \frac{m'n}{\omega\sin 1''}\,\frac{a}{\sin\gamma}\,\frac{dP_1}{d\Pi} = +\,0''{\cdot}4897, \qquad \frac{m'n}{\omega\sin 1''}\,\frac{a}{\sin\gamma}\,\frac{dP_1'}{d\Pi} = +\,0''{\cdot}6318,$$
$$\frac{a}{\sin\gamma}\,\frac{dP_1}{d\gamma} = \tfrac{1}{2}aM_9e\cos\varpi + \tfrac{1}{2}aM_8e'\cos\varpi'$$
$$+\,\tfrac{1}{2}aM_{11}e\cos(2\Pi - \varpi) + \tfrac{1}{2}aM_{12}e'\cos(2\Pi - \varpi'),$$
$$\frac{a}{\sin\gamma}\,\frac{dP_1'}{d\gamma} = \text{same with sines for cosines,}$$
$$\therefore\ \log\frac{a}{\sin\gamma}\,\frac{dP_1}{d\gamma} = \bar{1}{\cdot}1679648, \qquad \log\frac{a}{\sin\gamma}\,\frac{dP_1'}{d\gamma} = \bar{2}{\cdot}3363036$$
$$\therefore\ \frac{m'n}{\omega\sin 1''}\,\frac{a}{\sin\gamma}\,\frac{dP_1}{d\gamma} = -\,77''{\cdot}8903, \qquad \frac{m'n}{\omega\sin 1''}\,\frac{a}{\sin\gamma}\,\frac{dP_1'}{d\gamma} = -\,11''{\cdot}4768,$$
$$\frac{a'}{\sin\gamma'}\,\frac{dQ_1}{d\Pi'} = \frac{1}{\alpha}\,\frac{a}{\sin\gamma}\,\frac{dP_1}{d\Pi}, \qquad \frac{a'}{\sin\gamma'}\,\frac{dQ_1'}{d\Pi'} = \frac{1}{\alpha}\,\frac{a}{\sin\gamma}\,\frac{dP_1'}{d\Pi},$$
$$\therefore\ \frac{mn'}{\omega\sin 1''}\,\frac{a'}{\sin\gamma'}\,\frac{dQ_1}{d\Pi'} = +\,0''{\cdot}3500, \qquad \frac{mn'}{\omega\sin 1''}\,\frac{a'}{\sin\gamma'}\,\frac{dQ_1'}{d\Pi'} = +\,0''{\cdot}4515,$$
$$\frac{a'}{\sin\gamma'}\,\frac{dQ_1}{d\gamma'} = \frac{1}{\alpha}\,\frac{a}{\sin\gamma}\,\frac{dP_1}{d\gamma} + \left(\alpha - \frac{1}{4\alpha^2}\right)e'\cos\varpi',$$
$$\frac{a'}{\sin\gamma'}\,\frac{dQ_1'}{d\gamma'} = \text{same with sines for cosines,}$$
$$\therefore\ \log\frac{a'}{\sin\gamma'}\,\frac{dQ_1}{d\gamma'} = \bar{1}{\cdot}36299, \qquad \log\frac{a'}{\sin\gamma'}\,\frac{dQ_1'}{d\gamma'} = \bar{2}{\cdot}53314,$$
$$\therefore\ \frac{mn'}{\omega\sin 1''}\,\frac{a'}{\sin\gamma'}\,\frac{dQ_1}{d\gamma'} = -\,55''{\cdot}703, \qquad \frac{mn'}{\omega\sin 1''}\,\frac{a'}{\sin\gamma'}\,\frac{dQ_1'}{d\gamma'} = -\,8''{\cdot}2420,$$
$$\therefore\ \delta\gamma = +\,0''{\cdot}490\sin\lambda + 0''{\cdot}632\cos\lambda,$$
$$\delta\Pi = +\,77''{\cdot}890\sin\lambda + 11''{\cdot}477\cos\lambda,$$
$$\delta\gamma' = +\,0''{\cdot}350\sin\lambda + 0''{\cdot}451\cos\lambda,$$
$$\delta\Pi' = +\,55''{\cdot}703\sin\lambda + 8''{\cdot}242\cos\lambda.$$

28. The terms of the second order involving 2λ, are by (Art. 8),

$$R_2 = m'(P_2 \cos 2\lambda - P_2' \sin 2\lambda),$$

where
$$P_2 = N_1 e^2 \cos 2\varpi + N_2 e'^2 \cos 2\varpi' + N_3 ee' \cos(\varpi + \varpi') + N_4 \sin^2 \tfrac{1}{2}\gamma' \cos 2\Pi',$$

$$P_2' = \text{same with sines for cosines.}$$

R_2' differs from R_2 only in having m for m'.

Then the equations for the variations of the elements of Uranus, are

$$\delta(nt + \epsilon) = \left(\frac{3m'n^2 aP_2}{2\omega^2 \sin 1''} + \frac{m'na^2}{\omega \sin 1''}\frac{dP_2}{da} - \frac{m'nae}{4\omega \sin 1''}\frac{dP_2}{de}\right)\sin 2\lambda$$

$$+ (\text{same with } P_2' \text{ for } P_2) \cos 2\lambda,$$

$$\delta e = \left(-\frac{m'na}{2\omega \sin 1''}\frac{dP_2'}{de} + \frac{m'nae^2}{4\omega \sin 1''}\frac{dP_2'}{de} - \frac{m'naeP_2'}{2\omega \sin 1''}\right)\sin 2\lambda$$

$$- (\text{same with } P_2 \text{ for } P_2') \cos 2\lambda,$$

$$e\delta\varpi = \left(-\frac{m'na}{2\omega \sin 1''}\frac{dP_2}{de} + \frac{m'nae^2}{4\omega \sin 1''}\frac{dP_2}{de}\right)\sin 2\lambda$$

$$+ (\text{same with } P_2' \text{ for } P_2) \cos 2\lambda,$$

$$\delta n = -\frac{3m'n^2 aP_2'}{\omega}\sin 2\lambda + \frac{3m'n^2 aP_2}{\omega}\cos 2\lambda,$$

$$\delta a = \frac{2m'na^2 P_2'}{\omega}\sin 2\lambda - \frac{2m'na^2 P_2}{\omega}\cos 2\lambda,$$

$$\delta\gamma = -\left(\frac{m'n \tan\frac{1}{2}\gamma}{\omega \sin 1'' \sqrt{1-e^2}}\frac{a}{\sin\gamma}\frac{dP_2'}{d\gamma} + \frac{m'n \tan\frac{1}{2}\gamma\, aP_2'}{\omega \sin 1''} + \frac{m'nae \tan\frac{1}{2}\gamma}{2\omega \sin 1''}\frac{dP_2'}{de}\right)\sin 2\lambda$$

$$+ (\text{same with } P_2 \text{ for } P_2') \cos 2\lambda,$$

$$\delta\Pi = -\frac{m'n}{8\omega \sin 1''}\frac{a}{\sqrt{1-e^2}\sin^2\frac{1}{2}\gamma}\frac{dP_2'}{d\Pi}\sin 2\lambda$$

$$+ \frac{m'n}{8\omega \sin 1''}\frac{a}{\sqrt{1-e^2}\sin^2\frac{1}{2}\gamma}\frac{dP_2}{d\Pi}\cos 2\lambda,$$

and those for Neptune are

$$\delta(n't + \epsilon') = \left(-\frac{3mn'^2 a'P_2}{\omega^2 \sin 1''} + \frac{mn'a'^2}{\omega \sin 1''}\frac{dP_2}{da'} - \frac{mn'a'e'}{4\omega \sin 1''}\frac{dP_2}{de'}\right)\sin 2\lambda$$

$$+ (\text{same with } P_2' \text{ for } P_2) \cos 2\lambda,$$

$$\delta e' = \left(-\frac{mn'a'}{2\omega \sin 1''}\frac{dP_2'}{de'} + \frac{mn'a'e'^2}{4\omega \sin 1''}\frac{dP_2'}{de'} + \frac{mn'a'e'P_2'}{\omega \sin 1''}\right)\sin 2\lambda$$

$$- (\text{same with } P_2 \text{ for } P_2') \cos 2\lambda,$$

$$e'\delta\varpi' = \left(-\frac{mn'a'}{2\omega \sin 1''}\frac{dP_2}{de'} + \frac{mn'a'e'^2}{4\omega \sin 1''}\frac{dP_2}{de'}\right)\sin 2\lambda$$

$$+ (\text{same with } P_2' \text{ for } P_2) \cos 2\lambda,$$

$$\delta n' = \frac{6mn'^2 a'P_2'}{\omega}\sin 2\lambda - \frac{6mn'^2 a'P_2}{\omega}\cos 2\lambda,$$

$$\delta a' = -\frac{4mn'a'^2P_2'}{\omega}\sin 2\lambda + \frac{4mn'a'^2P_2}{\omega}\cos 2\lambda,$$

$$\delta\gamma' = \left(-\frac{mn'\tan\frac{1}{2}\gamma'}{\omega\sin 1''\sqrt{1-e'^2}}\,\frac{a'}{\sin\gamma'}\,\frac{dP_2'}{d\gamma'} + \frac{2mn'\tan\frac{1}{2}\gamma' a'P_2'}{\omega\sin 1''}\right.$$
$$\left. - \frac{mn'a'e'\tan\frac{1}{2}\gamma'}{2\omega\sin 1''}\,\frac{dP_2'}{de'}\right)\sin 2\lambda$$
$$-\,(\text{same with } P_2 \text{ for } P_2')\cos 2\lambda,$$

$$\delta\Pi' = -\frac{mn'a'}{8\omega\sin 1''\sqrt{1-e'^2\sin^2\frac{1}{2}\gamma'}}\,\frac{dP_2'}{d\Pi'}\sin 2\lambda + \frac{mn'a'}{8\omega\sin 1''\sqrt{1-e'^2\sin^2\frac{1}{2}\gamma'}}\,\frac{dP_2}{d\Pi'}\cos 2\lambda.$$

29.

$$\log aP_2 = \bar{3}\cdot 5412437\,-, \qquad \log aP_2' = \bar{4}\cdot 0623939,$$

$$\therefore\ \frac{3m'n^2aP_2}{2\omega^2\sin 1''} = -\,132''\cdot 9360, \qquad \frac{3m'n^2aP_2'}{2\omega^2\sin 1''} = +\,4''\cdot 4136,$$

$$\frac{3mn'^2a'P_2}{\omega^2\sin 1''} = -\,96''\cdot 9782, \qquad \frac{3mn'^2a'P_2'}{\omega^2\sin 1''} = +\,3''\cdot 2198,$$

$$\log a^2\frac{dP_2}{da} = \bar{2}\cdot 2149551\,-, \qquad \log a^2\frac{dP_2'}{da} = \bar{3}\cdot 2587211,$$

$$\therefore\ \frac{m'na^2}{\omega\sin 1''}\,\frac{dP_2}{da} = +\,8''\cdot 6791, \qquad \frac{m'na^2}{\omega\sin 1''}\,\frac{dP_2'}{da} = -\,0''\cdot 9599,$$

$$a'^2\frac{dP_2}{da'} = -\frac{1}{\alpha}\left(a^2\frac{dP_2}{da} + aP_2\right), \qquad a'^2\frac{dP_2'}{da'} = -\frac{1}{\alpha}\left(a^2\frac{dP_2'}{da} + aP_2'\right),$$

$$\therefore\ \frac{mn'a'^2}{\omega\sin 1''}\,\frac{dP_2}{da'} = -\,7''\cdot 5176, \qquad \frac{mn'a'^2}{\omega\sin 1''}\,\frac{dP_2'}{da'} = +\,0''\cdot 7297,$$

$$ae\frac{dP_2}{de} = 2aN_1e^2\cos 2\varpi + aN_3ee'\cos(\varpi+\varpi')$$

$$ae\frac{dP_2'}{de} = \text{same with sines for cosines},$$

$$\therefore\ \log ae\frac{dP_2}{de} = \bar{3}\cdot 7589679\,-, \qquad \log ae\frac{dP_2'}{de} = \bar{3}\cdot 0914348;$$

$$\therefore\ \frac{m'nae}{4\omega\sin 1''}\,\frac{dP_2}{de} = +\,0''\cdot 7593, \qquad \frac{m'nae}{4\omega\sin 1''}\,\frac{dP_2'}{de} = -\,0''\cdot 1633$$

$$a'e'\frac{dP_2}{de'} = \frac{1}{\alpha}\left\{2aN_2e'^2\cos 2\varpi' + aN_3ee'\cos(\varpi+\varpi')\right\},$$

$$a'e'\frac{dP_2'}{de'} = \text{same with sines for cosines.}$$

$$\therefore\ \frac{mn'a'e'}{4\omega\sin 1''}\,\frac{dP_2}{de'} = +\,0''\cdot 1025, \qquad \frac{mn'a'e'}{4\omega\sin 1''}\,\frac{dP_2'}{de'} = +\,0''\cdot 1089;$$

$$\therefore\ \delta(nt+\epsilon) = -\,125''\cdot 016\sin 2\lambda + 3''\cdot 617\cos 2\lambda,$$

$$\delta(n't+\epsilon') = +\ \ 89''\cdot 358\sin 2\lambda - 2''\cdot 599\cos 2\lambda,$$

30. By last Article,

$$\log a\frac{dP_2}{de} = \bar{1}{\cdot}0898429\ -, \qquad \log a\frac{dP_2'}{de} = \bar{2}{\cdot}4223098,$$

$$\therefore\ \frac{m'na}{2\omega\sin 1''}\,\frac{dP_2}{de} = +\ 32''{\cdot}5335, \qquad \frac{m'na}{2\omega\sin 1''}\,\frac{dP_2'}{de} = -\ 6''{\cdot}9952,$$

$$\frac{m'nae^2}{4\omega\sin 1''}\,\frac{dP_2}{de} = +\ \ 0''{\cdot}0354, \qquad \frac{m'nae^2}{4\omega\sin 1''}\,\frac{dP_2'}{de} = -\ 0''{\cdot}0076,$$

$$\frac{m'naeP_2}{2\omega\sin 1''} = +\ \ 0''{\cdot}0429, \qquad \frac{m'naeP_2'}{2\omega\sin 1''} = -\ 0''{\cdot}0014,$$

$$\log a'\frac{dP_2}{de'} = \bar{1}{\cdot}2892462\ -, \qquad \log a'\frac{dP_2'}{de'} = \bar{1}{\cdot}3157694\ -;$$

$$\therefore\ \frac{mn'a'}{2\omega\sin 1''}\,\frac{dP_2}{de'} = +\ 23''{\cdot}5020, \qquad \frac{mn'a'}{2\omega\sin 1''}\,\frac{dP_2'}{de'} = +\ 24''{\cdot}9820,$$

$$\frac{mn'a'e'^2}{4\omega\sin 1''}\,\frac{dP_2}{de'} = +\ \ 0''{\cdot}0009, \qquad \frac{mn'a'e'^2}{4\omega\sin 1''}\,\frac{dP_2'}{de'} = +\ \ 0''{\cdot}0010,$$

$$\frac{mn'a'e'P_2}{\omega\sin 1''} = +\ \ 0''{\cdot}0115, \qquad \frac{mn'a'e'P_2'}{\omega\sin 1''} = -\ \ 0''{\cdot}0004;$$

$$\therefore\ \delta e = +\ \ 6''.989\sin 2\lambda + 32''{\cdot}541\cos 2\lambda,$$

$$e\delta\varpi = -\ 32''{\cdot}498\sin 2\lambda +\ \ 6''{\cdot}988\cos 2\lambda,$$

$$\delta e' = -\ 24''{\cdot}981\sin 2\lambda + 23''{\cdot}490\cos 2\lambda,$$

$$e'\delta\varpi' = -\ 23''{\cdot}501\sin 2\lambda - 24''{\cdot}981\cos 2\lambda.$$

31.

$$\delta n = +\ 0''{\cdot}000037520\sin 2\lambda + 0''{\cdot}001130086\cos 2\lambda,$$

$$\delta a = -\ 0\ \ {\cdot}000011361\sin 2\lambda - 0\ \ {\cdot}000342197\cos 2\lambda,$$

$$\delta n' = -\ 0''{\cdot}000027371\sin 2\lambda - 0''{\cdot}000824410\cos 2\lambda,$$

$$\delta a' = +\ 0\ \ {\cdot}000025427\sin 2\lambda + 0\ \ {\cdot}000765867\cos 2\lambda.$$

32.

$$a\frac{dP_2}{d\Pi} = -\ 2aN_4\sin^2\tfrac{1}{2}\gamma\,\sin 2\Pi,$$

$$a\frac{dP_2'}{d\Pi} = \quad 2aN_4\sin^2\tfrac{1}{2}\gamma\,\cos 2\Pi;$$

$$\therefore\ \log a\frac{dP_2}{d\Pi} = \bar{4}{\cdot}1724278\ -, \qquad \log a\frac{dP_2'}{d\Pi} = \bar{4}{\cdot}1137830\ -,$$

$$\therefore\ \frac{mn'}{8\sqrt{1-e^2}\,\omega\sin 1''}\,\frac{a}{\sin^2\frac{1}{2}\gamma}\,\frac{dP_2}{d\Pi} = +\ 56''{\cdot}954, \qquad \frac{mn'}{8\sqrt{1-e^2}\,\omega\sin 1''}\,\frac{a}{\sin^2\frac{1}{2}\gamma}\,\frac{dP_2'}{d\Pi} = +\ 49''{\cdot}760,$$

$$\text{also}\ \frac{a}{\sin\gamma}\,\frac{dP_2}{d\gamma} = \frac{a}{4\sin^2\frac{1}{2}\gamma}\,\frac{dP_2'}{d\Pi}, \qquad \frac{a}{\sin\gamma}\,\frac{dP_2'}{d\gamma} = -\ \frac{a}{4\sin^2\frac{1}{2}\gamma}\,\frac{dP_2}{d\Pi};$$

$$\therefore\ \frac{m'n\tan\frac{1}{2}\gamma}{\sqrt{1-e^2}\,\omega\sin 1''}\,\frac{a}{\sin\gamma}\,\frac{dP_2}{d\gamma} = +\ 1''{\cdot}3087,$$

$$\frac{m'n \tan \frac{1}{2}\gamma}{\sqrt{1-e^2}\,\omega \sin 1''} \, \frac{a}{\sin\gamma} \, \frac{dP_2'}{d\gamma} = -1''\cdot 4979,$$

and, omitting the multiplier $\dfrac{1}{\sqrt{1-e'^2}}$ which $=1$ very nearly,

$$\frac{mn'}{8\omega \sin 1''} \, \frac{a'}{\sin^2 \frac{1}{2}\gamma'} \, \frac{dP_2}{d\Pi'} = +40''\cdot 6591, \qquad \frac{mn'}{8\omega \sin 1''} \, \frac{a'}{\sin^2 \frac{1}{2}\gamma'} \, \frac{dP_2'}{d\Pi'} = +35''\cdot 5233,$$

$$\frac{mn' \tan \frac{1}{2}\gamma'}{\omega \sin 1''} \, \frac{a'}{\sin\gamma'} \, \frac{dP_2}{d\gamma'} = +0''\cdot 9343, \qquad \frac{mn' \tan \frac{1}{2}\gamma'}{\omega \sin 1''} \, \frac{a'}{\sin\gamma'} \, \frac{dP_2'}{d\gamma'} = -1''\cdot 0694,$$

$$\frac{m'n \tan \frac{1}{2}\gamma a P_2}{\omega \sin 1''} = +0''\cdot 0242, \qquad \frac{m'n \tan \frac{1}{2}\gamma a P_2'}{\omega \sin 1''} = -0''\cdot 0008,$$

$$\frac{m'nae \tan \frac{1}{2}\gamma}{2\omega \sin 1''} \, \frac{dP_2}{de} = +0''\cdot 0200, \qquad \frac{m'nae \tan \frac{1}{2}\gamma}{2\omega \sin 1''} \, \frac{dP_2'}{de} = -0''\cdot 0043,$$

$$\frac{2mn' \tan \frac{1}{2}\gamma' a' P_2}{\omega \sin 1''} = +0''\cdot 0346, \qquad \frac{2mn' \tan \frac{1}{2}\gamma' a' P_2'}{\omega \sin 1''} = -0''\cdot 0011,$$

$$\frac{mn'a'e' \tan \frac{1}{2}\gamma'}{2\omega \sin 1''} \, \frac{dP_2}{de'} = +0''\cdot 0027, \qquad \frac{mn'a'e' \tan \frac{1}{2}\gamma'}{2\omega \sin 1''} \, \frac{dP_2'}{de'} = +0''\cdot 0029.$$

Therefore,

$$\begin{aligned}
\delta\gamma &= +\ \ 1''\cdot 503 \sin 2\lambda + \ \ 1''\cdot 353 \cos 2\lambda,\\
\delta\Pi &= -49''\cdot 760 \sin 2\lambda + 56''\cdot 954 \cos 2\lambda,\\
\delta\gamma' &= +\ \ 1''\cdot 065 \sin 2\lambda + \ \ 0''\cdot 902 \cos 2\lambda,\\
\delta\Pi' &= -35''\cdot 523 \sin 2\lambda + 40''\cdot 659 \cos 2\lambda.
\end{aligned}$$

33. For terms of the third order involving 3λ,

$$R_3 = m'(P_3 \cos 3\lambda - P_3' \sin 3\lambda),$$

where
$$\begin{aligned}
P_3 = {} & K_1 e^3 \cos 3\varpi + K_2 e'^3 \cos 3\varpi' + K_3 e^2 e' \cos(2\varpi + \varpi'),\\
& + K_4 e e'^2 \cos(2\varpi' + \varpi) + K_5 e \sin^2 \tfrac{1}{2}\gamma' \cos(2\Pi' + \varpi)\\
& + K_6 e' \sin^2 \tfrac{1}{2}\gamma' \cos(2\Pi' + \varpi'),
\end{aligned}$$

$P_3' =$ same with sines for cosines.

R_3' differs from R_3 only in having m for m'.

The equations for Uranus are

$$\delta(nt+\epsilon) = \left(\frac{m'n^2aP_3}{\omega^2 \sin 1''} + \frac{2m'na^2}{3\omega \sin 1''} \, \frac{dP_3}{da} - \frac{m'nae}{6\omega \sin 1''} \, \frac{dP_3}{de}\right) \sin 3\lambda,$$

$$+ (\text{same with } P_3' \text{ for } P_3) \cos 3\lambda,$$

$$\delta e = -\frac{m'na}{3\omega \sin 1''} \, \frac{dP_3'}{de} \sin 3\lambda + \frac{m'na}{3\omega \sin 1''} \, \frac{dP_3}{de} \cos 3\lambda,$$

$$e\delta\varpi = -\frac{m'na}{3\omega \sin 1''} \, \frac{dP_3}{de} \sin 3\lambda - \frac{m'na}{3\omega \sin 1''} \, \frac{dP_3'}{de} \cos 3\lambda,$$

$$\delta n = -\frac{3m'n^2aP_3'}{\omega} \sin 3\lambda + \frac{3m'n^2aP_3}{\omega} \cos 3\lambda,$$

$$\delta a = \frac{2m'na^2P_3'}{\omega}\sin 3\lambda - \frac{2m'na^2P_3}{\omega}\cos 3\lambda,$$

$$\delta\gamma = \frac{m'na}{3\omega\sin\gamma\sin 1''}\,\frac{dP_3}{d\Pi}\sin 3\lambda + \frac{m'na}{3\omega\sin\gamma\sin 1''}\,\frac{dP_3'}{d\Pi}\cos 3\lambda,$$

$$\delta\Pi = -\frac{m'na}{12\omega\sin^2\frac{1}{2}\gamma\sin 1''}\,\frac{dP_3'}{d\Pi}\sin 3\lambda + \frac{m'na}{12\omega\sin^2\frac{1}{2}\gamma\sin 1''}\,\frac{dP_3}{d\Pi}\cos 3\lambda\,;$$

and those for Neptune are,

$$\delta(n't+\epsilon') = \left(-\frac{2mn'^2a'P_3}{\omega^2\sin 1''} + \frac{2mn'a'^2}{3\omega\sin 1''}\,\frac{dP_3}{da'} - \frac{mn'a'e'}{6\omega\sin 1''}\,\frac{dP_3}{de'}\right)\sin 3\lambda,$$

$$+\ (\text{same with } P_3' \text{ for } P_3)\cos 3\lambda,$$

$$\delta e' = -\frac{mn'a'}{3\omega\sin 1''}\,\frac{dP_3'}{de'}\sin 3\lambda + \frac{mn'a'}{3\omega\sin 1''}\,\frac{dP_3}{de'}\cos 3\lambda,$$

$$e'\delta\varpi' = -\frac{mn'a'}{3\omega\sin 1''}\,\frac{dP_3}{de'}\sin 3\lambda - \frac{mn'a'}{3\omega\sin 1''}\,\frac{dP_3'}{de'}\cos 3\lambda,$$

$$\delta n' = \frac{6mn'^2a'P_3'}{\omega}\sin 3\lambda - \frac{6mn'^2a'P_3}{\omega}\cos 3\lambda,$$

$$\delta a' = -\frac{4mn'a'^2P_3'}{\omega}\sin 3\lambda + \frac{4mn'a'^2P_3}{\omega}\cos 3\lambda,$$

$$\delta\gamma' = \frac{mn'a'}{3\omega\sin\gamma'\sin 1''}\,\frac{dP_3}{d\Pi'}\sin 3\lambda + \frac{mn'a'}{3\omega\sin\gamma'\sin 1''}\,\frac{dP_3'}{d\Pi'}\cos 3\lambda,$$

$$\delta\Pi' = -\frac{mn'a'}{12\omega\sin^2\frac{1}{2}\gamma'\sin 1''}\,\frac{dP_3'}{d\Pi'}\sin 3\lambda + \frac{mn'a'}{12\omega\sin^2\frac{1}{2}\gamma'\sin 1''}\,\frac{dP_3}{d\Pi'}\cos 3\lambda.$$

34. $\log aP_3 = \bar{4}\cdot5430867\ -,\qquad \log aP_3' = \bar{5}\cdot3263359\,;$

$$\therefore\ \frac{m'n^2aP_3}{\omega^2\sin 1''} = -\,8''\cdot9001,\qquad \frac{m'n^2aP_3'}{\omega^2\sin 1''} = +\,0''\cdot5403,$$

$$\frac{2mn'^2a'P_3}{\omega^2\sin 1''} = -\,6''\cdot4927,\qquad \frac{2mn'^2a'P_3'}{\omega^2\sin 1''} = +\,0''\cdot3942.$$

$$\log a^2\frac{dP_3}{da} = \bar{3}\cdot36994\ -,\qquad \log a^2\frac{dP_3'}{da} = \bar{4}\cdot53725\,;$$

$$\therefore\ \frac{2m'na^2}{3\omega\sin 1''}\,\frac{dP_3}{da} = +\,0''\cdot8267,\qquad \frac{2m'na^2}{3\omega\sin 1''}\,\frac{dP_3'}{da} = -\,0''\cdot1215.$$

$$a'^2\frac{dP_3}{da'} = -\frac{1}{a}\left(a^2\frac{dP_3}{da} + aP_3\right),\qquad a'^2\frac{dP_3'}{da'} = -\frac{1}{a}\left(a^2\frac{dP_3'}{da} + aP_3'\right);$$

$$\therefore\ \frac{2mn'a'^2}{3\omega\sin 1''}\,\frac{dP_3}{da'} = -\,0''\cdot6788,\qquad \frac{2mn'a'^2}{3\omega\sin 1''}\,\frac{dP_3'}{da'} = +\,0''\cdot0922.$$

$$ae\frac{dP_3}{de} = 3aK_1e^3\cos 3\varpi + 2aK_3e^2e'\cos(2\varpi+\varpi')$$

$$+\ aK_4ee'^2\cos(2\varpi'+\varpi) + aK_5e\sin^2\tfrac{1}{2}\gamma'\cos(2\Pi'+\varpi),$$

$$ae\frac{dP_3'}{de} = \text{same with sines for cosines};$$

$$\therefore\ \log ae\frac{dP_3}{de} = \bar{4}\cdot 93050\ -, \qquad \log ae\frac{dP_3'}{de} = \bar{4}\cdot 30300;$$

$$\therefore\ \frac{m'nae}{6\omega\sin 1''}\,\frac{dP_3}{de} = +\,0''\cdot 0751, \qquad \frac{m'nae}{6\omega\sin 1''}\,\frac{dP_3'}{de} = -\,0''\cdot 0177.$$

$$a'e'\frac{dP_3}{de'} = \frac{1}{a}\left\{\begin{array}{l} 3aK_2e'^3\cos 3\varpi' + aK_3e^2e'\cos(2\varpi+\varpi') \\ +\,2aK_4ee'^2\cos(2\varpi'+\varpi) + aK_6e'\sin^2\tfrac{1}{2}\gamma'\cos(2\Pi'+\varpi') \end{array}\right\},$$

$$a'e'\frac{dP_3'}{de'} = \text{same with sines for cosines.}$$

$$\therefore\ \frac{mn'a'e'}{6\omega\sin 1''}\,\frac{dP_3}{de'} = +\,0''\cdot 0105, \qquad \frac{mn'a'e'}{6\omega\sin 1''}\,\frac{dP_3'}{de'} = +\,0''\cdot 0107,$$

$$\therefore\ \delta(nt+\epsilon) = -\,8''\cdot 148\sin 3\lambda + 0''\cdot 436\cos 3\lambda,$$
$$\delta(n't+\epsilon') = +\,5''\cdot 803\sin 3\lambda - 0''\cdot 313\cos 3\lambda.$$

35. By the last Article,

$$\log a\frac{dP_3}{de} = \bar{2}\cdot 2613758\ -, \qquad \log a\frac{dP_3'}{de} = \bar{3}\cdot 6338766;$$

$$\therefore\ \frac{m'na}{3\omega\sin 1''}\,\frac{dP_3}{de} = +\,3''\cdot 2194, \qquad \frac{m'na}{3\omega\sin 1''}\,\frac{dP_3'}{de} = -\,0''\cdot 7591.$$

$$\log a'\frac{dP_3}{de'} = \bar{2}\cdot 4754806\ -, \qquad \log a'\frac{dP_3'}{de'} = \bar{2}\cdot 4836492\ -,$$

$$\therefore\ \frac{mn'a'}{3\omega\sin 1''}\,\frac{dP_3}{de'} = +\,2''\cdot 4057, \qquad \frac{mn'a'}{3\omega\sin 1''}\,\frac{dP_3'}{de'} = +\,2''\cdot 4514,$$

$$\therefore\ \delta e = +\,0''\cdot 759\sin 3\lambda + 3''\cdot 219\cos 3\lambda,$$
$$e\delta\varpi = -\,3''\cdot 219\sin 3\lambda + 0''\cdot 759\cos 3\lambda,$$
$$\delta e' = -\,2''\cdot 451\sin 3\lambda + 2''\cdot 406\cos 3\lambda,$$
$$e'\delta\varpi' = -\,2''\cdot 406\sin 3\lambda - 2''\cdot 451\cos 3\lambda.$$

36.

$$\delta n = +\,0''\cdot 000006890\sin 3\lambda + 0''\cdot 000113489\cos 3\lambda,$$
$$\delta a = -\,0\ \cdot 000002086\sin 3\lambda - 0\ \cdot 000034365\cos 3\lambda,$$
$$\delta n' = -\,0''\cdot 000005026\sin 3\lambda - 0''\cdot 000082791\cos 3\lambda,$$
$$\delta a' = +\,0\ \cdot 000004669\sin 3\lambda + 0\ \cdot 000076912\cos 3\lambda.$$

37.

$$a\frac{dP_3}{d\Pi} = -\,2aK_5e\sin^2\tfrac{1}{2}\gamma\sin(2\Pi+\varpi) - 2aK_6e'\sin^2\tfrac{1}{2}\gamma\sin(2\Pi+\varpi'),$$

$$a\frac{dP_3'}{d\Pi} = \text{same with } -\text{ cosines for sines.}$$

$$\therefore\ \log a\frac{dP_3}{d\Pi} = \bar{5}\cdot 50893\ -, \qquad \log a\frac{dP_3'}{d\Pi} = \bar{5}\cdot 46359\ -;$$

$$\therefore\ \delta\gamma = +\,0''\cdot 216\sin 3\lambda + 0''\cdot 195\cos 3\lambda,$$
$$\delta\Pi = -\,7''\cdot 415\sin 3\lambda + 8''\cdot 231\cos 3\lambda,$$
$$\delta\gamma' = +\,0''\cdot 155\sin 3\lambda + 0''\cdot 139\cos 3\lambda,$$
$$\delta\Pi' = -\,5''\cdot 299\sin 3\lambda + 5''\cdot 883\cos 3\lambda.$$

38. We may now collect together the variations of the elements, so far as they depend upon the first power of the disturbing force. Putting ♅ = mean long. of Uranus, ♆ = mean long. of Neptune, and therefore writing ♅ − 2♆ for λ, the variations of the elements of Uranus will be

$$\begin{aligned}\delta(nt+\epsilon) = &+ 2783''\cdot825 \sin (2\text{♆}-\text{♅}) + 421''\cdot244 \cos (2\text{♆}-\text{♅})\\ &+ 125''\cdot016 \sin 2(2\text{♆}-\text{♅}) + 3''\cdot617 \cos 2(2\text{♆}-\text{♅})\\ &+ 8''\cdot148 \sin 3(2\text{♆}-\text{♅}) + 0''\cdot436 \cos 3(3\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta e = &- 85''\cdot105 \sin (2\text{♆}-\text{♅}) + 408''\cdot350 \cos (2\text{♆}-\text{♅})\\ &- 6''\cdot989 \sin 2(2\text{♆}-\text{♅}) + 32''\cdot541 \cos 2(2\text{♆}-\text{♅})\\ &- 0''\cdot759 \sin 3(2\text{♆}-\text{♅}) + 3''\cdot219 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}e\delta\varpi = &+ 409''\cdot040 \sin (2\text{♆}-\text{♅}) + 84''\cdot810 \cos (2\text{♆}-\text{♅})\\ &+ 32''\cdot498 \sin 2(2\text{♆}-\text{♅}) + 6''\cdot988 \cos 2(2\text{♆}-\text{♅})\\ &+ 3''\cdot219 \sin 3(2\text{♆}-\text{♅}) + 0''\cdot759 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta n = &- 0''\cdot00184260 \sin (2\text{♆}-\text{♅}) + 0''\cdot01228015 \cos (2\text{♆}-\text{♅})\\ &- 0''\cdot00003752 \sin 2(2\text{♆}-\text{♅}) + 0''\cdot00113009 \cos 2(2\text{♆}-\text{♅})\\ &- 0''\cdot00000689 \sin 3(2\text{♆}-\text{♅}) + 0''\cdot00011349 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta a = &+ 0\ \cdot00055795 \sin (2\text{♆}-\text{♅}) - 0\ \cdot00371851 \cos (2\text{♆}-\text{♅})\\ &+ 0\ \cdot00001136 \sin 2(2\text{♆}-\text{♅}) - 0\ \cdot00034220 \cos 2(2\text{♆}-\text{♅})\\ &+ 0\ \cdot00000209 \sin 3(2\text{♆}-\text{♅}) - 0\ \cdot00003436 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta\gamma = &- 0''\cdot582 \sin (2\text{♆}-\text{♅}) + 1''\cdot146 \cos (2\text{♆}-\text{♅})\\ &- 1''\cdot503 \sin 2(2\text{♆}-\text{♅}) + 1''\cdot353 \cos 2(2\text{♆}-\text{♅})\\ &- 0''\cdot216 \sin 3(2\text{♆}-\text{♅}) + 0''\cdot195 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta\Pi = &- 77''\cdot890 \sin (2\text{♆}-\text{♅}) + 11''\cdot477 \cos (2\text{♆}-\text{♅})\\ &+ 49''\cdot760 \sin 2(2\text{♆}-\text{♅}) + 56''\cdot954 \cos 2(2\text{♆}-\text{♅})\\ &+ 7''\cdot415 \sin 3(2\text{♆}-\text{♅}) + 8''\cdot231 \cos 3(2\text{♆}-\text{♅});\end{aligned}$$

and the variations of the elements of Neptune, are

$$\begin{aligned}\delta(n't+\epsilon') = &- 1996''\cdot662 \sin (2\text{♆}-\text{♅}) - 302''\cdot139 \cos (2\text{♆}-\text{♅})\\ &- 89''\cdot358 \sin 2(2\text{♆}-\text{♅}) - 2''\cdot599 \cos 2(2\text{♆}-\text{♅})\\ &- 5''\cdot803 \sin 3(2\text{♆}-\text{♅}) - 0''\cdot313 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta e' = &+ 89''\cdot999 \sin (2\text{♆}-\text{♅}) + 83''\cdot276 \cos (2\text{♆}-\text{♅})\\ &+ 24''\cdot981 \sin 2(2\text{♆}-\text{♅}) + 23''\cdot490 \cos 2(2\text{♆}-\text{♅})\\ &+ 2''\cdot451 \sin 3(2\text{♆}-\text{♅}) + 2''\cdot406 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}e'\delta\varpi' = &+ 83''\cdot024 \sin (2\text{♆}-\text{♅}) - 90''\cdot762 \cos (2\text{♆}-\text{♅})\\ &+ 23''\cdot501 \sin 2(2\text{♆}-\text{♅}) - 24''\cdot981 \cos 2(2\text{♆}-\text{♅})\\ &+ 2''\cdot406 \sin 3(2\text{♆}-\text{♅}) - 2''\cdot451 \cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta n' = &+ 0''{\cdot}00129446 \sin\ (2\Psi - \text{♅}) - 0''{\cdot}00900457 \cos\ (2\Psi - \text{♅})\\ &+ 0''{\cdot}00002737 \sin 2(2\Psi - \text{♅}) - 0''{\cdot}00082441 \cos 2(2\Psi - \text{♅})\\ &+ 0''{\cdot}00000503 \sin 3(2\Psi - \text{♅}) - 0''{\cdot}00008279 \cos 3(2\Psi - \text{♅}).\end{aligned}$$

$$\begin{aligned}\delta a' = &- 0\ {\cdot}00120254 \sin\ (2\Psi - \text{♅}) + 0\ {\cdot}00836513 \cos\ (2\Psi - \text{♅})\\ &- 0\ {\cdot}00002543 \sin 2(2\Psi - \text{♅}) + 0\ {\cdot}00076587 \cos 2(2\Psi - \text{♅})\\ &- 0\ {\cdot}00000467 \sin 3(2\Psi - \text{♅}) + 0\ {\cdot}00007691 \cos 3(2\Psi - \text{♅}).\end{aligned}$$

$$\begin{aligned}\delta\gamma' = &-\ 0''{\cdot}285 \sin\ (2\Psi - \text{♅}) +\ 0''{\cdot}083 \cos\ (2\Psi - \text{♅})\\ &-\ 1''{\cdot}065 \sin 2(2\Psi - \text{♅}) +\ 0''{\cdot}902 \cos 2(2\Psi - \text{♅})\\ &-\ 0''{\cdot}155 \sin 3(2\Psi - \text{♅}) +\ 0''{\cdot}139 \cos 3(2\Psi - \text{♅}).\end{aligned}$$

$$\begin{aligned}\delta\Pi' = &- 55''{\cdot}703 \sin\ (2\Psi - \text{♅}) +\ 8''{\cdot}242 \cos\ (2\Psi - \text{♅})\\ &+ 35''{\cdot}523 \sin 2(2\Psi - \text{♅}) + 40''{\cdot}659 \cos 2(2\Psi - \text{♅})\\ &+\ 5''{\cdot}299 \sin 3(2\Psi - \text{♅}) +\ 5''{\cdot}883 \cos 3(2\Psi - \text{♅}).\end{aligned}$$

The period of the principal part of the long inequality $= \dfrac{n}{2n' - n} \times$ periodic time of Uranus, or about 4047 years; and the periods of the smaller terms depending on 2λ and 3λ are respectively one half and one third of the period of the principal terms.

SECTION II.

VARIATIONS OF THE ELEMENTS DEPENDING ON THE SQUARE OF THE DISTURBING FORCE.

39. In order to obtain the terms depending on the square of the disturbing force we must substitute the first approximate values of the elements, $a + \delta a$, $e + \delta e$, &c. instead of their constant values a, e, &c. in the differential equations for the variations, and then integrate afresh rejecting the terms of one dimension in the masses.

First, let us consider the effect on the principal part of the long inequality in mean longitude.

Let ζ = this part of the inequality, then

$$\zeta = 3\int_t\int_t n^2 a \frac{dR}{d\epsilon}.$$

Then substituting $n + \delta n$, $a + \delta a$, &c.

$$\delta\zeta = 3\int_t\int_t \left\{n^2 a\delta \frac{dR}{d\epsilon} + \frac{dR}{d\epsilon}\,\delta\,(n^2 a)\right\},$$

$$n^2 a = \frac{1}{a^2}, \quad \therefore\ \delta\,(n^2 a) = -\,2n^2\delta a,$$

$$\delta a = -\,2na^2\int_t \frac{dR}{d\epsilon},$$

$$\therefore\ \delta\,(n^2 a) = 4n^3 a^2\int_t \frac{dR}{d\epsilon},$$

$$\therefore\ \delta\zeta = 3\int_t\!\!\int_t\left(n^2a\delta\,\frac{dR}{d\epsilon} + 4n^3a^2\,\frac{dR}{d\epsilon}\int_t\frac{dR}{d\epsilon}\right)$$

$$= 3n^2a\textstyle\int_t\int_t\delta\,\dfrac{dR}{d\epsilon} + 6n^3a^2\displaystyle\int_t\left(\int_t\frac{dR}{d\epsilon}\right)^2.$$

Now $$\delta\,\frac{dR}{d\epsilon} = \frac{d^2R}{da\,d\epsilon}\,\delta a + \frac{d^2R}{de\,d\epsilon}\,\delta e + \frac{d^2R}{d\epsilon^2}\,\delta\,(nt+\epsilon) + \&c.$$

$$+\frac{d^2R}{da'd\epsilon}\,\delta a' + \frac{d^2R}{de'd\epsilon}\,\delta e' + \frac{d^2R}{d\epsilon'd\epsilon}\,\delta\,(n't+\epsilon') + \&c.,$$

if, therefore, we put

$$\delta R = \frac{dR}{da}\,\delta a + \frac{dR}{de}\,\delta e + \frac{dR}{d\epsilon}\,\delta\,(nt+\epsilon) + \&c.$$

$$+\frac{dR}{da'}\,\delta a' + \frac{dR}{de'}\,\delta e' + \frac{dR}{d\epsilon'}\,\delta\,(n't+\epsilon') + \&c.,$$

then $\delta\,\dfrac{dR}{d\epsilon} = \dfrac{d\,(\delta R)}{d\epsilon}$, the brackets written round δR, signifying that δa, $\delta a'$, δe, $\delta e'$, &c. are to be considered constant in differentiating δR. Hence

$$\delta\zeta = 3n^2a\int_t\!\!\int_t\frac{d\,(\delta R)}{d\epsilon} + 6n^3a^2\int_t\left(\int_t\frac{dR}{d\epsilon}\right)^2.$$

40. The only parts of R which it will be necessary to retain are those of the first order involving λ, and those of the second and third orders involving 2λ and 3λ respectively, which we have represented by R_1, R_2, and R_3; and we shall retain only those portions of the variations of the elements depending on these terms of R, and which we shall represent by $\delta_1 a$, $\delta_2 a$, $\delta_3 a$, &c.; so that we may put $R = R + R_2 + R_3$ and $\delta a = \delta_1 a + \delta_2 a + \delta_3 a$, and similarly, for the other elements. It will be convenient to put $\delta\,(nt+\epsilon) = \zeta + \delta\epsilon$ where ζ is that part which is twice integrated, and therefore divided by ω^2, and $\delta\epsilon$ represents the other terms which are only once integrated.

If we consider those terms of δR only, which involve R_1, and the variations of the elements depending on R_1, except ζ_1 and ζ_1',

$$\delta R = \frac{dR_1}{da}\,\delta_1 a + \frac{dR_1}{de}\,\delta_1 e + \frac{dR_1}{d\varpi}\,\delta_1\varpi + \frac{dR_1}{d\epsilon}\,\delta_1\epsilon$$

$$+\frac{dR_1}{da'}\,\delta_1 a' + \frac{dR_1}{de'}\,\delta_1 e' + \frac{dR_1}{d\varpi'}\,\delta_1\varpi' + \frac{dR_1}{d\epsilon'}\,\delta_1\epsilon';$$

and if the values of $\delta_1 a$, &c. be substituted, and if we then differentiate considering $\delta_1 a$, $\delta_1 a'$, &c. constant, we shall find $\dfrac{d\,(\delta R)}{d\epsilon} = 0$.

Similarly, if we put

$$\delta R = \frac{dR_2}{da}\,\delta_2 a + \frac{dR_2}{de}\,\delta_2 e + \frac{dR_2}{d\varpi}\,\delta_2\varpi + \frac{dR_2}{d\epsilon}\,\delta_2\epsilon$$

$$+\frac{dR_2}{da'}\,\delta_2 a' + \frac{dR_2}{de'}\,\delta_2 e' + \frac{dR_2}{d\varpi'}\,\delta_2\varpi' + \frac{dR_2}{d\epsilon'}\,\delta_2\epsilon'$$

$$+ \frac{dR_2}{d\gamma'} (\delta_2\gamma + \delta_2\gamma') + \frac{dR_2}{d\Pi'} (\delta_2\Pi + \delta_2\Pi')\text{[*]}$$

we shall find $\dfrac{d(\delta R)}{d\epsilon} = 0$; and the same is true if for R_2 and δ_2 we substitute R_3 and δ_3 and so on. If therefore we reject all the above terms, and also all terms of the third order except such as will be divided by ω^4,

$$\begin{aligned}
\delta R = {} & \frac{dR_1}{de} (\delta_2 e + \delta_3 e) + \left(\frac{dR_2}{de} + \frac{dR_3}{de}\right) \delta_1 e \\
& + \frac{dR_1}{d\varpi} (\delta_2\varpi + \delta_3\varpi) + \left(\frac{dR_2}{d\varpi} + \frac{dR_3}{d\varpi}\right) \delta_1\varpi \\
& + \frac{dR_1}{d\epsilon} (\zeta_1 + \zeta_2) + \frac{dR_1}{d\epsilon'} (\zeta_1' + \zeta_2') + \frac{dR_2}{d\epsilon} \zeta_1 + \frac{dR_2}{d\epsilon'} \zeta_1' \\
& + \frac{dR_1}{de'} (\delta_2 e' + \delta_3 e') + \left(\frac{dR_2}{de'} + \frac{dR_3}{de'}\right) \delta_1 e' \\
& + \frac{dR_1}{d\varpi'} (\delta_2\varpi' + \delta_3\varpi') + \left(\frac{dR_2}{d\varpi'} + \frac{dR_3}{d\varpi'}\right) \delta_1\varpi' ;
\end{aligned}$$

to which we may add the following terms of the third and fourth orders, which will be divided by ω^3 and ω^4 respectively,

---

[*] The part of δR which is due to the variation of the inclination and node is correctly given by the expression

$$\frac{dR}{d\gamma'}(\delta\gamma + \delta\gamma') + \frac{dR}{d\Pi'}(\delta\Pi + \delta\Pi'),$$

when we consider only the principal terms of R containing any given argument $(kn - k'n')t$; and such terms only are considered in this section. But the expression is not true of all terms. Referring to the value of I given in Art. (2), we may put it in the form

$$\begin{aligned}
I = {} & \{\cos(\theta + \theta' - 2\,\text{☊}) - \cos(\theta - \theta')\} \sin^2 \tfrac{1}{2} i, \\
& + \{\cos(\theta + \theta' - 2\,\text{☊}') - \cos(\theta - \theta')\} \sin^2 \tfrac{1}{2} i', \\
& - \tfrac{1}{2}\{\cos(\theta + \theta' - \text{☊} - \text{☊}') - \cos(\theta - \theta' - \text{☊} + \text{☊}')\} \sin i \sin i',
\end{aligned}$$

omitting the term of the fourth order. Now the principal terms in R, involving the inclinations, of the form $(kn - k'n')t$ can arise only from the terms in I of the form $\cos(\theta + \theta' + Q)$ combined with the terms of the factor $\frac{1}{2}B_0 + B_1 \cos(\theta - \theta') + \&c.$, (see Art. 7); for the order of these will be $k \sim k'$; but the order of the terms of that form arising from the other parts of I must be at least $k \sim k' + 2$. Confining our attention, therefore, to the former,

$$\begin{aligned}
I = {} & \cos(\theta + \theta' - 2\,\text{☊}) \sin^2 \tfrac{1}{2} i + \cos(\theta + \theta' - 2\,\text{☊}') \sin^2 \tfrac{1}{2} i', \\
& - \tfrac{1}{2} \cos(\theta + \theta' - \text{☊} - \text{☊}') \sin i \sin i'.
\end{aligned}$$

Now let the planet be referred to its own plane at the epoch; then at any time t, i is entirely due to perturbation in the interval. Let Rp (fig. 9) be the plane of reference; rp the position of the orbit at the time t; then $i = \angle p$; $\text{☊} = \Pi' + Rp$; $i' = \gamma' + \delta\gamma' = \angle q$; $\text{☊}' = \Pi' + Rq = \Pi' + \delta\Pi'$; $\angle r = \gamma + \delta\gamma$; $Rr = -\delta\Pi$. It must be observed that Rp is not necessarily small, although it is due entirely to perturbation. Substituting these quantities in the above equation, we have

$$\begin{aligned}
\delta I = {} & 2 \sin(\theta + \theta' - 2\Pi') \sin^2 \tfrac{1}{2}\gamma' . \delta\Pi', \\
& + \cos(\theta + \theta' - 2\Pi') \sin \tfrac{1}{2}\gamma' . \delta\gamma', \\
& - \tfrac{1}{2}\{\cos(\theta + \theta' - 2\Pi') \cos Rp + \sin(\theta + \theta' - 2\Pi') \sin Rp\} \sin i \sin \gamma',
\end{aligned}$$

from which we must eliminate $\cos Rp$, $\sin Rp$, and $\sin i$.

By the triangle Rpr,

$$\cos Rp = \cos Rr \cos rp + \sin Rr \sin rp \cos r,$$

whence eliminating rp,

$$\cos Rp = \frac{\sin r}{\sin i} \cos rRp \cos Rr + \frac{\sin rRp}{\sin i} \cos r;$$

$$\begin{aligned}
\therefore \sin i \cos Rp & = -\sin(\gamma + \delta\gamma) \cos\gamma + \sin\gamma \cos(\gamma + \delta\gamma), \\
& = -\sin\delta\gamma = -\delta\gamma,
\end{aligned}$$

also $\sin i \sin Rp = \sin r . \sin Rr = -\sin\gamma . \delta\Pi$;

$\therefore$ substituting these in δI,

$$\begin{aligned}
\delta I = {} & 2 \sin(\theta + \theta' - 2\Pi') \sin^2 \tfrac{1}{2}\gamma' . \delta\Pi', \\
& + \cos(\theta + \theta' - 2\Pi') \sin \tfrac{1}{2}\gamma' . \delta\gamma', \\
& + \tfrac{1}{2} \cos(\theta + \theta' - 2\Pi') \sin\gamma' . \delta\gamma + \tfrac{1}{2} \sin(\theta + \theta' - 2\Pi') \sin^2 \gamma' . \delta\Pi, \\
= {} & 2 \sin(\theta + \theta' - 2\Pi') \sin^2 \tfrac{1}{2}\gamma' (\delta\Pi + \delta\Pi'), \\
& + \cos(\theta + \theta' - 2\Pi') \sin \tfrac{1}{2}\gamma' (\delta\gamma + \delta\gamma').
\end{aligned}$$

But this is the expression which would be obtained from the value of I given in Art. (3), {omitting $\cos(\theta - \theta')$} if γ' and Π' were supposed to receive the increments $\delta\gamma + \delta\gamma'$ and $\delta\Pi + \delta\Pi'$;

$$\therefore \delta I = \frac{dI}{d\gamma'}(\delta\gamma + \delta\gamma') + \frac{dI}{d\Pi'}(\delta\Pi + \delta\Pi');$$

$$\therefore \delta R = \frac{dR}{d\gamma'}(\delta\gamma + \delta\gamma') + \frac{dR}{d\Pi'}(\delta\Pi + \delta\Pi').$$

The expression is also true for the constant term of R of the second order.

$$\delta R = \frac{dR_3}{d\epsilon}\zeta_1 + \frac{dR_1}{d\epsilon}\zeta_3 + \frac{dR_3}{d\epsilon'}\zeta_1' + \frac{dR_1}{d\epsilon'}\zeta_3'$$

$$+ \frac{dR_3}{de}\delta_2 e + \frac{dR_2}{de}\delta_3 e + \frac{dR_3}{d\varpi}\delta_2\varpi + \frac{dR_2}{d\varpi}\delta_3\varpi$$

$$+ \frac{dR_3}{de'}\delta_2 e' + \frac{dR_2}{de'}\delta_3 e' + \frac{dR_3}{d\varpi'}\delta_2\varpi' + \frac{dR_2}{d\varpi'}\delta_3\varpi'.$$

41. We shall first calculate the terms which will be divided by ω^4.

$$\text{Let}\quad \delta R = \frac{dR_1}{d\epsilon}\zeta_1 + \frac{dR_1}{d\epsilon'}\zeta_1',$$

$$R_1 = m'(P\cos\lambda - P'\sin\lambda), \quad \text{(Art. 16.)}$$

$$\therefore\ \delta R = -m'(\zeta_1 - 2\zeta_1')(P\sin\lambda + P'\cos\lambda)$$

$$= -m'\left(1 + \frac{4m}{m'}a^2\right)\zeta_1(P\sin\lambda + P'\cos\lambda),$$

$$\text{since}\quad \frac{\zeta_1'}{\zeta_1} = -\frac{2m}{m'}a^2 \text{ nearly, see (Art. 9.)}$$

$$\therefore\ \frac{d(\delta R)}{d\epsilon} = -m'\left(1 + \frac{4m}{m'}a^2\right)\zeta_1(P\cos\lambda - P'\sin\lambda),$$

$$= -\frac{3m'^2n^2a}{\omega^2}\left(1 + \frac{4m}{m'}a^2\right)(P\sin\lambda + P'\cos\lambda)(P\cos\lambda - P'\sin\lambda),$$

$$= -\frac{3m'^2n^2a}{2\omega^2}\left(1 + \frac{4m}{m'}a^2\right)\left\{(P^2 - P'^2)\sin 2\lambda + 2PP'\cos 2\lambda\right\};$$

$$\therefore\ 3n^2a\int_t\!\!\int_t \frac{d(\delta R)}{d\epsilon} = \frac{9m'^2n^4a^2}{8\omega^4\sin 1''}\left(1 + \frac{4m}{m'}a^2\right)\left\{(P^2 - P'^2)\sin 2\lambda + 2PP'\cos 2\lambda\right\}.$$

$$\log\left(1 + \frac{4m}{m'}a^2\right) = 0{\cdot}39076, \qquad \log\frac{m'^2n^4}{\omega^4\sin 1''} = 3{\cdot}49819;$$

therefore by the values of $\log aP$ and $\log aP'$ in Art. (16), this part of

$$\delta\zeta = +12''{\cdot}169\sin 2\lambda - 3''{\cdot}750\cos 2\lambda,$$

and the corresponding part of $\delta\zeta' = -\frac{2m}{m'}a^2.\,\delta\zeta$ nearly,

$$\therefore\ \delta\zeta' = -8''{\cdot}877\sin 2\lambda + 2''{\cdot}736\cos 2\lambda.$$

42. Next let

$$\delta R = \frac{dR_1}{d\epsilon}\zeta_2 + \frac{dR_1}{d\epsilon'}\zeta_2' + \frac{dR_2}{d\epsilon}\zeta_1 + \frac{dR_2}{d\epsilon'}\zeta_1',$$

where $R_2 = m'(P_2\cos 2\lambda - P_2'\sin 2\lambda)$, Art. (28).

Then as in the last article,

$$3n^2a\int_{t'}\int_{t}\frac{d\,(\delta R)}{d\epsilon}=\frac{9m'^2n^4a^2}{4\omega^4\sin 1''}\left(1+\frac{4m}{m'}a^2\right)\left\{\begin{matrix}(PP_2-P'P_2')\sin 3\lambda+(P'P_2+PP_2')\cos 3\lambda\\ -7\,(PP_2+P'P_2')\sin\lambda+7\,(P'P_2-PP_2')\cos\lambda\end{matrix}\right\};$$

$$\begin{aligned}\therefore\ \delta\zeta=&+\ 2''\cdot279\sin 3\lambda-0''\cdot421\cos 3\lambda\\ &-16''\cdot115\sin\lambda-1''\cdot883\cos\lambda,\end{aligned}$$

and since $\delta\zeta'=-\dfrac{2m}{m'}a^2\delta\zeta$ nearly,

$$\begin{aligned}\delta\zeta'=&-\ 1''\cdot662\sin 3\lambda+0''\cdot307\cos 3\lambda\\ &+11''\cdot756\sin\lambda+1''\cdot374\cos\lambda.\end{aligned}$$

43. Again let

$$\delta R=\frac{dR_3}{d\epsilon}\zeta_1+\frac{dR_3}{d\epsilon'}\zeta_1'+\frac{dR_1}{d\epsilon}\zeta_3+\frac{dR_1}{d\epsilon'}\zeta_3',$$

where $R_3=m'(P_3\cos 3\lambda-P_3'\sin 3\lambda)$, Art. (33),

then, as before,

$$3n^2a\int_{t'}\int_{t}\frac{d\,(\delta R)}{d\epsilon}=-\frac{39m'^2n^4a^2}{4\omega^4\sin 1''}\left(1+\frac{4m}{m'}a^2\right)\left\{(PP_3+P'P_3')\sin 2\lambda+(PP_3'-P'P_3)\cos 2\lambda\right\}$$

$$+\frac{21m'^2n^4a^2}{8\omega^4\sin 1''}\left(1+\frac{4m}{m'}a^2\right)\left\{(PP_3-P'P_3')\sin 4\lambda+(PP_3'+P'P_3)\cos 4\lambda\right\};$$

$$\begin{aligned}\therefore\ \delta\zeta=&-1''\cdot006\sin 2\lambda-0''\cdot090\cos 2\lambda\\ &+0''\cdot266\sin 4\lambda-0''\cdot057\cos 4\lambda,\end{aligned}$$

and $\delta\zeta'=-\dfrac{2m}{m'}a^2\delta\zeta$ nearly;

$$\begin{aligned}\therefore\ \delta\zeta'=&+0''\cdot734\sin 2\lambda+0''\cdot066\cos 2\lambda\\ &-0''\cdot194\sin 4\lambda+0''\cdot042\cos 4\lambda.\end{aligned}$$

These are all the terms divided by ω^4 which are worth calculating.

44. Now let

$$\begin{aligned}\delta R=&\frac{dR_1}{de}\delta_2e+\frac{dR_1}{d\varpi}\delta_2\varpi+\frac{dR_1}{de'}\delta_2e'+\frac{dR_1}{d\varpi'}\delta_2\varpi'\\ &+\frac{dR_2}{de}\delta_1e+\frac{dR_2}{d\varpi}\delta_1\varpi+\frac{dR_2}{de'}\delta_1e'+\frac{dR_2}{d\varpi'}\delta_1\varpi'\\ &+\frac{dR_2}{de}\delta_3e+\frac{dR_2}{d\varpi}\delta_3\varpi+\frac{dR_2}{de'}\delta_3e'+\frac{dR_2}{d\varpi'}\delta_3\varpi'\\ &+\frac{dR_3}{de}\delta_2e+\frac{dR_3}{d\varpi}\delta_2\varpi+\frac{dR_1}{de'}\delta_2e'+\frac{dR_3}{d\varpi'}\delta_2\varpi'\end{aligned}$$

then $3n^2a\displaystyle\int_{t'}\int_{t}\frac{d\,(\delta R)}{d\epsilon}$

$$=+\frac{9m'^2n^3a^3}{2\omega^3\sin 1''}\left\{\begin{matrix}\left(\dfrac{dP}{de}\dfrac{dP_2}{de}+\dfrac{dP'}{de}\dfrac{dP_2'}{de}\right)\sin\lambda\\ +\left(\dfrac{dP}{de}\dfrac{dP_2'}{de}-\dfrac{dP'}{de}\dfrac{dP_2}{de}\right)\cos\lambda\end{matrix}\right\}$$

$$
\begin{aligned}
&-\frac{3mm'n^2n'aa'}{2\omega^3\sin 1''}\left\{\begin{array}{l}
\quad\left(\dfrac{dP}{de'}\dfrac{dP_2}{de'}+\dfrac{dP'}{de'}\dfrac{dP_2'}{de'}\right)\sin\lambda \\
+\left(\dfrac{dP}{de'}\dfrac{dP_2'}{de'}-\dfrac{dP'}{de'}\dfrac{dP_2}{de'}\right)\cos\lambda
\end{array}\right\} \\
&+\frac{6mm'n^2n'aa'}{\omega^3\sin 1''}\left\{\begin{array}{l}
\quad\left(\dfrac{dQ}{de'}\dfrac{dP_2}{de'}+\dfrac{dQ'}{de'}\dfrac{dP_2'}{de'}\right)\sin\lambda \\
+\left(\dfrac{dQ}{de'}\dfrac{dP_2'}{de'}-\dfrac{dQ'}{de'}\dfrac{dP_2}{de'}\right)\cos\lambda
\end{array}\right\} \\
&+\frac{5m'^2n^3a^2}{2\omega^3\sin 1''}\left\{\begin{array}{l}
\quad\left(\dfrac{dP_2}{de}\dfrac{dP_3}{de}+\dfrac{dP_2'}{de}\dfrac{dP_3'}{de}\right)\sin\lambda \\
+\left(\dfrac{dP_2}{de}\dfrac{dP_3'}{de}-\dfrac{dP_2'}{de}\dfrac{dP_3}{de}\right)\cos\lambda
\end{array}\right\} \\
&+\frac{5mm'n^2n'aa'}{2\omega^3\sin 1''}\left\{\begin{array}{l}
\quad\left(\dfrac{dP_2}{de'}\dfrac{dP_3}{de'}+\dfrac{dP_2'}{de'}\dfrac{dP_3'}{de'}\right)\sin\lambda \\
+\left(\dfrac{dP_2}{de'}\dfrac{dP_3'}{de'}-\dfrac{dP_2'}{de'}\dfrac{dP_3}{de'}\right)\cos\lambda
\end{array}\right\};
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \delta\zeta &= (-29''{\cdot}229+3''{\cdot}680-16''{\cdot}391-0''{\cdot}386-0''{\cdot}577)\sin\lambda, \\
&\quad+(+0''{\cdot}166-0''{\cdot}028+0''{\cdot}127+0''{\cdot}008+0''{\cdot}012)\cos\lambda, \\
&= -42''{\cdot}903\sin\lambda+0''{\cdot}285\cos\lambda,
\end{aligned}
$$

$$\delta\zeta'=-\frac{2m}{m'}a^2\delta\zeta \text{ nearly};$$

$$\therefore\ \delta\zeta'=+31''{\cdot}298\sin\lambda-0''{\cdot}208\cos\lambda.$$

45. Next let

$$
\begin{aligned}
\delta R &= \frac{dR_1}{de}\delta_3 e+\frac{dR_1}{d\varpi}\delta_3\varpi+\frac{dR_1}{de'}\delta_3 e'+\frac{dR_1}{d\varpi'}\delta_3\varpi' \\
&\quad+\frac{dR_3}{de}\delta_1 e+\frac{dR_3}{d\varpi}\delta_1\varpi+\frac{dR_3}{de'}\delta_1 e'+\frac{dR_3}{d\varpi'}\delta_1\varpi',
\end{aligned}
$$

then $3n^2a\int_{t}\int_{t}\dfrac{d(\delta R)}{d\epsilon}$

$$
\begin{aligned}
&=\frac{2m'^2n^3a^2}{\omega^3\sin 1''}\left\{\begin{array}{l}
\quad\left(\dfrac{dP}{de}\dfrac{dP_3}{de}+\dfrac{dP'}{de}\dfrac{dP_3'}{de}\right)\sin 2\lambda \\
+\left(\dfrac{dP}{de}\dfrac{dP_3'}{de}-\dfrac{dP'}{de}\dfrac{dP_3}{de}\right)\cos 2\lambda
\end{array}\right\} \\
&+\frac{2mm'n^2n'aa'}{\omega^3\sin 1''}\left\{\begin{array}{l}
\quad\left(\dfrac{dP}{de'}\dfrac{dP_3}{de'}+\dfrac{dP'}{de'}\dfrac{dP_3'}{de'}\right)\sin 2\lambda \\
+\left(\dfrac{dP}{de'}\dfrac{dP_3'}{de'}-\dfrac{dP'}{de'}\dfrac{dP_3}{de'}\right)\cos 2\lambda
\end{array}\right\}, \text{ nearly}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \delta\zeta &= (-1''{\cdot}936-0''{\cdot}737)\sin 2\lambda, \\
&\quad+(+0''{\cdot}049+0''{\cdot}022)\cos 2\lambda,
\end{aligned}
$$

$$= -2''\cdot 673 \sin 2\lambda + 0''\cdot 071 \cos 2\lambda.$$

$$\delta\zeta' = -\frac{2m}{m'} a^2 \delta\zeta;$$

$$\therefore\ \delta\zeta' = +1''\cdot 950 \sin 2\lambda - 0''\cdot 052 \cos 2\lambda.^*$$

46. The second term in the expression for $\delta\zeta$ in Art. (39), is of the order $\frac{e^2}{\omega^3}$ at least: we may therefore put $R = R_1 = m'(P \cos\lambda - P' \sin\lambda)$;

$$\therefore\ a^2n^2\left(\int_t \frac{dR_1}{d\epsilon}\right)^2 = \tfrac{1}{2}\left(\frac{m'na}{\omega}\right)^2 (P^2 + P'^2)$$

$$+ \tfrac{1}{2}\left(\frac{m'na}{\omega}\right)^2 \{(P^2 - P'^2)\cos 2\lambda - 2PP' \sin 2\lambda\};$$

$$\therefore\ 6n^3a^2 \int_t\left(\int_t \frac{dR_1}{d\epsilon}\right)^2 = 3\left(\frac{m'na}{\omega}\right)^2 (P^2 + P'^2)\, nt$$

$$+ \frac{3m'^2n^3a^2}{2\omega^3 \sin 1''} \{(P^2 - P'^2)\sin 2\lambda + 2PP' \cos 2\lambda\};$$

therefore neglecting the secular part,

$$\delta\zeta = -0''\cdot 137 \sin 2\lambda + 0''\cdot 042 \cos 2\lambda,$$

$$\delta\zeta' = +\frac{4m^2}{m'^2} a^{\frac{3}{2}} \delta\zeta \text{ nearly};$$

$$\therefore\ \delta\zeta' = -0''\cdot 143 \sin 2\lambda + 0''\cdot 044 \cos 2\lambda.$$

47. δR will also contain terms involving λ, which arise from terms in R involving $\lambda + lt$, combined with terms in the variations of the elements involving lt, and *vice versâ*; but these terms are not of much importance. We shall only calculate the following, which arises from the double combination of the arguments

$$2(nt + \epsilon) - 3(n't + \epsilon'), \quad \text{and} \quad (nt + \epsilon) - (n't + \epsilon').$$

$$\text{Let} \quad \phi = 2(nt + \epsilon) - 3(n't + \epsilon') + \varpi,$$

$$\phi' = 2(nt + \epsilon) - 3(n't + \epsilon') + \varpi',$$

$$\text{and } R = m'D_1e \cos\phi + m'D_2e' \cos\phi',$$

$$\text{where} \quad aD_1 = \frac{a}{2}\left(6b_3^{(\frac{1}{2})} + a\frac{db_3^{(\frac{1}{2})}}{da}\right),$$

$$aD_2 = -\frac{a}{2}\left(5b_2^{(\frac{1}{2})} + a\frac{db_2^{(\frac{1}{2})}}{da}\right);$$

$$\text{also let} \quad R_{,} = m'C \cos\psi, \quad \text{where } \psi = nt + \epsilon - (n't + \epsilon'),$$

---

[*] If the expansion of δR be carried a step further, we shall have the terms of the third order $\frac{d^2R_1}{de\,d\varpi}\delta_1e\delta_1\varpi + \frac{1}{2}\frac{d^2R_1}{d\varpi^2}(\delta_1\varpi)^2 +$ similar terms depending on δ_1e' and $\delta_1\varpi'$. Retaining only the principal part of the former, we shall have

$$\delta\zeta = -\frac{9m'^2n^3a^2}{4e\omega}\left(\frac{dP^2}{de^2} + \frac{dP'^2}{de^2}\right)\textstyle\int_t\int_t e\delta_1\varpi,$$

$$= +3''.89 \sin\lambda - 0''\cdot 81 \cos\lambda,$$

$$\text{and } \delta\zeta' = -\frac{2m}{m'}a^2.\delta\zeta = -2''.84 \sin\lambda,$$

rejecting the second term which is less than 1″. The terms depending on δ_1e', $\delta_1\varpi'$ are less than 1″, and may therefore be rejected.

$$\text{and}\quad aC = -\frac{a}{2}\left(b_1^{(\frac{1}{2})} - a\right),$$

$$\text{and}\quad R_{,}' = mC'\cos\psi,$$

$$\text{where}\quad a'C' = -\frac{1}{2}\left(b_1^{(\frac{1}{2})} - \frac{1}{a^2}\right).$$

Then the more important parts of the disturbance of mean longitude depending on $R_{,}$ and $R_{,}'$ are

$$\zeta = \frac{3m'n^2aC}{(n-n')^2}\sin\psi, \qquad \zeta' = -\frac{3mn'^2a'C'}{(n-n')^2}\sin\psi,$$

$$\text{and}\quad \delta R = \frac{dR}{d\epsilon}\zeta + \frac{dR}{d\epsilon'}\zeta'$$

$$= -\left\{\frac{6m'^2n^2aC}{(n-n')^2} + \frac{9mm'n'^2a'C'}{(n-n')^2}\right\}\sin\psi$$

$$\times\,(D_1e\sin\phi + D_2e'\sin\phi'),$$

$$\therefore\ \frac{d\,(\delta R)}{d\epsilon} = +\left\{\frac{6m'^2n^2aC}{(n-n')^2} + \frac{9mm'n'^2a'C'}{(n-n')^2}\right\}$$

$$\times\left\{D_1e\sin(\phi-\psi) + D_2e'\sin(\phi'-\psi)\right\},$$

neglecting the terms involving $\phi+\psi$, and $\phi'+\psi$.

$$\therefore\ 3n^2a\int_t\!\!\int_t \frac{d\,(\delta R)}{d\epsilon}$$

$$= -\left\{\frac{18m'^2n^2}{\omega^2\sin 1''}\left(\frac{n}{n-n'}\right)^2 aC + \frac{27mm'n^2}{\omega^2\sin 1''}\left(\frac{n'}{n-n'}\right)^2 a'C'\right\}$$

$$\times\, a\left\{D_1e\sin(\lambda+\varpi) + D_2e'\sin(\lambda+\varpi')\right\}.$$

Again, let
$$\delta R_{,} = \frac{dR_{,}}{d\epsilon}\zeta + \frac{dR_{,}}{d\epsilon'}\zeta',$$

where ζ and ζ' depend on R and R',

$$\text{and}\ \therefore\ \zeta = \frac{6m'n^2a}{(2n-3n')^2}(D_1e\sin\phi + D_2e'\sin\phi'),$$

$$\zeta' = -\frac{9mn'^2a'}{(2n-3n')^2}(D_1e\sin\phi + D_2e'\sin\phi');$$

$$\text{then as before,}\quad 3n^2a\int_t\!\!\int_t \frac{d\,(\delta R_{,})}{d\epsilon}$$

$$= +\left\{\frac{9m'^2n^2}{\omega^2\sin 1''}\left(\frac{n}{2n-3n'}\right)^2 aC + \frac{27mm'n^2}{2\omega^2\sin 1''}\left(\frac{n'}{2n-3n'}\right)^2 a'C\right\}$$

$$\times\, a\left\{D_1e\sin(\lambda+\varpi) + D_2e'\sin(\lambda+\varpi')\right\}.$$

Now $n = 2n'$ nearly, $\therefore\ \dfrac{n}{n-n'} = 2 = \dfrac{n}{2n-3n'}$ nearly,

and $\dfrac{n'}{n-n'} = 1 = \dfrac{n'}{2n-3n'}$ nearly;

therefore, combining the two expressions just found,

$$\delta\zeta = -\left\{\frac{36m'^2n^2aC}{\omega^2 \sin 1''} + \frac{27mm'n^2a'(C' - \frac{1}{2}C)}{\omega^3 \sin 1''}\right\}$$
$$\times\, a\left\{D_1 e \sin(\lambda + \varpi) + D_2 e' \sin(\lambda + \varpi')\right\}$$

$$= -\,0''{\cdot}770 \sin(\lambda + \varpi) + 0''{\cdot}211 \sin(\lambda + \varpi')$$
$$= +\,0''{\cdot}898 \sin\lambda - 0''{\cdot}004 \cos\lambda.$$

The corresponding terms of $\delta\zeta'$, are

$$3n'^2a'\int_{\iota}\int_{t} \frac{d(\delta R')}{d\epsilon'} = -\frac{3mn'^2a'}{2m'n^2a}\left\{3n^2a\int_{1}\int_{t}\frac{d(\delta R)}{d\epsilon}\right\},$$

and $$3n'^2a'\int_{\iota}\int_{t}\frac{d(\delta R'_{,})}{d\epsilon'} = -\frac{mn'^2a'C'}{m'n^2aC}\left\{3n^2a\,{}_{,}\int_{\iota}\int_{t}\frac{d(\delta R_{,})}{d\epsilon}\right\},$$

$$\therefore\ \delta\zeta' = -\,0''{\cdot}056 \sin(\lambda + \varpi) + 0''{\cdot}015 \sin(\lambda + \varpi')$$

which may be neglected.

These are all the terms of $\delta\zeta$ and $\delta\zeta'$, which appear to be worth retaining.

48. The effect of the square of the disturbing force upon n may be calculated from the value of $\delta\zeta$. Let $C \sin k\lambda + C' \cos k\lambda$ be any term of $\delta\zeta$, then

$$\delta'' n = \frac{d\delta\zeta}{dt} = k\omega \sin 1''(C \cos k\lambda - C' \sin k\lambda),$$

the coefficients C, C' being supposed to be expressed in seconds.

Hence $$\delta'' n = -\,0''{\cdot}0000080 \sin\lambda + 0''{\cdot}0002343 \cos\lambda$$
$$-\,0''{\cdot}0000319 \sin 2\lambda - 0''{\cdot}0000722 \cos 2\lambda$$
$$-\,0''{\cdot}0000291 \cos 3\lambda,$$

$$\delta'' n' = +\,0''{\cdot}0000058 \sin\lambda - 0''{\cdot}0001709 \cos\lambda$$
$$+\,0''{\cdot}0000232 \sin 2\lambda + 0''{\cdot}0000526 \cos 2\lambda$$
$$+\,0''{\cdot}0000212 \cos 3\lambda.$$

49. By (Art. 40),

$$\delta\epsilon = 2\int_t na^2 \frac{dR}{da} - \frac{1}{2}\int_t nae \frac{dR}{de}:$$

therefore, representing the variation due to the square of the disturbing force by $\delta''\epsilon$, we have

$$\delta''\epsilon = 2na^2\int_t \frac{d(\delta R)}{da} - \frac{na}{2}\int_t\left\{e\frac{d(\delta R)}{de} + \delta e\frac{dR}{de}\right\},$$

neglecting terms above the first order except such as will be divided by ω^3.

The portion of δR to be retained is

$$\delta R = \frac{dR_1}{de}(\delta_1 e + \delta_2 e) + \frac{dR_1}{d\varpi}(\delta_1\varpi + \delta_2\varpi) + \frac{dR_1}{de'}(\delta_1 e' + \delta_2 e') + \frac{dR_1}{d\varpi'}(\delta_1\varpi' + \delta\ \varpi')$$
$$+ \frac{dR_2}{de}\delta_1 e + \frac{dR_2}{d\varpi}\delta_1\varpi + \frac{dR_2}{de'}\delta_1 e' + \frac{dR_2}{d\varpi'}\delta_1\varpi' + \frac{dR_1}{d\epsilon}\zeta_1 + \frac{dR_1}{d\epsilon'}\zeta_1'.$$

First let $\delta R = \frac{dR_1}{d\epsilon}\zeta_1 + \frac{dR_1}{d\epsilon'}\zeta_1'$,

$$\text{then } 2na^2 \int_t \frac{d(\delta R)}{da} = -\frac{3m'^2n^2a^3}{\omega^2}\left(1 + \frac{4m}{m'}a^2\right)\left(P\frac{dP}{da} + P'\frac{dP'}{da}\right)nt$$

$$+\frac{3m'^2n^3a^3}{2\omega^3 \sin 1''}\left(1 + \frac{4m}{m'}a^2\right)\left\{\begin{array}{l} \left(P\frac{dP}{da} - P'\frac{dP'}{da}\right)\sin 2\lambda \\ + \left(P\frac{dP'}{da} + P'\frac{dP}{da}\right)\cos 2\lambda \end{array}\right\};$$

$$\therefore\ \breve{\delta}\epsilon = -0''{\cdot}969 \sin 2\lambda + 0''{\cdot}272 \cos 2\lambda,$$

omitting the secular perturbation.

The formula for Neptune is

$$2n'a'^2 \int_t \frac{d(\delta R')}{da'} = -\frac{3mm'nn'aa'^2}{\omega^2}\left(1 + \frac{4m}{m'}a^2\right)\left(P\frac{dQ}{da'} + P'\frac{dQ'}{da'}\right)nt$$

$$+\frac{3mm'n^2n'a'^2a}{2\omega^3 \sin 1''}\left(1 + \frac{4m}{m'}a^2\right)\left\{\begin{array}{l} \left(P\frac{dQ}{da'} - P'\frac{dQ'}{da'}\right)\sin 2\lambda \\ + \left(P\frac{dQ'}{da'} + P'\frac{dQ}{da'}\right)\cos 2\lambda \end{array}\right\};$$

$$\therefore\ \breve{\delta}\epsilon' = +0''{\cdot}933 \sin 2\lambda - 0''{\cdot}268 \cos 2\lambda.$$

Again, with the same value of δR,

$$-\tfrac{1}{2}nae \int_t \frac{d(\delta R)}{de}$$

$$= \frac{3m'^2n^2a^3}{4\omega^2}\left(1 + \frac{4m}{m'}a^2\right)e\left(P\frac{dP}{de} + P'\frac{dP'}{de}\right)nt$$

$$-\frac{3m'^2n^3a^3}{8\omega^3 \sin 1''}\left(1 + \frac{4m}{m'}a^2\right)e\left\{\begin{array}{l} \left(P\frac{dP}{de} - P'\frac{dP'}{de}\right)\sin 2\lambda \\ + \left(P\frac{dP'}{de} + P'\frac{dP}{de}\right)\cos 2\lambda \end{array}\right\},$$

$$\therefore\ \breve{\delta}\epsilon = +0''{\cdot}079 \sin 2\lambda - 0''{\cdot}030 \cos 2\lambda,$$

which as well as the corresponding part of $\breve{\delta}\epsilon'$ may be neglected.

Next let

$$\delta R = \frac{dR_1}{de}\delta_1 e + \frac{dR_1}{d\varpi}\delta_1\varpi + \frac{dR_1}{de'}\delta_1 e' + \frac{dR_1}{d\varpi'}\delta_1\varpi',$$

$$\text{then } 2na^2 \int_t \frac{d(\delta R)}{da}$$

$$= \frac{2m'^2na^3}{\omega}\left(\frac{dP}{de}\,\frac{d^2P}{deda} + \frac{dP'}{de}\,\frac{d^2P'}{deda}\right)nt + \frac{2mm'n'a^2a'}{\omega}\left(\frac{dQ}{de'}\,\frac{d^2P}{de'da} + \frac{dQ'}{de'}\,\frac{d^2P'}{de'da}\right)nt,$$

a secular perturbation of the epoch.

With the same value of δR

$$-\frac{na}{2}\int_t\left\{e\,\frac{d\,(\delta R)}{de}+\delta_1 e\,\frac{dR_1}{de}\right\}=-\frac{m'^2na^2}{2\omega}\left(\frac{dP^2}{de^2}+\frac{dP'^2}{de^2}\right)nt,$$

another secular perturbation of the epoch.

50. Now let

$$\delta R=\frac{dR_2}{de}\,\delta_1 e+\frac{dR_2}{d\varpi}\,\delta_1\varpi+\frac{dR_2}{de'}\,\delta_1 e'+\frac{dR_2}{d\varpi'}\,\delta_1\varpi'$$

$$+\frac{dR_1}{de}\,\delta_2 e+\frac{dR_1}{d\varpi}\,\delta_2\varpi+\frac{dR_1}{de'}\,\delta_2 e'+\frac{dR_1}{d\varpi'}\,\delta_2\varpi',$$

then $2na^2\int_t\frac{d\,(\delta R)}{da}$

$$=\frac{2m'^2n^2a^3}{\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{dP}{de}\,\frac{d^2P_2}{deda}+\frac{dP'}{de}\,\frac{d^2P_2'}{deda}\right)\sin\lambda\\+\left(\frac{dP}{de}\,\frac{d^2P_2'}{deda}-\frac{dP'}{de}\,\frac{d^2P_2}{deda}\right)\cos\lambda\end{array}\right\}$$

$$+\frac{2mm'nn'a^2a'}{\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{dQ}{de'}\,\frac{d^2P_2}{de'da}+\frac{dQ'}{de'}\,\frac{d^2P_2'}{de'da}\right)\sin\lambda\\+\left(\frac{dQ}{de'}\,\frac{d^2P_2'}{de'da}-\frac{dQ'}{de'}\,\frac{d^2P_2}{de'da}\right)\cos\lambda\end{array}\right\}$$

$$+\frac{m'^2n^2a^3}{\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{d^2P}{deda}\,\frac{dP_2}{de}+\frac{d^2P'}{deda}\,\frac{dP_2'}{de}\right)\sin\lambda\\+\left(\frac{d^2P}{deda}\,\frac{dP_2'}{de}-\frac{d^2P'}{deda}\,\frac{dP_2}{de}\right)\cos\lambda\end{array}\right\}$$

$$+\frac{mm'nn'a^2a'}{\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{d^2P}{de'da}\,\frac{dP_2}{de'}+\frac{d^2P'}{de'da}\,\frac{dP_2'}{de'}\right)\sin\lambda\\+\left(\frac{d^2P}{de'da}\,\frac{dP_2'}{de'}-\frac{d^2P'}{de'da}\,\frac{dP_2}{de'}\right)\cos\lambda\end{array}\right\}.$$

$$a^2\frac{d^2P_2}{deda}=2a^2\frac{dN_1}{da}\,e\cos 2\varpi+a^2\frac{dN_3}{da}\,e'\cos(\varpi+\varpi')\,;$$

$\therefore\ \log a^2\frac{d^2P_2}{deda}=\bar{1}\cdot 77175-,$ $\qquad\log a^2\frac{d^2P_2'}{deda}=\bar{1}\cdot 17260.$

$$a^2\frac{d^2P_2}{de'da}=2a^2\frac{dN_2}{da}\,e'\cos 2\varpi'+a^2\frac{dN_3}{da}\,e\cos(\varpi+\varpi')\,;$$

$\therefore\ \log a^2\frac{d^2P_2}{de'da}=\bar{1}\cdot 70082-,$ $\qquad\log a^2\frac{d^2P_2'}{de'da}=\bar{1}\cdot 68990-,$

$\log a^2\frac{d^2P}{deda}=0\cdot 33633-,$ $\qquad\log a^2\frac{d^2P'}{deda}=\bar{1}\cdot 65660,$

$$\log a^2 \frac{d^2P}{de'da} = \bar{1}\cdot 91155 -, \qquad \log a^2 \frac{d^2P'}{de'da} = \bar{1}\cdot 94481 -;$$

$$\begin{aligned}\therefore\ \delta\epsilon &= (+1''\cdot 306 + 0''\cdot 438 + 0''\cdot 378 + 0''\cdot 211)\sin\lambda,\\ &\quad + (-0''\cdot 053 - 0''\cdot 022)\cos\lambda,\\ &= +2''\cdot 333 \sin\lambda,\end{aligned}$$

the coefficient of the cosine may be neglected.

The corresponding formula for Neptune is

$$2n'a'^2 \int_t \frac{d(\delta R')}{da'}$$

$$\begin{aligned}
&= \frac{2mm'nn'aa'^2}{\omega^2 \sin 1''}\left\{\begin{aligned}&\left(\frac{dP}{de}\,\frac{d^2P_2}{deda'} + \frac{dP'}{de}\,\frac{d^2P_2'}{deda'}\right)\sin\lambda\\ &+\left(\frac{dP}{de}\,\frac{d^2P_2'}{deda'} - \frac{dP'}{de}\,\frac{d^2P_2}{deda'}\right)\cos\lambda\end{aligned}\right\}\\
&+ \frac{2m^2n'^2a'^3}{\omega^2 \sin 1''}\left\{\begin{aligned}&\left(\frac{dQ}{de'}\,\frac{d^2P_2}{de'da'} + \frac{dQ'}{de'}\,\frac{d^2P_2'}{de'da'}\right)\sin\lambda\\ &+\left(\frac{dQ}{de'}\,\frac{d^2P_2'}{de'da'} - \frac{dQ'}{de'}\,\frac{d^2P_2}{de'da'}\right)\cos\lambda\end{aligned}\right\}\\
&+ \frac{mm'nn'aa'^2}{\omega^2 \sin 1''}\left\{\begin{aligned}&\left(\frac{d^2Q}{deda'}\,\frac{dP_2}{de} + \frac{d^2Q'}{deda'}\,\frac{dP_2'}{de}\right)\sin\lambda\\ &+\left(\frac{d^2Q}{deda'}\,\frac{dP_2'}{de} - \frac{d^2Q'}{deda'}\,\frac{dP_2}{de}\right)\cos\lambda\end{aligned}\right\}\\
&+ \frac{m^2n'^2a'^3}{\omega^2 \sin 1''}\left\{\begin{aligned}&\left(\frac{d^2Q}{de'da'}\,\frac{dP_2}{de'} + \frac{d^2Q'}{de'da'}\,\frac{dP_2'}{de'}\right)\sin\lambda\\ &+\left(\frac{d^2Q}{de'da'}\,\frac{dP_2'}{de'} - \frac{d^2Q'}{de'da'}\,\frac{dP_2}{de'}\right)\cos\lambda\end{aligned}\right\}.
\end{aligned}$$

$$a'^2 \frac{d^2P_2}{deda'} = -\frac{1}{a}\left(a^2 \frac{d^2P_2}{deda} + a\frac{dP_2}{de}\right); \quad \therefore \log = 0\cdot 04856,$$

$$a'^2 \frac{d^2P_2'}{deda'} = -\frac{1}{a}\left(a^2 \frac{d^2P_2'}{deda} + a\frac{dP_2'}{de}\right); \quad \therefore \log = \bar{1}\cdot 43837 -,$$

$$a'^2 \frac{d^2P_2}{de'da'} = -\frac{1}{a}\left(a^2 \frac{d^2P_2}{de'da} + a\frac{dP_2}{de'}\right); \quad \therefore \log = \bar{1}\cdot 99162,$$

$$a'^2 \frac{d^2P_2'}{de'da'} = -\frac{1}{a}\left(a^2 \frac{d^2P_2'}{de'da} + a\frac{dP_2'}{de'}\right); \quad \therefore \log = \bar{1}\cdot 98839.$$

$$\log a'^2 \frac{d^2Q}{deda'} = 0\cdot 66346, \qquad \log a'^2 \frac{d^2Q'}{deda'} = \bar{1}\cdot 98373 -,$$

$$\log a'^2 \frac{d^2Q}{de'da'} = 0\cdot 61850, \qquad \log a'^2 \frac{d^2Q'}{de'da'} = 0\cdot 65176;$$

$$\begin{aligned}\therefore\ \delta\epsilon' &= (-1''\cdot 126 - 0''\cdot 394 - 0''\cdot 367 - 0''\cdot 491)\sin\lambda,\\ &= -2''\cdot 378 \sin\lambda,\end{aligned}$$

neglecting the coefficient of $\cos\lambda$ which is very small.

The value of $-\frac{na}{2}\int_t \left\{e\frac{d(\delta R)}{de} + \delta_1 e\frac{dR_2}{de} + \delta_2 e\frac{dR_1}{de}\right\}$ depending on this part of δR is very small, and may therefore be neglected.

51. Let F_0 be the constant term of R which is independent of the eccentricities and inclinations; and let $\delta R = \frac{dF_0}{da}\delta_1 a + \frac{dF_0}{da'}\delta_1 a'$,

$$\text{then } 2na^2\int_t \frac{d(\delta R)}{da} = -\frac{4m'n^2a^4}{\omega^2 \sin 1''}\frac{d^2F_0}{da^2}(P\sin\lambda + P'\cos\lambda)$$

$$+\frac{8mnn'a^2a'^2}{\omega^2\sin 1''}\frac{d^2F_0}{dada'}(Q\sin\lambda + Q'\cos\lambda),$$

$$aF_0 = -\frac{m'}{2}ab_0^{(\frac{1}{2})},$$

$$a^3\frac{d^2F_0}{da^2} = -\frac{m'}{2}a^3\frac{d^2b_0^{(\frac{1}{2})}}{da^2} = -m'aC_1 \text{ suppose,}$$

$$a^2a'\frac{d^2F_0}{dada'} = -\left(2a^2\frac{dF_0}{da} + a^3\frac{d^2F_0}{da^2}\right)$$

$$= +\frac{m'}{2}a\left(2a\frac{db_0^{(\frac{1}{2})}}{da} + a^2\frac{d^2b_0^{(\frac{1}{2})}}{da^2}\right) = m'aC_2 \text{ suppose}$$

then $\log aC_1 = \bar{1}\cdot76899$, $\log aC_2 = 0\cdot02293$.

Hence $\delta\epsilon = -0''\cdot431\sin\lambda + 0''\cdot063\cos\lambda$.

The formula for Neptune is,

$$2n'a'^2\int_t\frac{d(\delta R')}{da'} = -\frac{4mnn'a^2a'^2}{\omega^2\sin 1''}\frac{d^2F_0}{dada'}(P\sin\lambda + P'\cos\lambda),$$

$$+\frac{8m^2n'^2a'^4}{m'\omega^2\sin 1''}\frac{d^2F_0}{da'^2}(Q\sin\lambda + Q'\cos\lambda),$$

$$= -\frac{4mm'nn'aa'C_2}{\omega^2\sin 1''}(P\sin\lambda + P'\cos\lambda),$$

$$-\frac{8m^2n'^2a'^2C_2'}{\omega^2\sin 1''}(Q\sin\lambda + Q'\cos\lambda),$$

where $a'C_2' = \frac{1}{2}\left(2b_0^{(\frac{1}{2})} + 4a\frac{db_0^{(\frac{1}{2})}}{da} + a^2\frac{d^2b_0^{(\frac{1}{2})}}{da^2}\right)$; $\therefore \log a'C_2' = 0\cdot66762$;

$$\therefore \delta e' = +0''\cdot781\sin\lambda - 0''\cdot114\cos\lambda.$$

52. The above terms are of very little importance: but there are terms of the same kind arising from the action of the constant part of the disturbing force of Saturn and Jupiter upon Uranus and Neptune which are much more considerable. Let F_1 be the constant part of the disturbing function for Saturn and Uranus, m_1 the mass of Saturn, a_1 the $\frac{1}{2}$ major axis of its orbit $\alpha' = \frac{a_1}{a}$,

$$\text{then } aF_1 = -\frac{m_1}{2} b_0', \quad \text{and } \delta R = \frac{dF_1}{da} \delta_1 a,$$

$$\therefore\ 2na^2 \int_t \frac{d(\delta R)}{da} = -\frac{4m'n^2a^4}{\omega^2 \sin 1''} \frac{d^2F_1}{da^2} (P \sin\lambda + P' \cos\lambda),$$

$$a^3 \frac{d^2F_1}{da^2} = -\frac{m_1}{2} \left(2b_0' + 4a' \frac{db_0'}{da'} + a'^2 \frac{d^2b_0'}{da'^2}\right) = -m_1 aC \text{ suppose;}$$

$$\therefore\ 2na^2 \int_t \frac{d(\delta R)}{da} = +\frac{4m_1m'n^2a^2C}{\omega^2 \sin 1''} (P \sin\lambda + P' \cos\lambda).$$

According to Pontécoulant (Liv. VI. p. 375),

$$\log b_0' = 0{\cdot}33131, \qquad \log a' \frac{db_0'}{da'} = \bar{1}{\cdot}53098, \qquad \log a'^2 \frac{d^2b_0'}{da'^2} = \bar{1}{\cdot}76910,$$

$$m_1 = \frac{1}{3501{\cdot}6}; \qquad \therefore\ \log aC = 0{\cdot}49379, \qquad \log \frac{4m_1m'n^2}{\omega^2 \sin 1''} = 1{\cdot}46410;$$

$$\therefore\ \delta' e = -3''{\cdot}431 \sin\lambda + 0''{\cdot}517 \cos\lambda.$$

Similarly, if F_2 be the constant term of the disturbing function for Jupiter and Uranus, m_2 = mass of Jupiter, then $aF_2 = -\frac{m_2}{2} b_0''$,

$$a^3 \frac{d^2F_2}{da^2} = -\frac{m_2}{2} \left(2b_0'' + 4a'' \frac{db_0''}{da''} + a''^2 \frac{d^2b_0''}{da''^2}\right),$$

$$= -m_2 aD \text{ suppose.}$$

$$\delta R = \frac{dF_2}{da} \delta_1 a;$$

$$\therefore\ 2na^2 \int_t \frac{d(\delta R)}{da} = +\frac{4m_2m'n^2a^2D}{\omega^2 \sin 1''} (P \sin\lambda + P' \cos\lambda),$$

$$\log b_0'' = 0{\cdot}30928, \qquad \log a'' \frac{db_0''}{da''} = \bar{2}{\cdot}90360, \qquad \log a''^2 \frac{d^2b_0''}{da''^2} = \bar{2}{\cdot}97473,$$

$$m_2 = \frac{1}{1047{\cdot}9}, \qquad \log aD = 0{\cdot}35133, \qquad \log \frac{4m_2m'n^2}{\omega^2 \sin 1''} = 1.98805;$$

$$\therefore\ \delta' e = -8''{\cdot}259 \sin\lambda + 1''{\cdot}244 \cos\lambda.$$

For the action of Saturn on Neptune, the formula is

$$\delta' e' = 2n'a'^2 \int_t \frac{d(\delta R')}{da'} = -\frac{8m_1mn'^2a'^2C'}{\omega^2 \sin 1''} (Q \sin\lambda + Q' \cos\lambda),$$

$$\text{where } \quad a'C' = b_0' + 2a' \frac{db_0'}{da'} + \tfrac{1}{2} a'^2 \frac{d^2b_0'}{da'^2},$$

$$\text{and} \quad \log a' = \bar{1}{\cdot}50179, \qquad \log b_0' = 0{\cdot}31249,$$

$$\log a' \frac{db_0'}{da'} = \bar{1}{\cdot}05534 \qquad \log a'^2 \frac{d^2b_0'}{da'^2} = \bar{1}{\cdot}15275.$$

$$\text{Hence}\quad \log a'C' = 0{\cdot}37140, \quad \text{also } \log \frac{8m_1 m n'^2}{\omega^2 \sin 1''} = 1{\cdot}13239,$$

$$\therefore\ \breve{\delta}\epsilon' = +\,1''{\cdot}898 \sin\lambda - 0''{\cdot}274 \cos\lambda.$$

For the action of Jupiter on Neptune, we have

$$\breve{\delta}\epsilon' = 2n'a'^2 \int_t \frac{d\,(\delta R')}{da'} = -\frac{8m_2 m n'^2 a'^2 D'}{\omega^2 \sin 1''}\,(Q \sin\lambda + Q' \cos\lambda),$$

$$\text{where } a'D' = b_0'' + 2a''\,\frac{db_0''}{da''} + \tfrac{1}{2}a''^2\,\frac{d^2 b_0''}{da''^2},$$

$$\text{and}\quad \log a'' = \bar{1}{\cdot}23859, \qquad \log b_0'' = 0{\cdot}30433,$$

$$\log a''\,\frac{db_0''}{da''} = \bar{2}{\cdot}49205, \qquad \log a''^2\,\frac{d^2 b_0''}{da''^2} = \bar{2}{\cdot}52123, \qquad \log \frac{8m^2 m n'^2}{\omega^2 \sin 1''} = 1{\cdot}65634.$$

$$\text{Hence}\quad \log a'D' = 0{\cdot}32098,$$

$$\therefore\ \breve{\delta}\epsilon' = +\,5''{\cdot}647 \sin\lambda - 0''{\cdot}815 \cos\lambda.$$

53. The effect of the square of the disturbing force on the eccentricity and longitude of perihelion, will be given by the equations

$$\breve{\delta} e = \frac{na}{e} \int_t \left\{ \frac{d\,(\delta R)}{d\varpi} - \frac{\delta e}{e}\,\frac{dR}{d\varpi} \right\}, \qquad e\breve{\delta}\varpi = na \int_t \left\{ \frac{\delta e}{e}\,\frac{dR}{de} - \frac{d\,(\delta R)}{de} \right\},$$

(see Pontécoulant, Liv. VI. p. 213), omitting terms of the first and higher orders, except those terms of the first order which will be divided by ω^3. First let

$$\delta R = \frac{dR_1}{d\epsilon}\,\zeta_1 + \frac{dR_1}{d\epsilon'}\,\zeta_1',$$

$$\text{then}\quad \frac{na}{e} \int_t \frac{d\,(\delta R)}{d\varpi} = +\,\frac{3m'^2 n^2 a^2}{2\omega^2}\left(1 + \frac{4m}{m'}\,a^2\right)\left(P\,\frac{dP'}{de} - P'\,\frac{dP}{de}\right) nt$$

$$-\,\frac{3m'^2 n^3 a^2}{4\omega^3 \sin 1''}\left(1 + \frac{4m}{m'}\,a^2\right)\left\{\begin{array}{l} \left(P\,\dfrac{dP'}{de} + P'\,\dfrac{dP}{de}\right) \sin 2\lambda \\ -\left(P\,\dfrac{dP}{de} - P'\,\dfrac{dP'}{de}\right) \cos 2\lambda \end{array}\right\},$$

therefore, omitting the secular perturbation,

$$\breve{\delta} e = -\,1''{\cdot}267 \sin 2\lambda - 3''{\cdot}413 \cos 2\lambda.$$

$$\text{Also,}\quad -\,na \int_t \frac{d\,(\delta R)}{de} = +\,\frac{3m'^2 n^2 a^2}{2\omega^2}\left(1 + \frac{4m}{m'}\,a^2\right)\left(P\,\frac{dP}{de} + P'\,\frac{dP'}{de}\right) nt$$

$$-\,\frac{3m'^2 n^3 a^2}{4\omega^3 \sin 1''}\left(1 + \frac{4m}{m'}\,a^2\right)\left\{\begin{array}{l} \left(P\,\dfrac{dP}{de} - P'\,\dfrac{dP'}{de}\right) \sin 2\lambda \\ +\left(P\,\dfrac{dP'}{de} + P'\,\dfrac{dP}{de}\right) \cos 2\lambda \end{array}\right\},$$

$$\therefore\ e\breve{\delta}\varpi = +\,3''{\cdot}413 \sin 2\lambda - 1''{\cdot}267 \cos 2\lambda.$$

Similarly,

$$\frac{n'a'}{e'}\int_t \frac{d(\delta R')}{d\varpi'} = \frac{3mm'nn'aa'}{2\omega^2}\left(1+\frac{4m}{m'}a^2\right)\left(P\frac{dQ'}{de'} - P'\frac{dQ}{de'}\right)nt$$

$$-\frac{3mm'n^2n'aa'}{4\omega^3 \sin 1''}\left(1+\frac{4m}{m'}a^2\right)\left\{\begin{array}{c}\left(P\dfrac{dQ'}{de'} + P'\dfrac{dQ}{de'}\right)\sin 2\lambda \\ -\left(P\dfrac{dQ}{de'} - P'\dfrac{dQ'}{de'}\right)\cos 2\lambda\end{array}\right\},$$

$$\therefore\ \delta e' = +0''{\cdot}663 \sin 2\lambda - 0''{\cdot}829 \cos 2\lambda,$$

$$e'\delta\varpi' = +0''{\cdot}829 \sin 2\lambda + 0''{\cdot}663 \cos 2\lambda.$$

54. Now let

$$\delta R = \frac{dR_1}{de}\delta_1 e + \frac{dR_1}{d\varpi}\delta_1\varpi + \frac{dR_1}{de'}\delta_1 e' + \frac{dR_1}{d\varpi'}\delta_1\varpi',$$

$$\text{then}\quad \frac{d(\delta R)}{d\varpi} = 0;$$

$$\text{and}\quad -\frac{na}{e}\int_t \frac{\delta_1 e}{e}\frac{dR_1}{d\varpi} = +\frac{m'^2n^2a^2}{4e\omega^2\sin 1''}\left\{2\frac{dP}{de}\frac{dP'}{de}\sin 2\lambda - \left(\frac{dP^2}{de^2} - \frac{dP'^2}{de^2}\right)\cos 2\lambda\right\},$$

$$\therefore\ \delta e = -1''{\cdot}682 \sin 2\lambda - 4''{\cdot}155 \cos 2\lambda.$$

$$na\int_t\left\{\frac{\delta_1 e}{e}\frac{dR_1}{de} - \frac{d(\delta R)}{de}\right\} = \frac{m'^2n^2a^2}{2e\omega^2\sin 1''}\left\{\left(\frac{dP^2}{de^2} - \frac{dP'^2}{de^2}\right)\sin 2\lambda + 2\frac{dP}{de}\frac{dP'}{de}\cos 2\lambda\right\},$$

$$\therefore\ e\delta\varpi = +8''{\cdot}310 \sin 2\lambda - 3''{\cdot}363 \cos 2\lambda.$$

The formulæ for $\delta e'$ and $e'\delta\varpi'$, may be obtained from the above, by writing $mn'a'e'QQ'$ for $m'naePP'$; hence

$$\delta e' = \frac{m^2n'^2a'^2}{4e'\omega^2\sin 1''}\left\{2\frac{dQ}{de'}\frac{dQ'}{de'}\sin 2\lambda - \left(\frac{dQ^2}{de'^2} - \frac{dQ'^2}{de'^2}\right)\cos 2\lambda\right\}$$

$$= +2''{\cdot}058 \sin 2\lambda + 0''{\cdot}158 \cos 2\lambda.$$

$$e'\delta'\varpi' = -0''{\cdot}316 \sin 2\lambda + 4''{\cdot}117 \cos 2\lambda.$$

55. Next let δR have the same value as in (Art. 50), then

$$\delta e = \frac{na}{e}\int_t\left\{\frac{d(\delta R)}{d\varpi} - \frac{\delta_1 e}{e}\frac{dR_2}{d\varpi} - \frac{\delta_2 e}{e}\frac{dR_1}{d\varpi}\right\}$$

$$= \frac{m'^2n^2a^2}{\omega^2\sin 1''}\left\{\left(\frac{d^2P_2}{de^2}\frac{dP'}{de} - \frac{d^2P_2'}{de^2}\frac{dP}{de}\right)\sin\lambda + \left(\frac{d^2P_2}{de^2}\frac{dP}{de} + \frac{d^2P_2'}{de^2}\frac{dP'}{de}\right)\cos\lambda\right\}$$

$$+\frac{mm'nn'aa'}{\omega^2\sin 1''}\left\{\left(\frac{d^2P_2}{dede'}\frac{dQ'}{de'} - \frac{d^2P_2'}{dede'}\frac{dQ}{de'}\right)\sin\lambda + \left(\frac{d^2P_2}{dede'}\frac{dQ}{de'} + \frac{d^2P_2'}{dede'}\frac{dQ'}{de'}\right)\cos\lambda\right\}$$

$$+\frac{m'^2n^2a^2}{4e\omega^2\sin 1''}\left\{\left(\frac{dP_2}{de}\frac{dP'}{de} - \frac{dP_2'}{de}\frac{dP}{de}\right)\sin\lambda + \left(\frac{dP_2}{de}\frac{dP}{de} + \frac{dP_2'}{de}\frac{dP'}{de}\right)\cos\lambda\right\}$$

$$+\frac{m'^2n^2a^2}{4e\omega^2\sin 1''}\left\{\left(\frac{dP}{de}\frac{dP_2'}{de} + \frac{dP'}{de}\frac{dP_2}{de}\right)\sin 3\lambda - \left(\frac{dP}{de}\frac{dP_2}{de} - \frac{dP'}{de}\frac{dP_2'}{de}\right)\cos 3\lambda\right\},$$

$$a\frac{d^2P_2}{de^2} = 2aN_1\cos 2\varpi, \quad \therefore\ \log = 0{\cdot}32723\ -,$$

$$a\frac{d^2P_2'}{de^2} = 2aN_1\sin 2\varpi, \quad \therefore\ \log = \bar{1}{\cdot}96794,$$

$$a\frac{d^2P_2}{dede'} = aN_3\cos(\varpi+\varpi'), \quad \therefore\ \log = 0{\cdot}43638\ -,$$

$$a\frac{d^2P_2'}{dede'} = aN_3\sin(\varpi+\varpi'), \quad \therefore\ \log = 0{\cdot}28779\ -,$$

$$\begin{aligned}\text{hence}\quad \delta e &= (+\,0''{\cdot}509 + 0''{\cdot}214 + 0''{\cdot}004)\sin\lambda\\ &\quad + (2''{\cdot}433 + 1''{\cdot}025 + 0''{\cdot}722)\cos\lambda\\ &\quad - 0''{\cdot}293\sin 3\lambda - 0''{\cdot}660\cos 3\lambda\\ &= +\,0''{\cdot}727\sin\lambda + 4''{\cdot}180\cos\lambda\\ &\quad - 0''{\cdot}293\sin 3\lambda - 0''{\cdot}660\cos 3\lambda,\end{aligned}$$

$$e\delta\varpi = na\int_t\left\{\frac{\delta_1 e}{e}\,\frac{dR_2}{de} + \frac{\delta_2 e}{e}\,\frac{dR_1}{de} - \frac{d\,(\delta R)}{de}\right\}$$

$$\begin{aligned}&= -\frac{m'^2n^2a^2}{\omega^2\sin 1''}\left\{\left(\frac{dP}{de}\,\frac{d^2P_2}{de^2} + \frac{dP'}{de}\,\frac{d^2P_2'}{de^2}\right)\sin\lambda + \left(\frac{dP}{de}\,\frac{d^2P_2'}{de^2} - \frac{dP'}{de}\,\frac{d^2P_2}{de^2}\right)\cos\lambda\right\}\\ &\quad -\frac{mm'nn'aa'}{\omega^2\sin 1''}\left\{\left(\frac{dQ}{de'}\,\frac{d^2P_2}{dede'} + \frac{dQ'}{de'}\,\frac{d^2P_2'}{dede'}\right)\sin\lambda + \left(\frac{dQ}{de'}\,\frac{d^2P_2'}{dede'} - \frac{dQ'}{de'}\,\frac{d^2P_2}{dede'}\right)\cos\lambda\right\}\\ &\quad +\frac{m'^2n^2a^2}{2e\omega^2\sin 1''}\left\{\left(\frac{dP}{de}\,\frac{dP_2}{de} - \frac{dP'}{de}\,\frac{dP_2'}{de}\right)\sin 3\lambda + \left(\frac{dP}{de}\,\frac{dP_2'}{de} + \frac{dP'}{de}\,\frac{dP_2}{de}\right)\cos 3\lambda\right\},\end{aligned}$$

$$\begin{aligned}\therefore\ e\delta\varpi &= (-\,2''{\cdot}433 - 1''{\cdot}025)\sin\lambda + 1''{\cdot}320\sin 3\lambda\\ &\quad + (0''{\cdot}509 + 0''{\cdot}214)\cos\lambda - 0''{\cdot}586\cos 3\lambda\\ &= -\,3''{\cdot}458\sin\lambda + 0''{\cdot}723\cos\lambda\\ &\quad + 1''{\cdot}320\sin 3\lambda - 0''{\cdot}586\cos 3\lambda.\end{aligned}$$

The formulæ for $\delta e'$ and $e'\delta\varpi'$ may be obtained from the above, by writing $m'n'a'e'QQ'$ for $mnaePP'$ and *vice versâ*.

$$\begin{aligned}\therefore\ \delta' e' &= \frac{m^2n'^2a'^2}{\omega^2\sin 1''}\left\{\begin{aligned}&\left(\frac{d^2P_2}{de'^2}\,\frac{dQ'}{de'} - \frac{d^2P_2'}{de'^2}\,\frac{dQ}{de'}\right)\sin\lambda\\ +\ &\left(\frac{d^2P_2}{de'^2}\,\frac{dQ}{de'} + \frac{d^2P_2'}{de'^2}\,\frac{dQ'}{de'}\right)\cos\lambda\end{aligned}\right\}\\ &\quad + \frac{mm'nn'aa'}{\omega^2\sin 1''}\left\{\begin{aligned}&\left(\frac{d^2P_2}{dede'}\,\frac{dP'}{de} - \frac{d^2P_2'}{dede'}\,\frac{dP}{de}\right)\sin\lambda\\ +\ &\left(\frac{d^2P_2}{dede'}\,\frac{dP}{de} + \frac{d^2P_2'}{dede'}\,\frac{dP'}{de}\right)\cos\lambda\end{aligned}\right\}\\ &\quad + \frac{m^2n'^2a'^2}{4e'\omega^2\sin 1''}\left\{\begin{aligned}&\left(\frac{dP_2}{de'}\,\frac{dQ'}{de'} - \frac{dP_2'}{de'}\,\frac{dQ}{de'}\right)\sin\lambda\\ +\ &\left(\frac{dP_2}{de'}\,\frac{dQ}{de'} + \frac{dP_2'}{de'}\,\frac{dQ'}{de'}\right)\cos\lambda\end{aligned}\right\}\end{aligned}$$

$$+\frac{m^2n'^2a'^2}{4e'\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{dQ}{de'}\,\frac{dP_2'}{de'}+\frac{dQ'}{de'}\,\frac{dP_2}{de'}\right)\sin 3\lambda\\ -\left(\frac{dQ}{de'}\,\frac{dP_2}{de'}-\frac{dQ'}{de'}\,\frac{dP_2'}{de'}\right)\cos 3\lambda\end{array}\right\},$$

$$a'\,\frac{d^2P_2}{de'^2}=\frac{2aN_2}{a}\cos 2\varpi',\quad\therefore\ \log=\bar{1}\cdot 75774,$$

$$a'\,\frac{d^2P_2'}{de'^2}=\frac{2aN_2}{a}\sin 2\varpi',\quad\therefore\ \log=0\cdot 87320-,$$

$$\begin{aligned}\therefore\ \delta e'&=(-0''\cdot 784-1''\cdot 883)\sin\lambda\\ &\quad+(+0''\cdot 726+1''\cdot 744+1''\cdot 162)\cos\lambda\\ &\quad+1''\cdot 159\sin 3\lambda+0''\cdot 080\cos 3\lambda\\ &=-2''\cdot 667\sin\lambda+3''\cdot 632\cos\lambda\\ &\quad+1''\cdot 159\sin 3\lambda+0''\cdot 080\cos 3\lambda\,;\end{aligned}$$

$$\begin{aligned}e'\delta\varpi'=&-\frac{m^2n'^2a'^2}{\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{d^2P_2}{de'^2}\,\frac{dQ}{de'}+\frac{d^2P_2'}{de'^2}\,\frac{dQ'}{de'}\right)\sin\lambda\\ -\left(\frac{d^2P_2}{de'^2}\,\frac{dQ'}{de'}-\frac{d^2P_2'}{de'^2}\,\frac{dQ}{de'}\right)\cos\lambda\end{array}\right\}\\ &-\frac{mm'nn'aa'}{\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{d^2P_2}{dede'}\,\frac{dP}{de}+\frac{d^2P_2'}{dede'}\,\frac{dP'}{de}\right)\sin\lambda\\ -\left(\frac{d^2P_2}{dede'}\,\frac{dP'}{de}-\frac{d^2P_2'}{dede'}\,\frac{dP}{de}\right)\cos\lambda\end{array}\right\}\\ &+\frac{m^2n'^2a'^2}{2e'\omega^2\sin 1''}\left\{\begin{array}{l}\left(\frac{dQ}{de'}\,\frac{dP_2}{de'}-\frac{dQ'}{de'}\,\frac{dP_2'}{de'}\right)\sin 3\lambda\\ +\left(\frac{dQ}{de'}\,\frac{dP_2'}{de'}+\frac{dQ'}{de'}\,\frac{dP_2}{de'}\right)\cos 3\lambda\end{array}\right\};\end{aligned}$$

$$\begin{aligned}\therefore\ e'\delta\varpi'&=(-0''\cdot 726-1''\cdot 744)\sin\lambda-0''\cdot 160\sin 3\lambda,\\ &\quad+(-0''\cdot 784-1''\cdot 883)\cos\lambda+2''\cdot 319\cos 3\lambda,\\ &=-2''\cdot 470\sin\lambda-2''\cdot 667\cos\lambda,\\ &=-0''\cdot 160\sin 3\lambda+2''\cdot 319\cos 3\lambda.\end{aligned}$$

56. Let F = the constant part of the disturbing function of the second order in the eccentricities;

$$\therefore\ F=m'C_2(e^2+e'^2)+m'C_3ee'\cos(\varpi-\varpi'),$$

when $$aC_2=-\frac{a}{4}\left(a\,\frac{db_0^{(\frac{1}{2})}}{da}+\tfrac{1}{2}\,a^2\,\frac{d^2b_0^{(\frac{1}{2})}}{da^2}\right),$$

$$aC_3=-\frac{a}{2}\left(b_1^{(\frac{1}{2})}-a\,\frac{db_1^{(\frac{1}{2})}}{da}-\tfrac{1}{2}\,a^2\,\frac{d^2b_1^{(\frac{1}{2})}}{da^2}\right).$$

Then $$\delta F=\frac{dF}{de}\,\delta_1e+\frac{dF}{d\varpi}\,\delta_1\varpi+\frac{dF}{de'}\,\delta_1e'+\frac{dF}{d\varpi'}\,\delta_1\varpi',$$

and $$\delta e=\frac{na}{e}\int_t\left\{\frac{d(\delta F)}{d\varpi}-\frac{\delta_1e}{e}\,\frac{dF}{d\varpi}\right\},$$

$$= \frac{m'^2 n^2 a^2 C_3}{\omega^2 \sin 1''} \frac{e'}{e} \cos(\varpi - \varpi')\left\{\frac{dP'}{de}\sin\lambda - \frac{dP}{de}\cos\lambda\right\},$$

$$- \frac{mm'nn'aa'}{\omega^2 \sin 1''} C_3 M_2' \left\{\sin\varpi \sin\lambda - \cos\varpi \cos\lambda\right\},$$

$$= + 0''{\cdot}017 \sin\lambda + 0''{\cdot}081 \cos\lambda,$$

which may be rejected. We may also reject the corresponding terms of $e\delta\varpi$, $\delta e'$, $e'\delta\varpi'$, all of which are very small.

For the effect of Saturn on Uranus, the formula is

$$\delta e = \frac{m_1 m' n^2 a^2 C_3'}{\omega^2 \sin 1''} \frac{e_1}{e} \cos(\varpi - \varpi_1)\left\{\frac{dP'}{de}\sin\lambda - \frac{dP}{de}\cos\lambda\right\},$$

and a similar formula will give the effect due to the action of Jupiter: but these terms may be neglected; for all the coefficients are less than half a second.

57. The terms calculated in the preceding articles seem to be all of those due to the square of the disturbing force, which are of sufficient magnitude to be retained. But there are other larger terms of two dimensions in the masses, depending upon the constant corrections which must be applied to the values of the elements used in making the first approximation. Let Δe, $\Delta\varpi$, $\Delta e'$, $\Delta\varpi'$ be the corrections which must be applied to the values of e, ϖ, e', ϖ', used in the first section, so that $e + \Delta e$, &c. are accurately equal to the constants introduced in integrating the equations for $\frac{de}{dt}$, &c.: then, if e, ϖ, &c. were accurately the elements of the instantaneous ellipse at the epoch $t = 0$, Δe would be given by the equation

$$\Delta e = - \Sigma(\delta e)_{t=0},$$

$\Sigma(\delta e)$ being the sum of all the perturbations of the eccentricity produced by the action of all the planets: and similarly for $\Delta\varpi$, &c. We shall calculate the values of these corrections only in so far as they depend upon the long inequality.

Since e and ϖ (as deduced from observation) are given for the epoch of 1801, we must put $t = -(46 \times 365 + 11)$ days in the expressions for δe and $\delta\varpi$, in calculating the values of Δe and $\Delta\varpi$. If, then, λ'' = the value of λ for Jan. 1, 1801, $= 83^\circ\, 12'\, 35''$,

$$\begin{aligned} \Delta e = &- 85''{\cdot}105 \sin \lambda'' - 408''{\cdot}350 \cos \lambda'' \\ &- 6''{\cdot}989 \sin 2\lambda'' - 32''{\cdot}541 \cos 2\lambda'' \\ &- 0''{\cdot}759 \sin 3\lambda'' - 3''{\cdot}219 \cos 3\lambda'' \\ = &- 100''{\cdot}967. \end{aligned}$$

$$\begin{aligned} e\Delta\varpi = &+ 409''{\cdot}040 \sin \lambda'' - 84''{\cdot}810 \cos \lambda'' \\ &+ 32''{\cdot}498 \sin 2\lambda'' - 6''{\cdot}988 \cos 2\lambda'' \\ &+ 3''{\cdot}219 \sin 3\lambda'' - 0''{\cdot}759 \cos 3\lambda'' \\ = &+ 407''{\cdot}811. \end{aligned}$$

And if λ' = the value of λ at time $t = 0$,

$= 78^\circ\, 28''\, 36''{\cdot}4$.

$$\begin{aligned}\Delta e' &= +\ 89''{\cdot}999 \sin\ \lambda' - 83''{\cdot}276 \cos\ \lambda' \\ &\quad + 24''{\cdot}981 \sin 2\lambda' - 23''{\cdot}490 \cos 2\lambda' \\ &\quad + 2''{\cdot}451 \sin 3\lambda' - 2''{\cdot}406 \cos 3\lambda' \\ &= +102''{\cdot}292.\end{aligned}$$

$$\begin{aligned}e'\Delta\varpi' &= +\ 83''{\cdot}024 \sin\ \lambda' + 90''{\cdot}762 \cos\ \lambda' \\ &\quad + 23''{\cdot}501 \sin 2\lambda' + 24''{\cdot}981 \cos 2\lambda' \\ &\quad + 2''{\cdot}406 \sin 3\lambda' + 2''{\cdot}451 \cos 3\lambda' \\ &= +\ 82''{\cdot}322.\end{aligned}$$

To the above value of Δe may here be added the variation of e due to secular perturbation during the 46 years between 1801 and 1847, which was not taken account of in Art. (5), and which $= -2''{\cdot}51$. Again the value of the annual increase of ϖ including the secular perturbation produced by Neptune, and employing Bessel's value of precession, $= 53''.115$, which differs from the value used in Art. (5) by $+0''{\cdot}615$; and this difference amounts in 46 years to $+28''{\cdot}29$. The values, therefore, of the corrections will be, finally,

$$\Delta e = -103''{\cdot}48, \qquad \Delta\varpi = +2^\circ\, 26'\ 4''{\cdot}8,$$

$$\Delta e' = +102''{\cdot}29, \qquad \Delta\varpi' = +2^\circ\, 37'\, 21''{\cdot}0.$$

Hence for the corrected values of the elements, we have

$$\log e = \bar{2}{\cdot}6644323, \qquad \log e' = \bar{3}{\cdot}9645133,$$

$$\log \sin\varpi = \bar{1}{\cdot}2118325, \qquad \log\cos\varpi = \bar{1}{\cdot}9941623\ -,$$

$$\log\sin\varpi' = \bar{1}{\cdot}8830684, \qquad \log\cos\varpi' = \bar{1}{\cdot}8097403.$$

Upon recalculating those terms of $\delta(nt+\epsilon)$ and $\delta(n't+\epsilon')$, which depend upon the first powers of the eccentricities, with these corrected values of the elements, we shall find that the following terms must be added to those given in Art. (38.)

$$\delta\,(nt+\epsilon) = +7''{\cdot}159 \sin\lambda - 131''{\cdot}761 \cos\lambda,$$

$$\delta\,(n't+\epsilon') = -5''{\cdot}133 \sin\lambda + 94''{\cdot}505 \cos\lambda,$$

and the corresponding corrections of δe, $e\delta\varpi$, $\delta e'$, $e'\delta\varpi'$, are

$$\begin{aligned}\delta e &= -17''{\cdot}45 \sin\lambda + 3''{\cdot}26 \cos\lambda, \\ e\delta\varpi &= -\ 7''{\cdot}74 \sin\lambda - 16''{\cdot}71 \cos\lambda, \\ \delta e' &= -\ 3''{\cdot}70 \sin\lambda - 4''{\cdot}18 \cos\lambda, \\ e'\delta\varpi' &= +\ 8''{\cdot}41 \sin\lambda + 1''{\cdot}31 \cos\lambda.\end{aligned}$$

Again, $\Delta n = -0''{\cdot}00214598$ at 1801,

$\Delta n' = +0''{\cdot}00226816$ at 1847.

Hence $\omega + \Delta\omega = -0''{\cdot}8834086.$

Therefore upon recalculating the principal terms, using the multiplier $\dfrac{3m'(n+\Delta n)^2}{(\omega+\Delta\omega)^2 \sin 1''}$ instead of $\dfrac{3m'n^2}{\omega^2 \sin 1''}$ we shall find that the following terms must be added,

$$\delta\,(nt+\epsilon) = +43''{\cdot}74 \sin\lambda - 4''{\cdot}53 \cos\lambda,$$

and similarly,

$$\delta\,(n't+\epsilon') = -31''{\cdot}43 \sin\lambda + 3''{\cdot}04 \cos\lambda,$$

and the corresponding corrections of δe, $e\delta\varpi$, will be

$$\breve{\delta} e = -3''{\cdot}11 \cos\lambda, \qquad e\breve{\delta}\varpi = +3''{\cdot}11 \sin\lambda,$$

the corrections of $\delta e'$, $e'\delta\varpi'$ are less than 1″.

$$\text{Also} \quad \Delta a = +0{\cdot}00064982, \qquad \Delta a' = -0{\cdot}00210709.$$

$$\text{Hence} \quad \breve{\delta}(nt+\epsilon) = \frac{3m'n^2}{\omega^2 \sin 1''}\left\{\Delta aP \sin\lambda + \Delta aP' \cos\lambda\right\},$$

$$\text{where} \quad \Delta aP = \left\{aP + a^2 \frac{dP}{da}\right\}\left(\frac{\Delta a}{a} - \frac{\Delta a'}{a'}\right),$$

$$\Delta aP' = \left\{aP' + a^2 \frac{dP'}{da}\right\}\left(\frac{\Delta a}{a} - \frac{\Delta a'}{a'}\right);$$

$\therefore\ \breve{\delta}(nt+\epsilon) = -1''{\cdot}16 \sin\lambda$; the corresponding term of $\breve{\delta}(n't+\epsilon')$ is less than 1″.

58. Those terms of the variations of the elements which depend upon 2λ and 3λ may be corrected with sufficient accuracy by means of the following formulæ:

$$\begin{aligned}
\breve{\delta}(nt+\epsilon) = &\left\{\begin{matrix} -0{\cdot}02128\,\Delta e - 0{\cdot}00452\,e\,\Delta\varpi \\ -0{\cdot}02175\,\Delta e' + 0{\cdot}02323\,e'\Delta\varpi' \end{matrix}\right\} \sin 2\lambda \\
&+ \left\{\begin{matrix} +0{\cdot}00452\,\Delta e - 0{\cdot}02128\,e\,\Delta\varpi \\ -0{\cdot}02323\,\Delta e' - 0{\cdot}02175\,e'\Delta\varpi' \end{matrix}\right\} \cos 2\lambda \\
&+ \left\{\begin{matrix} -0{\cdot}00226\,\Delta e - 0{\cdot}00053\,e\,\Delta\varpi \\ -0{\cdot}00236\,\Delta e' + 0{\cdot}00240\,e'\Delta\varpi' \end{matrix}\right\} \sin 3\lambda \\
&+ \left\{\begin{matrix} +0{\cdot}00053\,\Delta e - 0{\cdot}00226\,e\,\Delta\varpi \\ -0{\cdot}00240\,\Delta e' - 0{\cdot}00236\,e'\Delta\varpi' \end{matrix}\right\} \cos 3\lambda.
\end{aligned}$$

$$\begin{aligned}
\breve{\delta}(n't+\epsilon') = &\left\{\begin{matrix} +0{\cdot}01532\,\Delta e + 0{\cdot}00326\,e\,\Delta\varpi \\ +0{\cdot}01566\,\Delta e' - 0{\cdot}01673\,e'\Delta\varpi' \end{matrix}\right\} \sin 2\lambda \\
&+ \left\{\begin{matrix} -0{\cdot}00326\,\Delta e + 0{\cdot}01532\,e\,\Delta\varpi \\ +0{\cdot}01673\,\Delta e' + 0\ 01566\,e'\Delta\varpi' \end{matrix}\right\} \cos 2\lambda \\
&+ \left\{\begin{matrix} +0{\cdot}00165\,\Delta e + 0{\cdot}00039\,e\,\Delta\varpi \\ +0{\cdot}00172\,\Delta e' - 0{\cdot}00175\,e'\Delta\varpi' \end{matrix}\right\} \sin 3\lambda \\
&+ \left\{\begin{matrix} -0{\cdot}00039\,\Delta e + 0{\cdot}00165\,e\,\Delta\varpi \\ +0{\cdot}00175\,\Delta e' + 0{\cdot}00172\,e'\Delta\varpi' \end{matrix}\right\} \cos 3\lambda,
\end{aligned}$$

which are obtained by differentiating the formulæ

$$\begin{aligned}
\delta(nt+\epsilon) = &\left(\frac{3m'n^2aP_2}{2\omega^2} + \frac{m'na^2}{\omega}\,\frac{dP_2}{da}\right) \sin 2\lambda \\
&+ \left(\frac{3m'n^2aP_2'}{2\omega^2} + \frac{m'na^2}{\omega}\,\frac{dP'_2}{da}\right) \cos 2\lambda,
\end{aligned}$$

$$\breve{\delta}(nt+\epsilon) = \frac{m'n^2aP_3}{\omega^2} \sin 3\lambda + \frac{m'n^2aP_3'}{\omega^2} \cos 3\lambda,$$

and the similar expressions for $n't+\epsilon'$ with respect to e, ϖ, e', and ϖ'.

When the values of Δe, $e\Delta\varpi$, &c., given in the last Art. are substituted, we have

$$\begin{aligned}
\breve{\delta}(nt+\epsilon) = &+0''{\cdot}047 \sin 2\lambda - 13''{\cdot}312 \cos 2\lambda, \\
&-0''{\cdot}027 \sin 3\lambda - 1''{\cdot}415 \cos 3\lambda.
\end{aligned}$$

$$\begin{aligned}\delta(n't+\epsilon') = & -0''{\cdot}032 \sin 2\lambda + 9''{\cdot}586 \cos 2\lambda, \\ & + 0''{\cdot}020 \sin 3\lambda + 1''{\cdot}032 \cos 3\lambda.\end{aligned}$$

The corresponding corrections of δe, $e\delta\varpi$, &c., are

$$\begin{aligned}\delta e &= -3''{\cdot}15 \sin 2\lambda + 0''.65 \cos 2\lambda, \\ e\delta\varpi &= -1''{\cdot}00 \sin 2\lambda - 3''{\cdot}08 \cos 2\lambda, \\ \delta e' &= -2''{\cdot}33 \sin 2\lambda - 2''{\cdot}52 \cos 2\lambda, \\ e'\delta\varpi' &= +3''{\cdot}86 \sin 2\lambda - 0''{\cdot}91 \cos 2\lambda.\end{aligned}$$

The corrections for Δn, $\Delta n'$ will be

$$\begin{aligned}\delta(nt+\epsilon) &= +2''{\cdot}017 \sin 2\lambda, \\ \delta(n't+\epsilon') &= -1''{\cdot}44 \sin 2\lambda.\end{aligned}$$

The whole of the corrections of δn, $\delta n'$, for Δe, $\Delta\varpi$, $\Delta e'$, $\Delta\varpi'$, Δn, $\Delta n'$, Δa, $\Delta a'$, will be

$$\begin{aligned}\delta n = & -0''{\cdot}0005917 \sin \lambda - 0''{\cdot}0001205 \cos \lambda \\ & -0''{\cdot}0001207 \sin 2\lambda - 0''{\cdot}0000086 \cos 2\lambda \\ & -0''{\cdot}0000180 \sin 3\lambda.\end{aligned}$$

$$\begin{aligned}\delta n' = & +0''{\cdot}0004360 \sin \lambda + 0''{\cdot}0000889 \cos \lambda \\ & +0''{\cdot}0000881 \sin 2\lambda + 0''{\cdot}0000061 \cos 2\lambda \\ & +0''{\cdot}0000132 \sin 3\lambda.\end{aligned}$$

59. Instead of the values of ϵ and ϵ' given in Articles (5) and (6), we must use $\epsilon+\Delta\epsilon$ and $\epsilon'+\Delta\epsilon'$ in calculating the values of ♅ and Ψ in the formulæ of Art. (38), where

$$\begin{aligned}\Delta\epsilon = & +2783''{\cdot}82 \sin \lambda'' - 421''{\cdot}24 \cos \lambda'' \\ & + 125{\cdot}02 \sin 2\lambda'' - 3{\cdot}62 \cos 2\lambda'' \\ & + 8{\cdot}15 \sin 3\lambda'' - 0{\cdot}44 \cos 3\lambda'' + (46 \times 365 + 11)\,\Delta n \\ = & +2739''{\cdot}88 - 36''{\cdot}05 = +2703''{\cdot}83 = +45'\,3''{\cdot}83,\end{aligned}$$

$$\begin{aligned}\text{and }\ \Delta\epsilon' = & -1996''.66 \sin \lambda' + 302''{\cdot}14 \cos \lambda' \\ & - 89{\cdot}36 \sin 2\lambda' + 2{\cdot}60 \cos 2\lambda' \\ & - 5{\cdot}80 \sin 3\lambda' + 0{\cdot}31 \cos 3\lambda' \\ = & -1928''{\cdot}81 = -32'\,8''{\cdot}81.\end{aligned}$$

And instead of nt, $n't$, the corrected values $(n+\Delta n)\,t$, $(n'+\Delta n')\,t$ must be used.

60. It remains now to correct the coefficients of the principal part of the long inequality for the secular variations of the eccentricities and longitudes of the perihelia.

Let E and H be the secular variations of e and ϖ in one day, resulting from the action of Jupiter, Saturn, and Neptune, and let E', H' be those of e' and ϖ' due to the action of Jupiter, Saturn, and Uranus. Then, substituting $e+Et$, $\varpi+Ht$, $e'+E't$, $\varpi'+H't$ for e, ϖ, e', ϖ', in terms of the first order in eccentricities, and integrating, neglecting squares and products of EH, $E'H'$, we have,

$$\begin{aligned}\delta(nt+\epsilon) = {} & \frac{3m'n^2aM_1e}{(\omega+H)^2\sin 1''}\left(1+\frac{E}{e}t\right)\sin(\lambda+\varpi)\\ & + \frac{6m'n^2aM_1E}{(\omega+H)^3\sin 1''}\cos(\lambda+\varpi)\\ & + \frac{3m'n^2aM_1eHt}{(\omega+H)^2}\cos(\lambda+\varpi)\\ & + \text{similar terms with } M_2e'\varpi'E'H' \text{ for } M_1e\varpi EH.\end{aligned}$$

Taking the values of E and H depending upon the action of Jupiter, Saturn, and Neptune, respectively,

$$\begin{aligned}365\cdot 25E &= -0''\cdot 006082 - 0''\cdot 046049 - 0''\cdot 002431,\\ &= -0''\cdot 054562,\\ 365\cdot 25H &= +1''\cdot 234 + 1''\cdot 188 + 0''\cdot 464 = +2''\cdot 886,\end{aligned}$$

and the values of E', H' depending on the action of Jupiter, Saturn, and Uranus,

$$\begin{aligned}365\cdot 25E' &= -0''\cdot 0011 + 0''\cdot 0031 + 0''\cdot 0093,\\ &= +0''\cdot 0113,\\ 365\cdot 25H' &= +0''\cdot 0050 - 0''\cdot 1766 + 0''\cdot 9504 = +0''\cdot 7788,\end{aligned}$$

then the corrections of the principal terms for the secular variations will be

$$\begin{aligned}\delta(nt+\epsilon) &= (-55''\cdot 58 + 0''\cdot 010\,t)\sin\lambda,\\ &\quad + (-12''\cdot 32 - 0''\cdot 043\,t)\cos\lambda,\\ \delta(n't+\epsilon') &= (+40''\cdot 70 - 0''\cdot 007\,t)\sin\lambda,\\ &\quad + (+\ 8''\cdot 96 + 0''\cdot 032\,t)\cos\lambda,\end{aligned}$$

t being here the number of Julian years from Jan. 1, 1847.

The corrections of δe and $e\delta\varpi$ will be, neglecting the terms involving t,

$$\begin{aligned}\delta e &= +0''\cdot 62\sin\lambda + 3''\cdot 75\cos\lambda,\\ e\delta\varpi &= -3''\cdot 53\sin\lambda + 2''\cdot 19\cos\lambda,\end{aligned}$$

and those of δn, $\delta n'$, will be

$$\begin{aligned}\delta n &= -0''\cdot 0000262\sin\lambda + 0''\cdot 0001181\cos\lambda,\\ \delta n' &= +0''\cdot 0000190\sin\lambda - 0''\cdot 0000865\cos\lambda.\end{aligned}$$

61. If we neglect all those terms in the preceding Articles, the coefficients of which are less than 1″, we shall have for the inequalities due to the square of the disturbing force, including the terms depending on the constant corrections of e, ϖ, e', ϖ', nn', aa', as well as the correction for the secular variations of e, ϖ, e', ϖ' *,

$$\begin{aligned}\delta(nt+\epsilon) = {} & (+\ 70''\cdot 33 - 0''\cdot 010\,t)\sin(2\Psi - \text{♅}) - (149''\cdot 25 + 0''\cdot 043\,t)\cos(2\Psi - \text{♅})\\ & - 10''\cdot 51\sin 2(2\Psi - \text{♅}) - 17''\cdot 06\cos 2(2\Psi - \text{♅})\\ & - \ 2''\cdot 28\sin 3(2\Psi - \text{♅}) - \ 1''\cdot 41\cos 3(3\Psi - \text{♅}).\end{aligned}$$

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* The term depending on the cube of the disturbing force in the note, p. 38, is also included.

$$\begin{aligned}\delta(n't+\epsilon') = {} & (-\,49''\cdot51 + 0''\cdot007\,t)\sin(2\text{♆}-\text{♅}) + (107''\cdot87 + 0''\cdot032\,t)\cos(2\text{♆}-\text{♅}) \\ & + 8''\cdot37\sin 2(2\text{♆}-\text{♅}) + 12''\cdot33\cos 2(2\text{♆}-\text{♅}) \\ & + 1''\cdot66\sin 3(2\text{♆}-\text{♅}) + 1''\cdot03\cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta e = {} & +\,17''\cdot45\sin(2\text{♆}-\text{♅}) + 8''\cdot08\cos(2\text{♆}-\text{♅}) \\ & + 6''\cdot10\sin 2(2\text{♆}-\text{♅}) - 7''\cdot57\cos 2(2\text{♆}-\text{♅})\end{aligned}$$

$$\begin{aligned}e\delta\varpi = {} & +\,11''\cdot62\sin(2\text{♆}-\text{♅}) - 14''\cdot52\cos(2\text{♆}-\text{♅}) \\ & - 10''\cdot72\sin 2(2\text{♆}-\text{♅}) - 7''\cdot71\cos 2(2\text{♆}-\text{♅}) \\ & - 1''\cdot32\sin 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta e' = {} & +\,6''\cdot37\sin(2\text{♆}-\text{♅}) \\ & \qquad\qquad - 2''\cdot52\cos 2(2\text{♆}-\text{♅}) \\ & - 1''\cdot16\sin 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}e'\delta\varpi' = {} & -\,5''\cdot94\sin(2\text{♆}-\text{♅}) - 1''\cdot36\cos(2\text{♆}-\text{♅}) \\ & - 3''\cdot86\sin 2(2\text{♆}-\text{♅}) + 4''\cdot12\cos 2(2\text{♆}-\text{♅}) \\ & \qquad\qquad + 2''\cdot32\cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta n = {} & +\,0''\cdot0006259\sin(2\text{♆}-\text{♅}) + 0''\cdot0002319\cos(2\text{♆}-\text{♅}) \\ & + 0''\cdot0001526\sin 2(2\text{♆}-\text{♅}) - 0''\cdot0000808\cos 2(2\text{♆}-\text{♅}) \\ & + 0''\cdot0000180\sin 3(2\text{♆}-\text{♅}) - 0''\cdot0000291\cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

$$\begin{aligned}\delta n' = {} & -\,0''\cdot0004608\sin(2\text{♆}-\text{♅}) - 0''\cdot0001685\cos(2\text{♆}-\text{♅}) \\ & - 0''\cdot0001113\sin 2(2\text{♆}-\text{♅}) + 0''\cdot0000587\cos 2(2\text{♆}-\text{♅}) \\ & - 0''\cdot0000132\sin 3(2\text{♆}-\text{♅}) + 0''\cdot0000212\cos 3(2\text{♆}-\text{♅}).\end{aligned}$$

The values of e, e' multiplying $\delta\varpi$, $\delta\varpi'$ in the above expressions, are the uncorrected values given in Arts. (5) and (6).

SECTION III.

GENERAL EXPLANATION OF THE MODE IN WHICH THE DISTURBING FORCES PRODUCE THE LONG INEQUALITY.

62. In attempting to explain the mode in which the long inequality is produced, we shall first briefly describe the nature of the disturbing forces when both orbits are supposed to be circular and the periodic time of the exterior planet double, or nearly double, of that of the interior. Then supposing each orbit in turn to become elliptical while the other remains circular, we shall shew how the alteration in the planet's place, and in the direction of its path due to the elliptic motion, gives rise to small additional disturbing forces which do not go

through all their changes in one synodic revolution, and the character and magnitude of which depend upon the position of the line of conjunctions relatively to the major axis. The tangential part of these additional forces produces the long inequality in the mean motions; but the eccentricities and perihelia are affected in a different manner; their long inequalities will be seen to depend upon the circular part of the disturbing forces (i. e. that part which is the same as if the orbits were circular).

Let then P and P' be the planets supposed to revolve in circular orbits about S (fig. 2); and let the periodic time of P' be twice that of P; A the point of conjunction, and BB' perpendicular to AA'. The disturbing forces acting on P are the attraction of P' on P in the direction PP', and the attraction of P' on S acting in a direction parallel to $P'S$. When both these forces are resolved along the tangent and normal, the tangential force vanishes when the angular separation of the planets $PSP' = 0^0$ or 180^0, and also when S and P are equally distant from P', i. e. when $PSP' = \cos^{-1}\frac{1}{2}a = 71^0\frac{1}{2}$ about; and is alternately retardative and accelerative. Now this force may be conceived to be made up of several forces, one depending upon the $\sin PSP'$, another upon the $\sin 2PSP'$, &c. A force depending on $\sin 2PSP'$ vanishes when $PSP' = 0^0$, 90^0, 180^0, 270^0, 360^0, and is at a maximum at points half way between these, and is alternately retardative and accelerative. While therefore P moves from A through B to A', it is retarded by this part of the tangential force (for the angle PSP' is always half of the angle described by P from conjunction); and while it moves through $A'B'A$, it is accelerated to the same amount. The tangential force which depends upon $\sin PSP'$ is retardative during one revolution of P, and accelerative in the next. The force depending on $2PSP'$ is the more important of the two, since its law of variation more nearly resembles that of the whole tangential force. In the same way the normal force may be considered to be made up of a constant part, and of a part depending upon $\cos PSP'$, another depending upon $\cos 2PSP'$, &c. The part depending upon $\cos 2PSP'$ acts outwards, while P moves from B' through A to B, and inwards during the other half of a revolution, and is at a maximum at A and A'. The force depending upon $\cos PSP'$ acts outwards during half a revolution of P before conjunction, and half a revolution after, arriving at a maximum at conjunction; and its direction is reversed in the next revolution of P.

63. Confining our attention at first to the forces depending upon the argument $2PSP'$ let us examine what alteration will be made in them when we suppose the orbit of P to become elliptical, while that of P' remains circular. They will be altered,

1°. By reason of P being sometimes before, sometimes behind its mean place.

2°. By the fluctuation of distance from S.

3°. The tangent to the elliptic orbit is not (generally) in the same direction as the tangent to the circular orbit; and, although this does not alter the value of the whole disturbing force, it alters the values of the tangential and normal components.

Since the momentary change of major axis is proportional to the tangential force multiplied by the linear velocity of the disturbed planet, a fluctuation in that velocity will produce changes in the major axis other than those which exist in a circular orbit, and may therefore give rise to a portion of the long inequality. We shall now examine separately the effects of these four causes of disturbance; for since the circular disturbing forces can only produce rapidly periodical variations of the major axis, it is in those circumstances, in which elliptic motion differs from circular, that the origin of the long inequality of the mean motions must be sought.

64. Let AA' (fig. 3) be the line of conjunctions, the planets coming to conjunction at A once in every revolution of P'; aa' the major axis of P's orbit, a the perihelion; and let the planets start from conjunction at A.

From A through B to A' the circular tangential force depending on $2PSP'$ is retardative, and at a maximum when P is at B. P moving from perihelion to aphelion is in advance of its mean place; therefore, when between A and B, it is nearer to that position, in which the tangential force is at a maximum, than it would have been if its motion had been uniform; and it is therefore acted upon by a greater retarding force. This is equivalent to being acted upon both by the circular tangential force, and an additional retarding force (which is indicated in the figure by a negative sign). From B to a' P is still in advance of its mean place; but the circular tangential force is now decreasing, and P is further from the position in which that force is at a maximum; therefore it is acted on by a smaller retarding force than it would have been if its motion had been uniform: this is equivalent to being acted on by an additional accelerating force (indicated by a + sign). Between a' and A' the circular tangential force is still retarding, but P is behind its mean place, and therefore the additional force is now retarding. Of course there is no additional force arising from this cause at a', since there the mean and true places of P coincide.

From A' through B' to A the circular tangential force is accelerating and at a maximum at B'. Therefore while P moves from A' to B', being behind its mean place, it is further from the position in which the tangential force is at a maximum, and therefore it is acted upon by an additional retarding force. From B' to a the true place of P is nearer to the position of maximum force, and therefore the additional force is accelerating: while from a to A the true place is before the mean, and the additional force is retarding, and vanishes at a. The additional force also vanishes when B or B' is half way between the mean and true places of P.

From this cause, therefore, in this position of the line of conjunction we have a preponderance of retarding force; and the amount of this excess depends upon the angle aSA, and is zero when AA' coincides with aa', and at a maximum when AA' coincides with bb', in which position the whole of the additional force is negative. Now since the periodic time of P' is somewhat less than twice that of P, each conjunction takes place at a point a little in advance of the preceding, so that the point A performs a complete revolution in about 4047 years. While therefore A revolves from a through b to a' the additional force is retarding; and it is accelerating throughout the second half of the revolution of A. Consequently during the former half of a complete revolution of the point of conjunction the major axis is continually decreasing and the mean angular velocity increasing, and *vice versâ* during the latter half.

Suppose now a conjunction to take place when P is at perihelion. Since A has just completed a revolution the mean angular velocity is reduced to its minimum value; therefore P will be continually in every revolution dropping behind its undisturbed elliptic place. But, as A is also very slowly revolving from a to b, the effect of disturbing force is to restore the mean angular velocity to its undisturbed value; and this value is attained when A has arrived at b. As A passes from b to a' the mean angular velocity is increasing beyond its undisturbed value, and therefore the planet is regaining the longitude it had lost: and when A arrives at a' the disturbance in longitude is reduced to zero. Therefore throughout the former half of the revolution of A from the perihelion the planet P is behind its undisturbed place, and similarly it is before that place throughout the latter half.

65. The fluctuation of distance of P from S produces an analogous alteration in the tangential force. That part of the force which depends upon the argument $2PSP'$ is smaller when SP is smaller, and *vice versâ*. While P is moving from b' to A (fig. 4) the circular tangential force is accelerating, and SP is less than its mean value. P is therefore acted on by a less accelerating force than it would be if its orbit were circular (radius = mean distance): therefore the additional tangential force is retarding. While P moves from A to b the circular tangential force is retarding, and SP less than its mean value: therefore the additional force is accelerating. Similarly it is retarding from b to A' and accelerating from A' to b'. Therefore there is from this cause also a preponderating force, which is retarding while A revolves through aba' and accelerating while it revolves through $a'b'a$, coming to a maximum when A is at b and b' and vanishing when it is at a and a'. It is therefore always of the same sign as the force investigated in the last article, and concurs with it in producing the same perturbation.

66. We have hitherto considered only that alteration of the tangential force which is due to the deviation in the position of P. But the tangential force in the ellipse will also differ from that in the circle on account of the difference of the direction in which the disturbing forces must be resolved. The forces which are normal and tangential in the circle (and which we have hitherto treated as if they were normal and tangential in the elliptic orbit too) are radial and transversal in the ellipse. These forces must therefore be both resolved along the tangent in order to obtain the correct value of the tangential force. We might have supposed the disturbing forces to be always resolved along the normal and tangent, but in that case we could not have determined the effect due to the deviation of P from its mean place separately from the effect produced by the change in the direction of the tangent.

First, the radial force will contribute a portion of the tangential force. That part of it which depends upon the argument $2PSP'$ acts outwards while P moves through $B'AB$ and inwards during the other half of the revolution. When P is anywhere in the arc aAB (fig. 5) the radius vector makes an acute angle with the tangent which is turned towards the perihelion a. The radial force, therefore, when resolved along the tangent produces an accelerating tangential force which does not exist in the circular orbit. When

P is between *B* and *a′* the radial force acts inwards and the angle between the radius vector and the tangent is still acute, therefore the additional force is retarding. Similarly through the arc *a′A′B′* the angle between the radius vector and the tangent being always acute towards perihelion, and the radial force acting inwards, the additional force is accelerating; while through *B′A*, the radial force acts outwards and therefore the additional force is retarding. Here, then, we have an excess of positive force which tends to produce a disturbance of a directly opposite character to that previously examined and which goes through its changes in the same time.

67. Again, the circular tangential force (or transversal force) must be resolved along the tangent to the elliptic orbit, and the difference which this resolution produces may, as before, be represented by supposing a small force to act in addition to the circular tangential force. Let ψ equal the angle between the normal and the radius vector, and let T equal the circular tangential force depending upon $2PSP'$. Then the additional force is $(\cos\psi - 1)T$. Now T is negative through the arc *Aa′A′*, therefore the additional force is positive; and T is positive through the arc *A′aA*, therefore the additional force is then negative: and it vanishes at *aAa′A′*. The additional force in the arc *aA* is equal and opposite to that in the arc *A′a′*, and the force in the arc *ABa′* is equal and opposite to that in the arc *A′B′a*; this force therefore completely compensates itself in one revolution of *P*, and therefore produces no part of the long inequality.

68. The momentary change of major axis is greater, *cæteris paribus*, when the linear velocity of the disturbed planet is greater. Between *A* and *b* (fig. 6) the velocity is above the mean value, and therefore the effect of the retarding force is greater than it would have been if the velocity had been uniform. Through the arc *ba′A′* the velocity is below the mean, and the retarding force produces a less effect. Through *A′b′* the velocity is still below the mean value; therefore the accelerating force produces a less effect: while through *b′aA* it produces a greater effect because the velocity is above the mean value. Upon the whole, therefore, the retarding force produces a less effect and the accelerating force produces a greater effect than in the circular orbit. There is therefore from this cause a long inequality of opposite character to that investigated in articles (64) and (65).*

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* The terms explained in Arts. 64—68, may be traced in the following manner. Let ϕ, ϕ' be the longitudes of PP' in their circular orbits, a the radius of P's orbit, and V its velocity $= an$. Then,

$$\frac{d(R)}{dt} = \frac{dR}{d\phi}\,\frac{d\phi}{dt} = \frac{dR}{ad\phi}\,V = -\text{tan force} \times \text{velocity}.$$

Now let the orbit of P be supposed to become elliptical, while that of P' remains circular, and let θr be the long. and rad. vector of P. Then,

$$\frac{d(R)}{dt} = \frac{dR}{d\theta}\,\frac{d\theta}{dt} + \frac{dR}{dr}\,\frac{dr}{dt} = \left(\frac{dR}{rd\theta}\,\frac{rd\theta}{ds} + \frac{dR}{dr}\,\frac{dr}{ds}\right)\frac{ds}{dt}.$$

But $\theta = \phi + \delta\phi$, and $r = a + \delta a$, and $\frac{ds}{dt} = V + \delta V$, where $\delta\phi = 2e\sin\beta$, $\delta a = -ae\cos\beta$, $\delta V = nae\cos\beta$, omitting terms above the first order;

$$\therefore \frac{dR}{rd\theta} = \frac{dR}{ad\phi} + \frac{d^2R}{ad\phi^2}\,\delta\phi + \frac{d}{da}\left(\frac{dR}{ad\phi}\right)\delta a,$$

$$r\frac{d\theta}{ds} = \cos\angle \text{ between normal and rad. vector} = 1 - \tfrac{1}{2}e^2\sin^2\beta,$$

$$\frac{dr}{ds} = \sin \text{ of same } \angle = e\sin\beta;$$

$$\therefore \frac{dR}{dr}\,\frac{dr}{ds} = \frac{dR}{da}\,e\sin\beta.$$

69. The perturbations of eccentricity and perihelion produced by these small additional tangential forces are very nearly compensated in each revolution of *P*. For these forces are in all cases symmetrically situated in the opposite halves of *P*'s orbit so that the effect on the eccentricity about perihelion is destroyed by the equal and opposite effect produced by a force of the *same* sign about aphelion; and the effect on the longitude of perihelion while *P* is approaching *S* is destroyed by the effect while *P* is receding from *S*. And similarly the effects of the corresponding normal forces may be shewn to destroy each other. But it is otherwise with the circular parts of the disturbing forces. These forces are of opposite signs in the opposite halves of the orbit, and therefore their effects are added together, instead of destroying each other.

The tangential force depending on $2PSP'$ is negative through the arc AbA'. From a point near b (since e is small) to A' this force is acting about aphelion and therefore increases the eccentricity, but in the arc Ab diminishes it. Again, the tangential force is positive through $A'b'A$, and therefore in the arc $A'b'$ diminishes the eccentricity, and in the arc $b'A$ increases it. Also the points of maximum force, and the points at which it acts most advantageously, all lie in the arcs bA', $b'A$, in which the eccentricity is increased. Hence in this position of the point of conjunctions, the eccentricity is on the average increased in every revolution of P; and it is evident that it continues to increase while A revolves through aba', and continually decreases during the next half of the revolution of A. When therefore A is at a, the eccentricity is at its minimum value; it is restored to its mean value when A

Hence in the elliptic orbit,

$$\frac{d(R)}{dt}=\frac{dR}{ad\phi}V+\frac{d^2R}{ad\phi^2}V\delta\phi+\frac{d}{da}\left(\frac{dR}{ad\phi}\right)V\delta a+\frac{dR}{da}Ve\sin\beta$$
$$+\frac{dR}{ad\phi}\delta V,$$

of which the first term is the same as the expression in the circular orbit, and the remaining four are the terms from which the long inequality of the mean motion arises, so far as it depends on the ellipticity of *P*'s orbit alone.

Now let

$$R=-m'A_2\cos 2PSP'=-m'A_2\cos(2\phi-2\phi');$$
$$\therefore \frac{d^2R}{ad\phi^2}V\delta\phi=+8m'nA_2\cos(2\phi-2\phi')e\sin\beta,$$
$$=-4m'nA_2e\sin(2\phi-2\phi'-\beta),\ \&c.,$$
$$=-4m'nA_2e\sin(\lambda+\varpi),\ \&c.;$$

this is the term explained in Art. (64.)

$$\frac{d}{da}\left(\frac{dR}{ad\phi}\right)V\delta a=\left(-\frac{dR}{a^2d\phi}+\frac{1}{a}\frac{d^2R}{dad\phi}\right)V\delta a,$$
$$=\left(2m'\frac{A_2}{a^2}-2m'\frac{dA_2}{ada}\right)\sin(2\phi-2\phi')na.ae\cos\beta,$$
$$=\left(m'nA_2-m'na\frac{dA_2}{da}\right)\sin(\lambda+\varpi),\ \&c.,$$

which is the term explained in Art. (65).

$$\frac{dR}{da}Ve\sin\beta=-m'na\frac{dA_2}{da}\cos(2\phi-2\phi')e\sin\beta,$$
$$=+\tfrac{1}{2}m'na\frac{dA_2}{da}e\sin(\lambda+\varpi),\ \&c.$$

this is the term due to the portion of the radial force resolved along the tangent, explained in Art. (66).

$$\frac{dR}{ad\phi}\delta V=+2m'A_2\sin(2\phi-2\phi')ne\cos\beta,$$
$$=+m'nA_2e\sin(\lambda+\varpi),\ \&c.$$

the term due to the fluctuation of velocity, explained in Art. (68). Since $\frac{rd\theta}{ds}=1$ nearly the alteration of the circular tan. force due to the alteration in the direction of the tangent does not give rise to any term of the long inequality of the first order, as explained in Art. (67).

Collecting the terms together, we have,

$$\frac{d(R)}{dt}=-\left(2m'nA_2+\tfrac{1}{2}m'na\frac{dA_2}{da}\right)e\sin(\lambda+\varpi),$$
$$=-m'nM_1e\sin(\lambda+\varpi),\quad \text{by Art. (8)};$$
$$\therefore \frac{da}{dt}=-2a^2\frac{d(R)}{dt}=+2m'na^2M_1e\sin(\lambda+\varpi),$$

which agrees with the formula of Art. (16) when regard is had only to the ellipticity of *P*'s orbit.

It appears from the above, that the effects of the causes 1° and 2° (Art. 63) preponderate over those of the other two.

Suppose (as in Art. 64) that at the time $t=0$, the planets are in conjunction at a, then $\epsilon=\epsilon'=\varpi$;

$$\therefore \lambda+\varpi=-(2n't-nt)=-aSA$$

at the time of any conjunction; and

$$\frac{da}{dt}=-2m'na^2M_1e\sin(2n't-nt),$$

Now while aSA varies discontinuously from 0° to 180°, $2n't-nt$ varies continuously between the same limits; therefore $\frac{da}{dt}$ is negative, as it ought to be, according to Art. (64): so that the explanation agrees with the analysis in the sign of the inequality.

is at *b*, reaches its maximum when *A* is at *a′*, and then decreases, reaching its mean value when *A* is at *b′*, and its minimum when *A* is at *a*.

The same force causes the perihelion to recede in the arcs *Aba′*, *A′b′a*, and to advance in the arcs *aA*, *a′A′*. The perihelion therefore is in its mean place when *A* coincides with it; has accomplished its greatest recess when *A* has arrived at *b*; has returned to its mean place when *A* is at *a′*; attains its greatest advance when *A* is at *b′*; and again returns to its mean place at the end of the revolution of *A*.

The normal force produces perturbations of the eccentricity and perihelion of the same sign as those produced by the tangential force, and which respectively reach their maxima and minima simultaneously with them[*].

70. Changes in the eccentricity and longitude of perihelion necessitate a change in the mean longitude, which the planet is supposed to have at the epoch; but this element is not affected in any other way. The true longitude of the planet at any time *t* depends upon the mean longitude at epoch, the mean motion during the interval from the epoch up to the time *t*, and the eccentricity and longitude of perihelion of the instantaneous ellipse at that time. Suppose the interval between the epoch and the time *t* to be divided into a great number of very small portions, and suppose the planet to move during each of these portions of time in the instantaneous ellipse which belongs to the beginning of that portion. By indefinitely diminishing the length of these divisions of time, and increasing their number, we may make this hypothetical motion of the planet approximate as nearly as we please to the real motion. Now the mean motion during the time *t* which has elapsed since the epoch, may be estimated in either of two ways. We may either put it equal to the mean angular velocity in the instantaneous ellipse at the time *t*, multiplied by that time (nt): or we may multiply the mean angular velocity of the instantaneous ellipse at the beginning of each small portion of time, by the length of that portion, and take the sum of all these products from the epoch up to the instant in question ($\int ndt$). We shall suppose it to be estimated in the latter way. This sum, then, at the time *t*, depends upon the values of the mean angular velocity during all the interval from the epoch *up to* the time *t*, but not upon its value *at* that instant. Consider now the subdivision of time immediately preceding the completion of the time *t*. The mean longitude at epoch is so adjusted, that, with the eccentricity and perihelion belonging to the beginning of this portion of time, and with the mean motion calculated for each instant in the above manner, the planet may have its true longitude at every instant, during this small interval of time, and therefore both at the beginning and end of it. At the time *t*, the planet proceeds to move in a different ellipse;

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[*] $\frac{dR}{d\varpi} = \frac{dR}{d\theta}\frac{d\theta}{d\varpi} + \frac{dR}{dr}\frac{dr}{d\varpi}$: $\frac{d\theta}{d\varpi} = -2e\cos\beta$, $\frac{dr}{d\varpi} = -ae\sin\beta$,

therefore taking the circular part of the disturbing forces only,

$$\frac{dR}{d\varpi} = -\frac{dR}{d\phi}2e\cos\beta - \frac{dR}{da}ae\sin\beta,$$

$$= -2m'A_2\sin(2\phi - 2\phi').2e\cos\beta$$

$$+ m'\frac{dA_2}{da}\cos(2\phi - 2\phi').ae\sin\beta,$$

$$= -\left(2m'A_2 + \tfrac{1}{2}m'a\frac{dA_2}{da}\right)e\sin(\lambda + \varpi), \text{ \&c.},$$

$$= -m'M_1e\sin(\lambda + \varpi), \text{ \&c.};$$

$$\therefore \frac{de}{dt} = \frac{na}{e}\frac{dR}{d\varpi} = -m'naM_1\sin(\lambda + \varpi),$$

which agrees with the formula of Art. (16). The principal part of the perturbation of the eccentricity therefore depends upon the circular disturbing forces; and the same is true of the longitude of perihelion.

and the mean longitude at epoch must be again adjusted, so that with the new eccentricity and perihelion the planet may still have its true longitude at the same instant of time t. But the mean motion calculated for that instant is the same as it was for the same instant in the previous ellipse. Therefore, the variation of the epoch depends upon the alteration of the eccentricity and perihelion, but not upon that of the mean angular velocity. If, however, the mean motion were estimated in the former of the two ways described above, the mean longitude at epoch would be different, and its variation would depend upon that of the mean angular velocity, as well as upon that of the eccentricity and perihelion.

71. Suppose a normal force to act outwards on P during a whole revolution. While P moves from perihelion to aphelion the eccentricity is increased, and is diminished in the other half of the orbit. While P is about perihelion the apsides recede, and while about aphelion they advance. Therefore while P is moving from perihelion to mean distance the equation of the centre corresponding to a given mean longitude is increased, both by the increase of the eccentricity and because the point at which the equation is at a maximum is brought nearer to the planet by the recess of perihelion. If therefore the mean longitude were unaltered the planet would be in advance of its true place. Therefore the mean longitude at epoch must be diminished in order that the planet may come to its true place at the right time. And similarly the equation of the centre is increased, and therefore the epoch must be diminished, while P moves from the extremity l of the latus rectum through the upper focus H (fig. 7) to aphelion. While P moves from aphelion to the other extremity l' of the latus rectum, the equation of the centre is diminished by the diminution of the eccentricity, and because the point at which it is at a maximum is kept further from the planet by the advance of perihelion: and since the equation is now negative the planet would be in advance of its true place if the epoch were not diminished. Similarly, the epoch must be diminished while P moves from mean distance to perihelion. Between points on either side at which the equation of the centre is at a maximum, and l or l' the diminution of epoch due to the alteration of eccentricity is partially neutralized by the effect due to alteration of perihelion. A normal force acting outwards therefore in any part of the orbit requires the epoch to be diminished on account of its action on the eccentricity and perihelion; and similarly a normal force acting inwards requires it to be increased.

72. Now by the same reasoning as in (Art. 64), it may be shewn that there is a small additional normal force acting inwards in the arcs ABa', $A'B'a$, and outwards in the arcs aA, $a'A'$. And by the same reasoning as in (Art. 65), it follows that there is a normal force acting inwards in the arcs $B'Ab$, $BA'b'$, and outwards in the arcs bB, $b'B'$. In both cases therefore the inward force exceeds the outward force by a quantity depending on the angle aSA. In consequence of the action of this force the epoch will be at a maximum when A is at b, and at a minimum at b', and is restored to its mean value, when A is at a or a'.

Again, the circular tangential force must be resolved along the normal to the ellipse, and will contribute a small normal force, which does not exist in the circular orbit. This force acts outwards through the arcs ABa' and $A'B'a$, and inwards in the arcs Aa, $A'a'$, and therefore produces a long inequality of epoch of an opposite sign to the former.

73. The tangential forces produce but little effect on the epoch. Suppose an accelerating tangential force to act during the whole of a revolution. Then the eccentricity is increased throughout the arc $l'al$ (fig. 7), and diminished in $la'l'$. The perihelion advances in the arc ala', and recedes in the arc $a'l'a$. Therefore in the arc ab the equation of the centre in increased by increase of eccentricity, but diminished by advance of perihelion, so that the effects of these changes on the epoch tend to neutralize each other. And the same tendency towards compensation may be shewn to exist in the other parts of the orbit. The analytical expression however shews that the compensation is not exact on account of the orbit having an eccentricity independent of perturbation; and that the circular tangential force produces a part of the long inequality of epoch. The circular normal force also produces a portion of the long inequality for the same reason*.

74. If we neglect powers of the eccentricity above the first, the disturbing forces depending on the arguments PSP', $3PSP'$, &c. cannot give rise to any portion of the long inequality when the orbit of P alone is considered to be elliptical. For the forces which produce the long inequality of the mean motion investigated in Arts. 64—68, arise from the fact that a portion of the circular disturbing forces goes through its changes in nearly the same time as the fluctuations in the motion of the planet due to the ellipticity of its orbit; that is, nearly in the periodic time of the planet. And it is because the period of the force is not exactly equal to the periodic time of the planet, that the inequality is at length compensated: for by this means the force and the irregularity in the motion of the planet are in the lapse of time presented to each other in opposite phases, so that the additional force becomes of an opposite character. And this compensation will be effected more rapidly for those forces which depend upon the angles PSP', $3PSP'$, &c., the periods of which differ very greatly from that of P. Thus, the period of the force depending on PSP' is nearly equal to twice that of P; so that this force is of opposite signs in successive revolutions of P; and therefore the additional force in one revolution is of opposite

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* Let H, H' (fig. 7), be two consecutive positions of the upper focus under the action of a normal force directed outwards; then

$$2ade = HH' \sin PHS = HH' \sin\beta, \text{ neglecting } e^2, \&\text{c.}$$

$$2aed\varpi = -HH' \cos PHS = -HH' \cos\beta,$$

$$\theta = nt + \epsilon + 2e\sin\beta + \frac{5}{4}e^2 \sin 2\beta\,;$$

$$\therefore\ 0 = d\epsilon\,(1 + 2e\cos\beta) + 2de\sin\beta - 2ed\varpi\cos\beta + \frac{5}{2}ede\sin 2\beta - \frac{5}{2}e^2 d\varpi\cos 2\beta\,;$$

$$\therefore\ d\epsilon = \frac{2ed\varpi\,(\cos\beta + \frac{5}{4}e\cos 2\beta) - 2de\,(\sin\beta + \frac{5}{4}e\sin 2\beta)}{1 + 2e\cos\beta},$$

$$= -\frac{HH'}{a}\,\frac{1 + \frac{5}{4}e\cos\beta}{1 + 2e\cos\beta} = -\frac{HH'}{a}\left(1 - \frac{3}{4}e\cos\beta\right),$$

HH' is proportional to the disturbing force; hence both the elliptical and circular forces produce long inequalities of epoch.

Let HH'' (fig. 7), be consecutive positions of the upper focus under the action of a retarding tangential force; then

$$2eda + 2ade = -HH''\cos\beta\,;$$

$$\therefore\ 2ade = -HH''(\cos\beta - 2e),$$

$$2aed\varpi = -HH''\sin\beta\,;$$

$$\therefore\ d\epsilon = -\frac{3}{4}\,\frac{HH''e\sin\beta}{a(1 + 2e\cos\beta)} = -\frac{3}{4}\,\frac{HH''}{a}\,e\sin\beta,$$

HH'' is proportional to force × velocity; hence the circular tan. force produces a part of the long inequality.

sign to that in the next. The force depending on *3PSP'* goes through its changes three times in two revolutions of *P*, or once and a half in one revolution; therefore in two successive revolutions, the planet is, at every point of its orbit, presented to equal and opposite values of this force; so that the additional forces are equal and opposite. Similar reasoning will apply to the forces depending on all higher multiples of *PSP'*, and will shew that all disturbances must be compensated in one synodical revolution of *P* and *P'* at the most; though it is not to be inferred that they require so much as one synodical revolution.

75. Next, let us suppose the orbit of *P'* to be elliptical, while that of *P* remains circular; and consider the alteration which this will make in those circular forces the period of which is nearly equal to the period of *P'*; that is, the forces depending on the argument *PSP'*. This part of the circular tangential force is retarding during one revolution of *P*, and accelerating in the next.

Let *aa'* (fig. 5) be the line of apses of the orbit of *P'*; *a* the perihelion; and let the planets start from conjunction at *A*.

Then the circular tangential force is retarding on *P*, while *P'* moves through *ABA'*, and accelerating while *P'* moves through *A'B'A*, and arrives at a maximum when the mean place of *P'* is at *B* and *B'*. While *P'* moves from *A* to *B* it is before its mean place; therefore *PSP'* is less than it would otherwise have been, and therefore the retarding force is less, or, in other words, there is an additional accelerating force on *P*. While *P'* describes *Ba'* it is still before its mean place, and therefore *PSP'* is less than it would have been; but the retarding force is past its maximum, so that a diminution of *PSP'* increases the force: therefore there is an additional retarding force. Through the arc *a'A'* the angle *PSP'* is increased, and therefore the retarding force diminished; or the additional force is now positive. Similarly, it is positive in the arc *A'B'*, negative in *B'a*, and positive in *aA*. Hence there is a preponderance of force (which is positive in the assumed position of *A*) depending on the position of the line of conjunction with respect to the apses of the orbit of *P'*, which produces a long inequality in the mean motion and epoch of *P*.

76. Another inequality of the same sign arises from the fluctuation in the distance of *P'* from *S*. While *P'* describes the arc *b'B'A* (fig. 6), the tangential accelerating force depending on *PSP'* is increased by the diminution of *SP'*; and in the arc *bBA'* the force, which is then retarding, is diminished by the increase of *SP'*. Therefore in each case there is an additional accelerating force. Similarly, in the arcs *Ab*, *A'b'*, there is an additional retarding force; so that there is a preponderance of force of the same sign as that investigated in the last Article*.

---

* When the orbit of *P'* is alone supposed to be elliptical

$$\frac{d(R)}{dt} = \frac{dR}{ad\phi} V + \frac{d^2R}{ad\phi d\phi'} V\delta\phi' + \frac{d}{da'}\left(\frac{dR}{ad\phi}\right) V\delta a',$$

$$R = -m'\left(A_1 - \frac{a}{a'^2}\right)^{\cos}(\phi - \phi');$$

$$\therefore \frac{d^2R}{ad\phi d\phi'} V\delta\phi' = -m'n\left(A_1 - \frac{a}{a'^2}\right)\cos(\phi-\phi').2e'\sin\beta',$$

$$= +m'n\left(A_1 - \frac{a}{a'^2}\right)e'\sin(\lambda+\varpi'), \&c.$$

$$\frac{d}{da'}\left(\frac{dR}{ad\phi}\right) V\delta a' = -m'n\left(\frac{dA_1}{da'} + \frac{2a}{a'^3}\right)\sin(\phi-\phi').a'e'\cos\beta',$$

$$= +\frac{m'n}{2}\left(A_1 + a\frac{dA_1}{da} - \frac{2a}{a'^2}\right)e'\sin(\lambda+\varpi'), \&c.$$

For the position of the line of conjunction assumed in the figures $\lambda+\varpi'$ is negative; therefore both the above terms indicate a preponderance of accelerating force.

The forces depending on the argument PSP' cannot produce any long inequality of the eccentricity and perihelion, because the effect produced in one revolution of P is destroyed in the next. And in the same way the forces depending upon $3PSP'$, &c. cannot require more than one synodical revolution of P and P' for compensation.

77. We have now considered all the long inequalities of P depending upon the first powers of the eccentricities. Those which are due to the ellipticity of P''s orbit are of opposite sign to the principal of those depending upon the ellipticity of the orbit of P, (see Arts. 64, 65), but the latter may be expected to preponderate (as they do) both because the eccentricity of P's orbit is by far the greater of the two, and because the circular force depending on $2PSP'$ is greater than that depending on PSP', (Art. 62). We may now therefore proceed to the case of the exterior planet disturbed by the interior.

78. Since the whole of the above reasoning may be applied *mutatis mutandis* to the explanation of the perturbations of P' by P, it will not be necessary to go into the same degree of detail in considering this case. We must first describe the disturbing forces on the supposition that the orbits are circular.

Let PP' (fig. 8) be the planets, and suppose, for simplicity, that P' remains fixed while P revolves with an angular velocity equal to the difference between the angular velocities of P and P'. Then the disturbing forces acting on P' are the attraction of P in the direction $P'P$ and the attraction of P on S acting in direction parallel to PS. When these forces are resolved along the tangent, the tangential force is zero when the angular separation of the planets is 0°, and also when P' and S are equally distant from P; i. e. at an angle $\cos^{-1}\dfrac{1}{2a}$, or about $38^\circ\frac{1}{2}$ (at the points C and C_1); and again vanishes when $PSP' = 180^\circ$. It is accelerating from A to C, then retarding from C to A', accelerating from A' to C_1, and again retarding from C_1 to conjunction; and is at a maximum at points between these. The normal force acts inwards about A, and outwards about A', vanishing only at two points D, D_1, which are nearer to A' than to A.

As before, each of these forces may be replaced by an infinite number of forces depending on the arguments PSP', $2PSP'$, $3PSP'$, &c. The whole tangential and normal forces however do not resemble those parts depending only on $2PSP'$ in their laws of variation so nearly as in the former case.

79. We shall first consider the alteration produced in the tangential force depending on $2PSP'$ by the variation of P from its mean place, supposing the orbit of P' to be circular. Now this force is positive when the corresponding force acting on P is negative, and *vice versâ*. Therefore by precisely the same reasoning as in Art. 64 it follows that there is an additional positive force acting on P' while P describes the arcs aAB, $a'A'B'$, and an additional negative force while it describes the arcs $B'a$, Ba'. And exactly as in Art. 65 the fluctuation in the distance of P from S produces an additional force acting on P' which is of opposite sign to the corresponding force acting on P, and is therefore positive while P describes the arcs

$b'aA$, $ba'A'$, and negative while it describes the arcs Ab, $A'b'$. The resulting inequalities of P' are therefore of opposite sign to the corresponding inequalities of P.

Other terms are produced by the alteration of the tangential force depending on PSP' due to the deviation of P' from circular motion. Now this force is of the *same* sign as the corresponding force acting on P, and therefore the additional force arising from the variation in the longitude of P' will be of the same sign as the corresponding force acting on P, and therefore the disturbance will be of the same sign. But it is otherwise with the alteration of this force due to the fluctuation in the radius vector of P'. The tangential force depending on PSP' consists of two parts, one part arising from the attraction of P on S, and the other from the attraction of P on P'. Now it is this latter part only which is altered by the fluctuation of the radius vector of P'; and this part is accelerating while P' moves through ABA', and retarding while it moves through $A'B'A$, and is increased by a diminution of SP'. The resulting inequality is therefore of *opposite* sign to the corresponding term for P investigated in Art. 76.

Again the radial force acts inwards while P' describes $B'AB$, and outwards in the other half of the orbit. Therefore resolving this along the tangent the additional force is retarding through the arcs aAB, $a'A'B'$, and accelerating through $B'a$, Ba'. Therefore there is a preponderance of negative force diminishing the axis and increasing the mean motion.

Lastly, the velocity is above the mean in the arc $b'B'A$ where the tangential force is accelerating, and below the mean in the arc bBA' where it is retarding, therefore the effect of accelerating force preponderates increasing the major axis and diminishing the mean motion. These are all the perturbations of the mean motion of P' of the first order.

The effect of the normal force upon the epoch may be investigated as in Art. 72.

80. By the same reasoning as that in Art. 69 it may be shewn that the tangential force depending on PSP' increases the eccentricity (in the position of A assumed in the figures) and causes the perihelion to recede. And the normal force depending on PSP' produces a perturbation of these elements of an opposite character.

81. Since the disturbing forces alter the mean longitudes of the planets, the position of the line of conjunction will be different from what it would have been if there had been no disturbing force. But the disturbing forces themselves have been shewn to depend upon the position of the line of conjunction with respect to the apses. These forces therefore will be altered by their own action; and this alteration gives rise to another long inequality which we shall now examine.

Let the planets start from conjunction at the perihelion of P's orbit. Then (taking account only of the disturbance arising from the ellipticity of the orbit of P) P's angular velocity is least and P''s greatest; P immediately begins to lose in longitude and P' to gain; the progressive motion of the line of conjunctions therefore is accelerated, and the disturbing force (which is greatest when A is at b) is increased; *i. e.* there is an additional force tending to increase the mean motion of P and diminish that of P'. When A is near b this new force vanishes. When it has passed b the place of conjunction is still in advance of its undisturbed place, and therefore the force producing the long inequalities is less than it would have been; or there is an additional force tending to diminish the mean motion of P and increase that of P'. This force vanishes at a' since the mean longitudes, and therefore the point of conjunction, are restored to their undisturbed state. Consequently, the new force goes through all its changes while A performs half a revolution, and therefore in half the periodic time of the principal disturbing force.

82. It is clear that the long inequality in the longitude of the perihelion will produce a similar force (of much greater importance, because the change of place of perihelion is much greater than the change in the place of conjunction); for this perturbation of perihelion alters the relative position of the perihelion and line of conjunction in the same manner as the perturbation of mean longitudes does. But the resulting disturbance is nearly neutralized by another arising from the variation of the eccentricity. For when A is at a the eccentricity is least (Art. 69), and is less than its mean value while A revolves from a to b, at which point it is restored to its mean value. On this account therefore the disturbing force is diminished while it has been increased by the simultaneous recess of perihelion.

83. Since the perturbations of e and ϖ depend upon the position of A with respect to the perihelion, the shifting of A will produce inequalities in those elements depending on the square of the disturbing force similar to those produced in the mean motions.

A similar explanation may be given of those terms of the square of the disturbing force which depend upon the ellipticity of the orbit of P'; and the explanation of the perturbations of P by P' may be applied to those of P' by P.

Fig. 1.

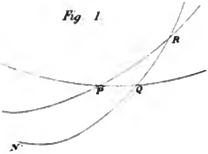


Fig. 2.

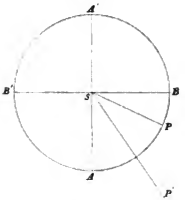


Fig. 3.

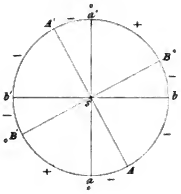


Fig. 4.

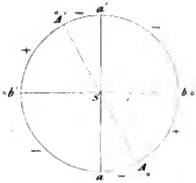


Fig. 5.

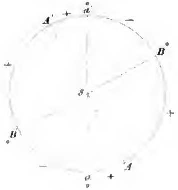


Fig. 6.

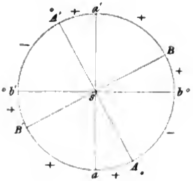


Fig. 7.

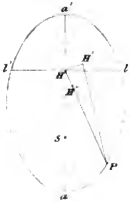


Fig. 8.

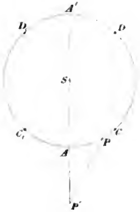


Fig. 9.

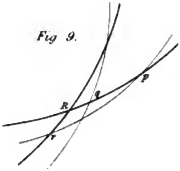


Fig 1.

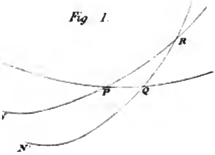


Fig 2.

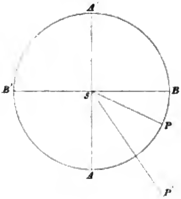


Fig. 3.

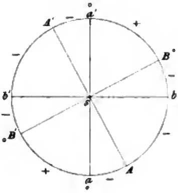


Fig 4.

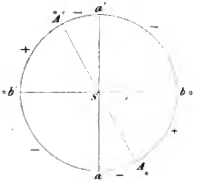


Fig 5.

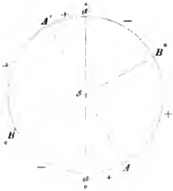


Fig 6.

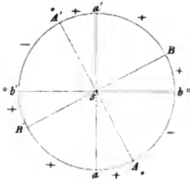


Fig 7.

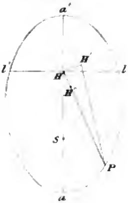


Fig 8.

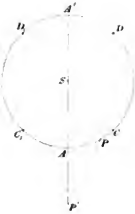
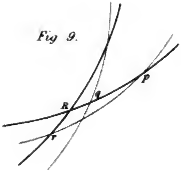


Fig 9.



XVIII. *On the Geology of some parts of Suffolk, particularly of the Valley of the Gipping.* *By* J. B. Phear, Esq. M.A. F.G.S. *Fellow and Assistant Tutor of Clare Hall, Cambridge.*

[Read *Feb.* 27, 1854.]

The general features of the county of Suffolk point it out as a district where some portion, at least, of the Drift phenomena may be advantageously studied. It is bounded on the West and North-west by chalk uplands, on the South and East by the sea, and the area thus enclosed is occupied by a mass of drift-clay which several main valleys with their ramifications cut into undulations of some depth and abruptness.

The county is singularly divided from Norfolk by a marked depression through which the Waveney flows eastwards and the Ouse westwards; the courses of all the streams, with the exception of the Lark, which joins the Ouse and finally discharges itself at Lynn, run towards the South-east coast and are approximately normal to it: of these the most considerable is the Orwell or Gipping, and it is worthy of remark that its valley is parallel to, and almost continuous with, all the important part of the valley of the Lark on the other side of the county: both seem to be perpendicular to the general strike of the chalk in this locality. The few miles of high ground which intervenes between them consists of chalk very partially covered with drift of any kind, and scarcely anywhere with any amount of drift-clay. It is impossible not to be struck, at a first glance, with these transverse sections of the chalk-range.

On pursuing the streams towards the coast, the central mass of clay is found almost entirely to cease upon reaching a line (hereafter to be more particularly defined), whose maximum distance from the sea may be roughly stated at twelve miles: London clay here comes to the surface, covered in many places (and almost universally upon nearing the coast-line) by crag and other marine and freshwater deposits, and still more superficially by a layer of unfossiliferous sand, which may probably be associated with the drift.

The name drift is generally made to include a great variety of materials, such as sand, gravel, rough clay, of every degree of coarseness, but all sharing the common characteristic of an almost entire absence of stratification or apparent law in their arrangement. The clay is very far from being homogeneous, but is generally of a colour varying from blue to grey; sometimes however it is brown, and at others the excessive prevalence in it of chalk nodules gives it a yellowish white hue: perhaps these variations of colour point to real distinctions in the conditions of deposit; and it may be mentioned as an approximate law that the deeper blues seem to be found at the greater distances from the chalk, and that the sterile yellows form some of the high grounds. It is very full of all kinds of rock, from lias to chalk inclusive, and is especially rich in specimens of septaria belonging to the Oxford and Kimmeridge clays and the

Gault: these different fragments, or boulders as it is usual to call them, have every variation in magnitude, from five or six feet in diameter downwards; they are always much worn in appearance, but can never be confounded with pebbles, as they are rarely very smooth, and invariably exhibit at least one distinct face, meeting the other surface at small though blunted angles. Belemnites, ammonites, gryphea, single pieces of saurian remains, and the more substantial fossils, occur separately in great abundance; and I have found a fragile star-shaped selenite alone in the mass of clay; but with these exceptions fossils are only to be looked for in the boulders: these however are so plentiful, that the ordinary drains in any field of the clay district, or even any small stone-heap, would go far to furnish a cabinet with specimens from the secondary formation.

The drift-gravel is coarse, very unequal in size, very heterogeneous, and generally angular; the greater portion is flint from the chalk, and the whole perhaps may be well described by saying that the first sight of it suggests the idea of its being the result of washing away the above described drift-clay from its insoluble contents. Single fossils more frequently occur in it than in the drift-clay, but are always much broken and worn. Tertiary remains are so frequently found in gravels of this character apparently *in situ* (a case which I believe never happens in the clay), as to point to a probable want of synchronism in the two deposits.

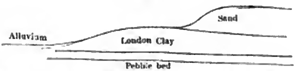
The sands are unfossiliferous.

The habit of viewing this mass as a whole has in a great degree vitiated the observations of local fossil collectors, which if made with careful attention to the relative positions of the beds of gravel, clays, &c., would have afforded invaluable help towards the attainment of an accurate knowledge of the formation; as it is, they necessarily claim attention only to result in throwing additional embarrassment upon an inquiry which is already sufficiently beset with difficulties of its own:—not the least of these is the paucity of sections to be obtained; the Eastern Union Railway when being made might have been turned to good account in this way, but no accurate notes of its cuttings have been preserved, and they are now far too weather-worn or covered with vegetation to admit of being examined. Wells are not often sunk, brick-pits are very rarely found in any but the thin diluvial clays, and the gravel, sand, and clay-lump pits are too shallow to be separately of much value; the only promising cases therefore are either those furnished by the occurrence of several of these pits scattered upon the flank of one of the larger valleys, or by the sea-coast. Accordingly the observations mentioned in the first part of this paper will be arranged in an order following the upward course of the Gipping.

The name Gipping is given to the prolongation of the Orwell estuary; it extends from Ipswich in a N.W. direction to beyond Stowmarket, a distance of at least 12 miles, and throughout its length is continually throwing out side valleys, which are again subdivided into smaller branches: its transverse section exhibits flat boggy meadows, often half a mile in width, terminated by rising ground which attains considerable height as it recedes.

At Ipswich the sides approach more nearly to each other, and at the back of the town rise some steep sand-hills, forming the grounds of Christ-Church park and Brook's Hall: Stoke hills, also sand, are opposite to these, and are traversed by a tunnel of the Eastern Union Railway.

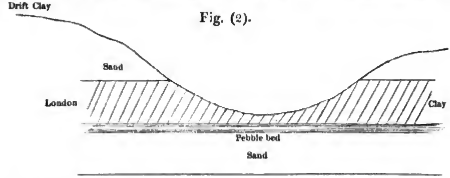
Fig. (1).



Adjoining the Norwich road on the right as it leaves Ipswich, and just at the foot of the sand-hills, is a large brick-pit: it is worked in London-clay, which is here about twenty or thirty feet in thickness, resting upon Mr Prestwich's bed of black pebbles. The partial section of the main valley is therefore that given by fig. (1).

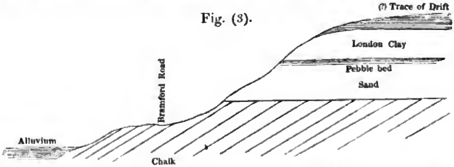
A few yards farther on, the turnpike-road passes the opening of a short hollow running up to the right, along the sides of which water oozes out very plentifully, evidently at the line of junction of the sand and London-clay; some works in progress for the purpose of collecting this water afford a good section, represented by fig. (2). The upper sand is very compact, and of a pure red colour at the top, but very white below, unfossiliferous, and containing a few angular flints. The never-failing pebble-bed is here below the London-clay, and below that again is a thick mass of sand. The surface of the high ground towards the north is drift-clay.

Fig. (2).



Ascending the valley by following the Bramford road, we find a brickkiln some way up the hill to our right, and approached by a narrow lane; this is in the London clay, and has been described by Mr Prestwich, who refers to it, as situated on the west of Whitton street.

Fig. (3).

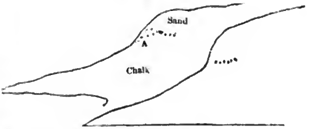


At the lower part of the lane are two chalk-pits, and half way up a sand-pit capped by the pebble-bed, which also appears at the bottom of the clay-pit: fig. (3) gives the partial sections of the main valley at this point.

From this point, (and indeed lower, for the road followed has cut off a very prominent bluff to the left), the side of the main valley which we are tracing, as also both sides of its Coddenham tributary, exhibits a very marked line of sand-cliffs resting upon a lower shelf of chalk; fig. (4) shews their general contour.

At the bottom of one of these bluffs {say at A, fig. (4)}, just above Bramford paper-mill, is a small chalk-pit, where a thin stratum of London clay seems to rest immediately upon the chalk, itself being overlaid by sand and fine laminated clays; but this section is too small and obscure to be worth much. What is the constitution of the rising ground to the back I have found no opportunity of ascertaining.

Fig. (4).



From Claydon to the opening of the Coddenham valley the sand-cliffs are concealed by the plantations and gardens of Shrubland; a pit is worked in them for economical purposes, and is said to exhibit a section of at least forty or fifty feet of pure red and white sand resting on the chalk. At this point the tongue between the Gipping and its Coddenham branch is very narrow, and is formed entirely of this sand with its chalk foundation.

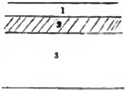
Throughout the whole length of the Coddenham valley until it dies away just beyond Hemingston Hall, the sand-cliffs form a very decided feature, especially on the north side; they generally shew themselves in the shape of rabbit-warrens or plantations: the hollow running up from Coddenham limekilns to Coddenham Row is a very good and picturesque example of their form. There is but little opportunity for examining their structure, and I have only seen the small section afforded by a pit exactly opposite Hemingston Hall: fig. (5) represents it:

(1) Irregular gravel containing portions of gryphea, laminæ of Inoceramus (?) and other traces of worn drift.

(2) Irregular layers of sand (full of small chalk-stones) and small pebbles.

(3) Pure white sand, the bottom of which is not seen.

Fig. (5).



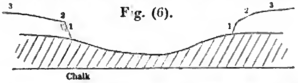
The surrounding country rises very slowly and almost imperceptibly from the top of these cliffs, but at a very short distance from their edge the soil passes into most unmistakeable drift-clay. In the annexed transverse section of the valley the figures refer to the superficial phenomena.

(1) Pure white sand.

(2) Gravel.

(3) Drift-clay.

Fig. (6).

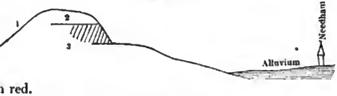


In a lateral depression at Hemingston (marked Dunstans on the Ordnance map) is a brickfield worked in a layer of pure grey clay, stratified and full of small chalk pebbles, about five feet thick: this bason is lower than the sand-cliffs, but not so low as the bottom of the valley: the workmen represent its floor to be a coarse chalky clay (probably chalk-rubble), and speak of some "joint-bones" being sometimes found there in groups.

At Bridge Place, where the Ipswich and Norwich turnpike-road crosses the mouth of the Coddenham valley, on the left ascending Creeting hill, is a large deep chalk-pit, in the side of which towards the valley is an irregular opening of considerable size, stopped up by a mass of rough pebbly gravel and reddish sand: this opening has all the appearance of being the transverse section of a cavern, and the materials filling it are the same, and follow the same order of arrangement, as those of the adjoining sand-hills, although the level is decidedly lower than the foot of them; these cliffs however are very undefined at this spot, and do not become well marked again until we reach Bosmere, where they once more start into great prominence, and continue so as far as the Creeting and Needham road. In this length there are several small sand and gravel-pits worked, while at their base are as many chalk-pits: the evidence of all these gives the section fig. (7) as the partial transverse section of the Gipping valley at this point.

(1) Drift-clay.
(2) Irregular pebbly flint-gravel with worn boulders, shells oxidized, lenticular nodules, &c.
(3) Pure white sand capped with red.

Fig. (7).*



These sand-hills terminate in a bluff looking towards Creeting Chapel, at the base of which instead of chalk, as is the case with the front towards Needham, pure drift-clay occurs: it is well exhibited in a pond placed in the angle which the Creeting and Needham Road makes at this spot; a plantation of fir-trees covers the abrupt rise on the other side of the road, but a pit worked in it gives a very good section of the compact sand.

The declination of surface represented at 1 on the left of fig. (7) is caused by a small lateral valley, which passes round the just mentioned bluff, and reduces the breadth of the sand-ridge to a few hundred yards only: it is remarkahle that both flanks of this hollow are formed of pure drift-clay; but that on the left hand which leads up to Creeting Chapel exhibits towards the top patches of sand and gravel.

In returning from this point to trace the right bank of the Gipping between Ipswich and Needham, I ought to say that my materials are not so complete as in the case of the opposite side of the valley, but as far as they go they present phenomena in exact accordance with those which we have just been reviewing.

The village of Stoke is on the side of the river Gipping opposite to Ipswich, and is placed just at its junction with the tidal estuary of the Orwell; the parsonage-house fronts the Orwell, and is situated exactly like Hemingston Hall, at the foot of a sand-cliff, which runs under the windmill and extends itself above the tunnel-mouth on the side looking towards Sproughton, here forming the Stoke sand-hills already alluded to (p. 434). The shelf projecting at the base of these hills is probably London clay, for there is a brickkiln in the village; however I have not been able to examine this, nor have I ascertained the nature of the section made by the tunnel.

The platform reaching through Goldrood and Birkfield to the valley of Bourn Brook is of the same red and white sand as the Stoke hills: the flank of this valley is very abrupt and cut into deep hollows, the upper part shews the sand very evidently, but at a certain small distance down (say 30 or 40 feet) a regular line of springs and ponds manifest themselves, probably indicating the junction with the London clay, or still more likely with the red crag, which exhibits itself in a pit close by, in an unusually hard and concrete state.

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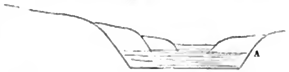
* In this figure the spectator is supposed to be looking down the valley.

Of the interval between Stoke and Blakenham I know but little; but in the neighbourhood of the latter place the line of cliffs or abrupt elevation is marked in many places; and I believe that universally the ground between this line and the bottom of the valley is chalk, the line itself is generally sand or gravel, whilst at a short distance down the drift-clay sets in. The Offton valley, which is celebrated for its chalk-pits, would probably afford illustrations of the same arrangement.

Between Darmsden Chapel and Baylham Church a dry lateral hollow runs up, the sketch of which is given by fig. (8); its sides, which are most cliff-like, are covered with wood, and offer no section, but they seem to be sandy. They apparently correspond exactly in the level of their base with the line at Shrubland on the opposite side of the Gipping.

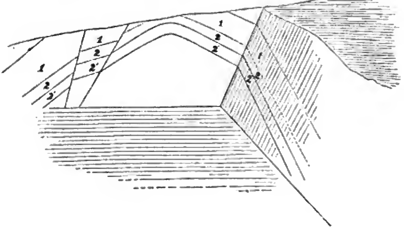
Fig. (8).



At the right-hand shoulder, A, entering this hollow is an old chalk-pit giving a section of about half the height of the cliff, at the top of which is a mass of large pebbles and sand. Just above the top of (A) in the still rising ground is an old marl-pit, which *seems* to be a brownish blue drift-clay, and certainly a little further back the soil is the heaviest drift-clay.

Between Darmsden and Needham is a gravel-pit, whose situation, when viewed from the opposite Creeting sand-hills, is seen to be in a certain ill-defined line of abruptness, which is a continuation of the preceding, and extends up both sides of the Barking valley, where it again changes into well-marked cliffs, covered with woods. Just below this line is Mount's chalk-pit, and others occur in similar situations towards Barking. This gravel has the same characteristics as the No. (2) of fig. (7), and is quite undistinguishable from it: whether it rests upon sand or not cannot be seen; but just below occurs, in a cutting of the E. U. R. the Gallows Hill pit, supplying the information of fig. (8): the two sections given by it are at right angles to each other, and that which is shaded looks towards the valley:

Fig. (9).



(1) All above ...... is a dark rough gravel, interspersed with loamy clay and sand, similar in all respects to that of the Darmsden pit above.

(2) Below ...... the layers are whiter and distinctly stratified.

(2′) A layer of fine whitish loam or brick-earth, with a bed of small chalky gravel containing broken belemnites, worn ostrea, echini, &c.

Below (2′) is a perfectly white pure sand to the bottom of the section; the greatest depth exposed is between 60 and 70 feet.

The strata are much bent, and many slips can be seen.

We have now sketched both sides of the Gipping valley from Ipswich to Needham; this portion has a character of its own different from the remainder, arising from the fact that the chalk does not again make its appearance, nor do the abrupt sand-cliffs extend any further: there is however no kind of discontinuity in the form of the valley itself.

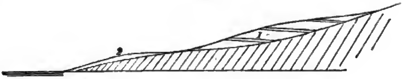
Resuming the upward course of the left bank we come to the opening of a large valley running up to Stonham. Following this we find on the right side, immediately under Creeting Chapel, a sand-pit, which is not now worked; when fresh cut it exhibited a section of about 20 feet high, shewing alternate horizontal strata of sand and fine clay, exactly like the (2) and (2′) of fig. (9). The flank of the valley against which this mass appears to be placed is, from the evidence of drains and ditches, a pure drift-clay extending down to at least the level of the base of this pit, when it is covered over or separated from the flat boggy meadows by a sort of gravelly or shingly bank: a little further on (at Woolney Hall), still on the right ascending side, this low gravelly portion contains a mass of soft white brick-clay.

Across the stream, and just at the level of the preceding pit, is a large sand-pit immediately below Hill-farm. The section exhibited by it is perhaps 20 feet high: nearly all this, beginning at the top, is occupied by alternate layers of fine brick-clay and white chalky sand; the latter contains broken belemnites, ostrea, small black spots like powdered carbon or extremely decayed wood, and is full of fragments of white shells lying flat: it has exactly the appearance of ordinary beach-sand; and I have even found tolerably preserved specimens of recent murex, and one or two pieces of bivalves so coloured as to resemble the red crag fossils. At the base the sand seems to be purer and more continuous. There are also considerable slips in the strata.

On this side, throughout the whole extent of the valley, traces of sand and gravel are well known by farmers and surveyors to occur along the level of this pit and below it, but above (and the ground rises considerably) the stiff tenacious drift-clay always sets in.

Recrossing the stream again we find on Deerbolt's farm several pits situated at different levels up to the before-mentioned superior limit; they are respectively worked for gravel, sand, and brick-earth: the highest exposes alternate layers of fine brick-clay and sand capped by an unstratified mass of earthy rubble; the workmen represent the base of this to be a stiff chalky clay, like the subsoil of the land above (pure drift-clay); the lower pits give thin layers of sand and brick-earth upon a floor of angular flints, so that fig. (10) would seem to be the section at this point.

Fig. (10).



(1) Alternate layers of brick-earth and sand.

(2) Angular flints.

(A) Boggy meadows.

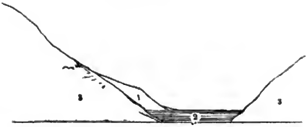
(B) Drift-clay.

In (2) a group of bones of some large mammal were lately found; they were apparently *in situ*, but were very fragmentary and corroded.

At Stonham Brewery the valley divides, one branch passing up to Aspal Stonham and Mickfield, the other running to the left under Little Stonham church: both these rapidly become shallow and lose the presence of the peaty meadows. The wells of the Brewery are sunk through the flat alluvial deposit, and reach a gravelly bed with a perpetual spring of hard water at a depth of between 10 and 15 feet.

A short distance up the left branch on its right flank the well of E. Stonham Rectory passes through 40 feet to a bed of gravel, where there is again a never-failing spring of hard water. The drift-clay here comes down to the bottom of the valley, but along both its sides are traces of a gravelly bank, and about a mile further up is a pit exhibiting 10 or 12 feet of a sandy dark red gravel, with unbroken pebbles resting upon at least the same depth of a marly sand impervious to water: this mass is bounded laterally by drift-clay: fig. (11) is the section of the valley at this place.

Fig. (11).



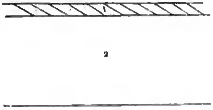
(1) Dark red sandy gravel.

(2) Marly sand.

(3) Drift-clay.

The remaining portion of the Gipping valley from Needham upwards has not yet supplied me with much evidence of a definite kind. On the left-hand side of the road from Needham to Stowmarket, just as it leaves the former town, is a pit whose section is given by (fig. 12).

Fig. (12).



(1) Thin layer of chalky gravel with broken belemnites, &c.

(2) White brick-clay, perhaps 20 feet thick

On the other side of the road adjoining Hawk Mill is another pit of the same kind, in which some depth of sand overlies the brick-earth.

The boggy meadows in this part have generally a narrower breadth than before, and are bordered by mounds which display a gravelly sandy soil, commonly called light land, but very often the drift-clay is shewn by drains and ditches to come down to the bottom on both sides; the before-mentioned gravel, &c. only lie against it.

Just before the railway enters Stowmarket it cuts through a spur which comes down from the Stowupland platform, giving a good section 60 or 70 feet high of pure *blue* clay-drift, resting upon a compact mass of sand; the sand is of a ferruginous red colour at the top but white below, and I found in it one or two small pieces of shells resembling worn bits of ostrea. The line of junction of the clay and sand, as far as it can be traced, is very irregular; and at both extremities of the cutting the sand apparently gets confused with a mass of singularly rough gravel full of boulders of all sizes, which is also overlaid by the blue clay: the opposite side of the cutting is of small altitude, and is composed entirely of this irregular rubble.

On the same side the ground at the back of Thorney Hall, which rises rapidly, is faced with sand, though both the roads ascending it cut deep enough to expose the concealed drift-clay.

The town of Stowmarket covers the opposite side at this point, and almost reaches up to the level where the subsoil is unmistakeable drift-clay: a brickyard however intervenes, occupying an ascending hollow, but only shews two layers distinctly stratified of a greyish brick-clay differing from each other in tint; this is evidently reconstructed drift. The workmen spoke of some kind of organic remains being found here, but had none to shew me.

A few years ago an artesian well was sunk at a brewery in the middle of the town, and the following fig. (13) has been given me as its section:

Fig. (13).

Gravel	7 feet
Sand	2
Wet loamy brick-clay .	4
Common brick-clay . .	13
Light-blue clay . . Dark do. . . . . . Dark-blue clay with lumps of chalk . . .	27
Bed of sand with *recent shells*	18
	71
Chalk	250
Green sand (?) . .	9
Total	330 feet

The foregoing are the most prominent details of the information which I have been able to gather for myself with regard to the geological phenomena of the Gipping valley: they seem capable of being very concisely summed up:

The upper part of the valley and its branches is channelled in the drift-clay, from Needham to about Bramford in the chalk, and the remaining portion, including the Orwell itself, in the London clay. It is fringed at a high level on both sides, at least between Ipswich and Needham, with a line of sand-cliffs. At Stoke, the easterly end, this sand overflows, as it were, the edge of the valley, and forms the light lands of the higher grounds between the Stour and Orwell estuaries: indeed it is not improbable that the sandy heaths, which cover the whole coast-district of the county from Saxmundam to Harwich, and thence extend into Essex, may be continuations of the same. If the prolongation of the same cliffs in the opposite or N. W. direction is less distinctly marked beyond the limits of the chalk, it is still extended far enough to cover the drift-clay at Creeting (p. 435), and thus to suggest very strongly its identity with the patches along the Stonham valley: it is in one (p. 437 Hill-farm) of these latter only that I have found any fossils *in situ;* they are very ill preserved, but Prof. M^cCoy has pronounced some of them to be recent.

The question now offers itself, what is the proper relative position of this sand? is it the sand upon which (as will be afterwards mentioned) the drift-clay always seems to rest, exposed by the depth of the valley sections, or is it merely the edge of a mass which once filled the whole breadth of the valley and has since been swept away by denudation?

The first hypothesis is that which most naturally suggests itself, and I confess that I cannot with any satisfaction dismiss it; but on the other hand the results of my observations do not at all encourage me to hold it. The circumstances, which appear to me inconsistent with it, are: 1st, that the clay is nowhere seen to rest upon *this* sand; had it ever been above it, the probabilities are that it would have been left by the denuding cause, at some spots at least, throughout the many miles of sand-ridge in the shape of a sharply-defined cap; but the fact is, that it appears nowhere in an immediately superior position, and where it does manifest itself close in the rear of the sand, its thickness is not perceptibly less than at remoter distances: 2ndly, that at Creeting (p. 437) if the sand is not exhibited actually resting upon the clay, it is above it and in contact with it in several directions: 3rdly, that the gravels and coarse sands bear every appearance of being due to the destruction of the clay itself: 4thly, that these gravels and coarse sands with some immediately underlying laminæ of fine clay invariably cap the sands in question, while nothing of the kind occurs between the drift-clay, and *its* underlying sand.

It may be added that in the Stonham valley the sands are undoubtedly resting in more than one place upon the drift-clay, and they resemble in regularity of stratification and every other characteristic those just alluded to: this circumstance strongly suggests their being continuous with the ridge, and therefore supports the second view.

The section of the Stowmarket well (p. 439) no doubt complicates the case, as its sand with recent shells greatly resembling the sand of the Stonham valley (p. 437), lies between a so-called drift-clay and the chalk; but may not this clay "with lumps of chalk" be reconstructed drift? that which is worked in the brick-yard at the head of the town most undoubtedly is so, and yet exhibits a considerable amount of chalk nodules. The chalk of this well must rise very nearly to the level of the river, but at no place nearer than Needham does it actually exhibit itself at the surface; indeed the sand and gravel which forms the foundation of the valley, has not, I believe, been fathomed.

I have little doubt but that the line of the valley corresponds with a fissure in the chalk; if so the disturbance producing the fracture must have taken place subsequently to the deposition of the drift-clay, otherwise there would be no reason for the coincidence: it is weighty evidence in favour of a still later deposition of the sand, that it cannot be said to have suffered such general disturbance; its strata are very easily distinguished, and although full of such partial slips and displacements, as would occur in cliffs subject to wear at the base, are upon the whole perfectly horizontal.

If the preceding observations have been correctly made they would seem best to accord with the supposition that the Gipping valley was formed subsequently to the deposition of the drift-clay, and for the greater part of its course cut completely through the clay: that it afterwards formed the estuary of a very sandy-bottomed sea; that as the rising of the land or other causes made the water shallower the deposits assumed the character of fine clay, then succeeded a small chalky gravel with broken belemnites, ostrea, &c. (especially indicative of a tranquil beach), and finally came a large mass of drift, boulders and flints—the whole apparently the produce of drift denudation. As the land still rose this deposit was itself cut into, the greater portion carried away, and only its edges left. The floor of the valley thus a second time excavated is very generally covered over with a sandy gravelly rubble in which bones are sometimes found (p. 438, (2) fig. 10); but the larger portion even of this is concealed by alluvial deposit and peat of the meadows.

Whether or not the real order of events at all resemble this, the continuous line of cliffs on both sides of the valley, some of them carved in the chalk itself (fig. 8), points almost irresistibly to a period when the sea level must have remained with some permanence at a height much above the present high-water mark.

It is probable that the chalk which comes so near to the surface as to be exposed by the channelling of the Gipping valley was the ridge bounding the London clay on the North: the drift-clay clearly overlies it here, and therefore cannot have been stopped in its S. E. course by the same ridge, although it is not now seen to any great extent between this point and the coast: none of the cliff sections (certainly not those at Harwich, Walton on the Naze, and Aldborough) except Lowestoft appear to shew any undisturbed drift-clay at all.

I have not examined any of the other valleys of the county in so much detail as I have that of the Gipping, but the following very patent facts connected with them are of much importance in any attempt to arrange the various phenomena of the Suffolk Drift. The sections given are most of them from Mr Clarke's memoir printed in the *Transactions of the Geological Society* (2nd Series, Vol. V.); a most valuable paper, but unfortunately not sufficiently definite in regard to the localities quoted. The nomenclature is taken from the Ordnance maps.

The *Offton Stream* rises about Great Bricett in deep drift-clay, cuts at Offton into the chalk, which is there largely quarried, and continues in it until it joins the Gipping at

Bramford. At Willisham Hall above Offton the chalk is only covered by two feet of clay, itself being 192 feet above the valley, while at Little Blakenham there is 157 feet of clay, the chalk which forms the floor of the valley exhibiting much disturbance; elsewhere in Willisham there is 157 feet of clay above chalk.

The *Elmsett* and *Flowton* brooks rise towards Naughton in deep clay. At *Flowton Hall* on the flank of the valley a well gives 94 feet diluvial clay resting on the chalk: at *Hintlesham* 15 feet drift-clay and, at a higher level, 21 of mixed clay and sand: *Copdock* on high ground 15 feet of drift-clay, above 45 of sand, and at a lower level 30 feet of sand and gravel: at *Belstead* (adjoining) 15 feet of stiff loam on 35 feet of crag.

The *Brett* rises beyond Thorpe in thick drift-clay; a branch of it originates in the same neighbourhood, and runs round by Cockfield, Lavenham, Brent Eleigh, and joins the main stream at Chelsworth, while another comes from *Hitcham* through Bildeston.

At *Bradfield* there is 111 feet of drift-clay upon at least 40 feet of red sand, at *Cockfield* 126 feet of pyritous blue drift-clay. At *Lavenham* parsonage on the higher ground 45 feet of drift-clay. At *Bildestone* 60 feet of drift-clay. At *Monk's Eleigh* 62 feet of drift-clay on the flank of the valley with chalk rising seven feet above the level of the stream where it is quarried. Close to Chelsworth is a chalk-pit on the Ordnance map some way up the flank of the valley, and again on the side opposite to *Corford Union House* also high up. At *Whatfield* on the neighbouring high ground there is 40 feet of drift-clay upon chalk, and then 66 feet to the permanent water-level. At *Kersey* is a chalk-pit on the flank of the valley, but on the higher ground there is 75 feet of drift-clay on sand. At *Hadleigh* the London clay lies on the chalk to a depth of at least 100 feet, and from this point to the Stour the brickkilns on the Ordnance map point out the continuance of the London clay. At *Shelly* the London-clay formation is cut through and the chalk is exposed at the level of the river; from here to Harwich the chalk seems to slope about six feet per mile.

A branch of the Stour rises at *Depden* just two miles S. of *Chevington*, passes *Hawkedon*, close to *Hartest*, through *Boxstead*, and joins the Stour just above Sudbury. At Hartest we find 300 feet of blue drift-clay, and close to Sudbury the chalk is laid bare, forming the flanks of the Stour valley: below this again the brickkilns bespeak the region of London clay.

The streams just traced all lie on the West of the Gipping, and it may be worth while here to make a *résumé* of their evidence. Between Clare and Bury the drift-clay seems to average 100 feet, but N. of an imaginary line drawn through Lidgate, Ousden, Ickworth, Rougham, Stowlangtoft, &c., the chalk comes very close to the surface. S. & S. E. of this line at Bradfield, Norton, Elmswell, Drinkstone, Wetherden, Haughley, Finborough, &c., the drift-clay ranges from 100 to 50 feet, *always* resting upon sand.

Just on the line between Clare and Claydon the chalk is continuously laid bare by the river-courses, and the drift-clay gradually thins off on passing this line towards the S.: in the same direction the London-clay formation, often capped by crag and sand, increases to a considerable thickness.

On the other side of the Gipping we have the Deben, rising nearly in the center of Suffolk at Aspal, and passing through Debenham, Cretingham, Wickham Market, to Woodbridge.

At Aspal two wells, whose position relative to the stream I do not know, give 90 and 45 feet respectively of drift-clay resting upon a red sand. At *Debenham* and *Winston* are brickkilns a short way up the flanks in reconstructed stratified drift-clay lying upon a basis of drift-clay. At Framsden and Cretingham, an equal height up are sand-pits, which probably rest upon drift-clay, as this is seen below them. (There seems a possibility of the sand, though distinctly stratified, being in some places within the drift-clay.) The lower level of the valley from this point downwards is entirely sand and gravel, as an inspection of the Ordnance map will shew: the higher lands however are drift-clay; and much doubt may be entertained whether this sand is merely a manifestation of that which all the above-quoted sections prove to exist below the drift-clay, or whether it is the result of a much later estuary action within the valley itself.

In the valleys of the *Ore* and *Alde* it shews itself in the same way, and at *Benhall* pottery I have seen it come out most unmistakeably from under the drift-clay itself. The drift-clay is very thin and almost disappears, except on high ground, everywhere to the S. E. of a line passing from *Ipswich* through *Woodbridge*, *Saxmundham*, to *Dunwich:* so thin is it along this line that farmers are in the habit of draining their high lands by "tapping," i. e. by making a perforation through the clay at some convenient spot, which serves as a conduct pipe to carry away any amount of water.

It is remarkable that this supposed line is very nearly the N. W. boundary of the crag-deposit; some fossiliferous clays, as those of *Chillesford* and the neighbourhood of *Tunstall* and Iken (investigated by Mr Prestwich), overlay portions of the crag; but the whole district is generally covered with a sand which is most probably a part of that just alluded to as underlying the drift-clay, left behind in the denudation of the latter, and going far by its own reconstruction to produce the sand-deposits in the present river-valleys.

We are not without evidence of considerable disturbances posterior to the crag. A section of the coast from *Harwich* to the mouth of the *Deben* given by Mr Clarke exhibits dislocations of the strata of the crag and London clay.

The strata of the chalk are not in all cases readily made out, but in many places they are distinct enough to shew that the present valleys are due to elevation and fracture. At *Offton* where the valley runs N. W. to S. E. a chalk-pit on the N. E. side shews the strata rising to the S.W. At *Blakenham*, lower down the same stream, they rise to the S. At *Shelly* on the west side of the valley they rise to the N. E. At Bramford the beds of the London clay on the chalk dip to the S. E., probably indicating a dip of the chalk in the same direction; these pits are on the east side of the valley.

It may not be out of place at this point to remark, that a somewhat hasty glance at the coast sections of Norfolk (from Wells to Happisburgh) has led me to believe that the drift-clay there exhibited differs greatly in character from that of Suffolk: it is generally a pretty nearly homogeneous clay with small chalk "nibs" plentifully sprinkled through it, and has a certain rough species of stratification about it; the multitudes of secondary boulders, or rock fragments, which characterise the Suffolk clay, are almost entirely wanting in it. This Norfolk clay is also thickly covered by very irregular masses of sand and gravel, of which I have seen no distinct evidence in Suffolk excepting on the north side of Lowestoft;

its absence may be ascribed to denudation, and probably it would be instructive to trace its edge closely.

I imagine that there is moreover another sand and gravel superior to this. There certainly exists an extensive platform of these materials, resting against the chalk towards Holkham, Wighton, Wells, &c., passing inland by Holt, &c., and terminating in singularly abrupt escarpments towards the coast, as at Cley, Weybourne, Sherringham, &c. These escarpments strongly resemble the sand-cliffs of the Gipping valley in form, and like them generally rest upon a chalk shelf: no doubt a great difference in the powers of resisting denudation between two contiguous strata produces this appearance: they also resemble the Gipping sand-hills in the regularity of their stratification, and in the fact of their being capped by a large coarse gravel. As a feature in the landscape they contribute very much to the beauty of the Glandford valley, as well as of the neighbourhood of Sherringham and Cromer.

RECAPITULATION.

A N. E. and S. W. line of chalk bounds the drift-clay on one side of the county. On the S. E. of this the clay almost immediately manifests an average thickness of about 100 feet.

The clay gradually thins off on approaching a line, which passes through the heads of the estuaries and approximately forms a boundary to both the London clay and the Crag.

The sand underlying the drift-clay passes out over the London clay and Crag.

The phenomena of Norfolk, though they do not readily range themselves side by side with those of Suffolk, very probably accord with a southerly denudation.

Valleys of denudation expose a ridge of chalk along a certain well-defined line, passing it approximately at right angles, and exhibit evidence of having been due to elevatory transverse fracture.

Evidence of disturbance at a date posterior to the crag is afforded by the coast-sections.

The just-mentioned ridge of chalk bounds the London clay towards the north, but does not stop the drift-clay, as patches of the latter appear south of it.

A distinct line of sand-cliffs of uncertain geological date exists along the Gipping valley, marking a period when the sea remained permanently at that level.

XIX. *On the Transformation of Surfaces by Bending.*

By JAMES CLERK MAXWELL, ESQ. B.A. *Trinity College, Cambridge.*

[Read *March* 13, 1854.]

EUCLID has given two definitions of a surface, which may be taken as examples of the two methods of investigating their properties.

That in the first book of the Elements is—

"A superficies is that which has only length and breadth."

The superficies differs from a line in having breadth as well as length, and the conception of a third dimension is excluded without being explicitly introduced.

In the eleventh book, where the definition of a solid is first formally given, the definition of the superficies is made to depend on that of the solid—

"That which bounds a solid is a superficies."

Here the conception of three dimensions in space is employed in forming a definition more perfect than that belonging to plane Geometry.

In our analytical treatises on geometry a surface is defined by a function of three independent variables equated to zero. The surface is therefore the boundary between the portion of space in which the value of the function is positive, and that in which it is negative; so that we may now define a surface to be the boundary of any assigned portion of space.

Surfaces are thus considered rather with reference to the figures which they limit than as having any properties belonging to themselves.

But the conception of a surface which we most readily form is that of a portion of matter, extended in length and breadth, but of which the thickness may be neglected. By excluding the thickness altogether, we arrive at Euclid's first definition, which we may state thus—

"A surface is a lamina of which the thickness is diminished so as to become evanescent."

We are thus enabled to consider a surface by itself, without reference to the portion of space of which it is a boundary. By drawing figures on the surface, and investigating their properties, we might construct a system of theorems, which would be true independently of the position of the surface in space, and which might remain the same even when the form of the solid of which it is the boundary is changed.

When the properties of a surface with respect to space are changed, while the relations of lines and figures in the surface itself are unaltered, the surface may be said to preserve its identity, so that we may consider it, after the change has taken place, as the same surface.

When a thin material lamina is made to assume a new form it is said to be *bent*. In certain cases this process of bending is called *development*, and when one surface is bent so as to coincide with another it is said to be *applied* to it.

By considering the lamina as deprived of rigidity, elasticity, and other mechanical properties, and neglecting the thickness, we arrive at a mathematical definition of this kind of transformation.

"The operation of bending is a continuous change of the form of a surface, without extension or contraction of any part of it."

The following investigations were undertaken with the hope of obtaining more definite conceptions of the nature of such transformations by the aid of those geometrical methods which appear most suitable to each particular case. The order of arrangement is that in which the different parts of the subject presented themselves at first for examination, and the methods employed form parts of the original plan, but much assistance in other matters has been derived from the works of GAUSS[1], LIOUVILLE[2], BERTRAND[3], PUISEUX[4], &c. references to which will be given in the course of the investigation.

I.

On the Bending of Surfaces generated by the motion of a straight line in space.

If a straight line can be drawn in any surface, we may suppose that part of the surface which is on one side of the straight line to be fixed, while the other part is turned about the straight line as an axis.

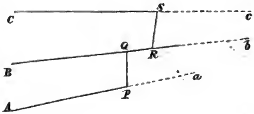
In this way the surface may be bent about any number of generating lines as axes successively, till the form of every part of the surface is altered.

The mathematical conditions of this kind of bending may be obtained in the following manner.

Let the equation of the generating line be expressed so that the constants involved in it are functions of one independent variable u, by the variation of which we pass from one position of the line to another.

If in the equation of the generating line Aa, $u=u_1$, then in the equation of the line Bb we may put $u=u_2$, and from the equations of these lines we may find by the common methods the equation of the shortest line PQ between Aa and Bb, and its length, which we may call $\delta\zeta$. We may also find the angle between the directions of Aa and Bb, and let this angle be $\delta\theta$.

In the same way from the equation of Cc, in which $u=u_3$, we may deduce the equation of RS, the shortest line between Bb and Cc, its length $\delta\zeta_2$, and the angle $\delta\theta_2$ between the directions of Bb and Cc. We may also find the value of QR, the distance between the points at which PQ and RS cut Bb. Let $QR = \delta\sigma$, and let the angle between the directions of PQ and RS be $\delta\phi$.



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[1] *Disquisitiones generales circa superficies curvas.* Presented to the Royal Society of Gottingen, 8th October, 1827. *Commentationes Recentiores*, Tom. VI.

[2] Liouville's *Journal*, XII.

[3] *Ibid.* XIII.

[4] *Ibid.*

Now suppose the part of the surface between the lines Aa and Bb to be fixed, while the part between Bb and Cc is turned round Bb as an axis. The line RS will then revolve round the point R, remaining perpendicular to Bb, and Cc will still be at the same distance from Bb, and will make the same angle with it. Hence of the four quantities $\delta\zeta_2$, $\delta\theta_2$, $\delta\sigma$ and $\delta\phi$, $\delta\phi$ alone will be changed by the process of bending. $\delta\phi$, however, may be varied in a perfectly arbitrary manner, and may even be made to vanish.

For, PQ and RS being both perpendicular to Bb, RS may be turned about Bb till it is parallel to PQ, in which case $\delta\phi$ becomes $= 0$.

By repeating this process, we may make all the "shortest lines" parallel to one another, and then all the generating lines will be parallel to the same plane.

We have hitherto considered generating lines situated at finite distances from one another; but what we have proved will be equally true when their distances are indefinitely diminished. Then in the limit

$$\frac{\delta\zeta}{u_2 - u_1} \text{ becomes } \frac{d\zeta}{du},$$

$$\frac{\delta\theta}{u_2 - u_1} \;.\;.\;.\;.\;.\; \frac{d\theta}{du},$$

$$\frac{\delta\sigma}{u_2 - u_1} \;.\;.\;.\;.\;.\; \frac{d\sigma}{du},$$

$$\frac{\delta\phi}{u_2 - u_1} \;.\;.\;.\;.\;.\; \frac{d\phi}{du}.$$

All these quantities being functions of u, ζ, θ, σ, and ϕ, are functions of u and of each other; and if the forms of these functions be known, the positions of all the generating lines may be successively determined, and the equation to the surface may be found by integrating the equations containing the values of ζ, θ, σ, and ϕ.

When the surface is bent in any manner about the generating lines, ζ, θ, and σ remain unaltered, but ϕ is changed at every point.

The form of ϕ as a function of u will depend on the nature of the bending; but since this is perfectly arbitrary, ϕ may be any arbitrary function of u. In this way we may find the form of any surface produced by bending the given surface along its generating lines.

By making $\phi = 0$, we make all the generating lines parallel to the same plane. Let this plane be that of xy, and let the first generating line coincide with the axis of x, then ζ will be the height of any other generating line above the plane of xy, and θ the angle which its projection on that plane makes with the axis of x. The ultimate intersections of the projections of the generating lines on the plane of xy will form a curve, whose length, measured from the axis of x, will be σ.

Since in this case the quantities ζ, θ, and σ are represented by distinct geometrical quantities, we may simplify the consideration of all surfaces generated by straight lines by reducing them by bending to the case in which those lines are parallel to a given plane.

In the class of surfaces in which the generating lines ultimately intersect, $\frac{d\zeta}{du} = 0$, and

ζ constant. If these surfaces be bent so that $\phi = 0$, the whole of the generating lines will lie in one plane, and their ultimate intersections will form a plane curve. The surface is thus reduced to one plane, and therefore belongs to the class usually described as "developable surfaces." The form of a developable surface may be defined by means of the three quantities θ, σ and ϕ. The generating lines form by their ultimate intersection a curve of double curvature to which they are all tangents. This curve has been called the cuspidal edge. The length of this curve is represented by σ, its absolute curvature at any point by $\frac{d\theta}{d\sigma}$, and its torsion at the same point by $\frac{d\phi}{d\sigma}$.

When the surface is developed, the cuspidal edge becomes a plane curve, and every part of the surface coincides with the plane. But it does not follow that every part of the plane is capable of being bent into the original form of the surface. This may be easily seen by considering the surface when the position of the cuspidal edge nearly coincides with the plane curve but is not confounded with it. It is evident that if from any point in space a tangent can be drawn to the cuspidal edge, a sheet of the surface passes through that point. Hence the number of sheets which pass through one point is the same as the number of tangents to the cuspidal edge which pass through that point; and since the same is true in the limit, the number of sheets which coincide at any point of the plane is the same as the number of tangents which can be drawn from that point to the plane curve. In constructing a developable surface of paper, we must remove those parts of the sheet from which no real tangents can be drawn, and provide additional sheets where more than one tangent can be drawn.

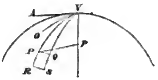
In the case of developable surfaces we see the importance of attending to the position of the lines of bending; for though all developable surfaces may be produced from the same plane surface, their distinguishing properties depend on the form of the plane curve which determines the lines of bending.

II.

On the Bending of Surfaces of Revolution.

In the cases previously considered, the bending in one part of the surface may take place independently of that in any other part. In the case now before us the bending must be simultaneous over the whole surface, and its nature must be investigated by a different method.

The position of any point P on a surface of revolution may be determined by the distance PV from the vertex, measured along a generating line, and the angle AVO which the plane of the generating line makes with a fixed plane through the axis. Let $PV = s$ and $AVO = \theta$. Let r be the distance (Pp) of P from the axis, r will be a function of s depending on the form of the generating curve.



Now consider the small rectangular element of the surface at P. Its length $PR = \delta s$, and its breadth $PQ = r\delta\theta$, where r is a function of s.

If in another surface of revolution r' is some other function of s, then the length and breadth of the new element will be δs and $r'\delta\theta'$, and if

$$r' = \mu r, \qquad \text{and} \quad \theta' = \frac{1}{\mu}\theta,$$

$$r'\delta\theta = rd\theta,$$

and the dimensions of the two elements will be the same.

Hence the one element may be applied to the other, and the one surface may be applied to the other surface, element to element, by bending it. To effect this, the surface must be divided by cutting it along one of the generating lines, and the parts opened out, or made to overlap, according as μ is greater or less than unity.

To find the effect of this transformation on the form of the surface we must find the equation to the original form of the generating line in terms of s and r, then putting $r' = \mu r$, the equation between s and r' will give the form of the generating line after bending.

When μ is greater than 1 it may happen that for some values of s, $\frac{dr}{ds}$ is greater than $\frac{1}{\mu}$. In this case

$$\frac{dr'}{ds} = \mu\frac{dr}{ds} \text{ is greater than } 1;$$

a result which indicates that the curve becomes impossible for such values of s and μ.

The transformation is therefore impossible for the corresponding part of the surface. If, however, that portion of the original surface be removed, the remainder may be subjected to the required transformation.

The theory of bending when applied to the case of surfaces of revolution presents no geometrical difficulty, and little variety; but when we pass to the consideration of surfaces of a more general kind, we discover the insufficiency of the methods hitherto employed, by the vagueness of our ideas with respect to the nature of bending in such cases. In the former case the bending is of one kind only, and depends on the variation of one variable; but the surfaces we have now to consider may be bent in an infinite variety of ways, depending on the variation of three variables, of which we do not yet know the nature or interdependence.

We have therefore to discover some method sufficiently general to be applicable to every possible case, and yet so definite as to limit each particular case to one kind of bending easily understood.

The method adopted in the following investigations is deduced from the consideration of the surface as the limit of the inscribed polyhedron, when the size of the sides is indefinitely diminished, and their number indefinitely increased.

A method is then described by which such a polyhedron may be inscribed in any surface so that all the sides shall be triangles, and all the solid angles composed of six plane angles.

The problem of the bending of such a polyhedron is a question of trigonometry, and equations might be found connecting the angles of the different edges which meet in each solid angle of the polyhedron. It will be shown that the conditions thus obtained would be equivalent to three equations between the six angles of the edges belonging to each solid angle. Hence three additional conditions would be necessary to determine the value of every such angle, and the problem would remain as indefinite as before. But if by any means we can reduce the number of edges meeting in a point to four, only one condition would be necessary to determine them all, and the problem would be reduced to the consideration of one kind of bending only.

This may be done by drawing the polyhedron in such a manner that the planes of adjacent triangles coincide two and two, and form quadrilateral facets, four of which meet in every solid angle. The bending of such a polyhedron can take place only in one way, by the increase of the angles of two of the edges which meet in a point, and the diminution of the angles of the other two.

The condition of such a polyhedron being inscribed in any surface is then found, and it is shown that when two forms of the same surface are given, a perfectly definite rule may be given by which two corresponding polyhedrons of this kind may be inscribed, one in each surface.

Since the *kind* of bending completely defines the nature of the quadrilateral polyhedron which must be described, the lines formed by the edges of the quadrilateral may be taken as an indication of the kind of bending performed on the surface.

These lines are therefore defined as "*Lines of Bending.*"

When the lines of bending are given, the forms of the quadrilateral facets are completely determined; and if we know the angle which any two adjacent facets make with one another, we may determine the angles of the three edges which meet it at one of its extremities. From each of these we may find the angles of three other edges, and so on, so that the form of the polyhedron after bending will be completely determined when the angle of one edge is given. The bending is thus made to depend on the change of one variable only.

In this way the angle of any edge may be calculated from that of any given edge; but since this may be done in two different ways, by passing along two different sets of edges, we must have the condition that these results may be consistent with each other. This condition is satisfied by the method of inscribing the polyhedron. Another condition will be necessary that the *change* of the angle of any edge due to a small change of the given angle, produced by bending, may be the same by both calculations. This is the condition of "Instantaneous Lines of Bending." That this condition may continue to be satisfied during the whole process we must have another, which is the condition for "Permanent Lines of Bending."

The use of these lines of bending in simplifying the theory of surfaces is the only part of the present method which is new, although the investigations connected with them naturally led to the employment of other methods which had been used by those who have already treated of this subject. A statement of the principal methods and results of these mathematicians will save repetition, and will indicate the different points of view under which the subject may present itself.

The first and most complete memoir on the subject is that of M. GAUSS, already referred to.

The method which he employs consists in referring every point of the surface to a corresponding point of a sphere whose radius is unity. Normals are drawn at the several points of the surface toward the same side of it, then lines drawn through the centre of the sphere in the direction of each of these normals intersect the surface of the sphere in points corresponding to those points of the original surface at which the normals were drawn.

If any line be drawn on the surface, each of its points will have a corresponding point on the sphere, so that there will be a corresponding line on the sphere.

If the line on the surface return into itself, so as to enclose a finite area of the surface; the corresponding curve on the sphere will enclose an area on the sphere, the extent of which will depend on the form of the surface.

This area on the sphere has been defined by M. Gauss as the measure of the "entire curvature" of the area on the surface. This mathematical quantity is of great use in the theory of surfaces, for it is the only quantity connected with curvature which is capable of being expressed as the sum of all its parts.

The sum of the entire curvatures of any number of areas is the entire curvature of their sum, and the entire curvature of any area depends on the form of its boundary only, and is not altered by any change in the form of the surface within the boundary line.

The curvature of the surface may even be discontinuous, so that we may speak of the entire curvature of a portion of a polyhedron, and calculate its amount.

If the dimensions of the closed curve be diminished so that it may be treated as an element of the surface, the ultimate ratio of the entire curvature to the area of the element on the surface is taken as the measure of the "specific curvature" at that point of the surface.

The terms "entire" and "specific" curvature when used in this paper are adopted from M. Gauss, although the use of the sphere and the areas on its surface formed an essential part of the original design. The use of these terms will save much explanation, and supersede several very cumbrous expressions.

M. Gauss then proceeds to find several analytical expressions for the measure of specific curvature at any point of a surface, by the consideration of three points very near each other.

The co-ordinates adopted are first rectangular, x and y, or x, y and z, being regarded as independent variables.

Then the points on the surface are referred to two systems of curves drawn on the surface, and their position is defined by the values of two independent variables p and q, such that by varying p while q remains constant, we obtain the different points of a line of the first system, while p constant and q variable defines a line of the second system.

By means of these variables, points on the surface may be referred to lines on the surface itself instead of arbitrary co-ordinates, and the measure of curvature may be found in terms of p and q when the surface is known.

In this way it is shown that the specific curvature at any point is the reciprocal of the product of the principal radii of curvature at that point, a result of great interest.

From the condition of bending, that the length of any element of the curve must not be altered, it is shown that the specific curvature at any point is not altered by bending.

The rest of the memoir is occupied with the consideration of particular modes of describing the two systems of lines. One case is when the lines of the first system are geodesic, or "shortest" lines having their origin in a point, and the second system is drawn so as to cut off equal lengths from the curves of the first system.

The angle which the tangent at the origin of a line of the first system makes with a fixed line is taken as one of the co-ordinates, and the distance of the point measured along that line as the other.

It is shewn that the two systems intersect at right angles, and a simple expression is found for the specific curvature at any point.

M. LIOUVILLE (*Journal*, Tom. XII.) has adopted a different mode of simplifying the problem. He has shewn that on every surface it is possible to find two systems of curves intersecting at right angles, such that the length and breadth of every element into which the surface is thus divided shall be equal, and that an infinite number of such systems may be found. By means of these curves he has found a much simpler expression for the specific curvature than that given by M. GAUSS.

He has also given, in a note to his edition of MONGE, a method of testing two given surfaces in order to determine whether they are applicable to one another. He first draws on both surfaces lines of equal specific curvature, and determines the distance between two corresponding consecutive lines of curvature in both surfaces.

If by assuming the origin properly these distances can be made equal for every part of the surface, the two surfaces can be applied to each other. He has developed the theorem analytically, of which this is only the geometrical interpretation.

When the lines of equal specific curvature are equidistant throughout their whole length, as in the case of surfaces of revolution, the surfaces may be applied to one another in an infinite variety of ways.

When the specific curvature at every point of the surface is positive and equal to a^2, the surface may be applied to a sphere of radius a, and when the specific curvature is negative $= -a^2$ it may be applied to the surface of revolution which cuts at right angles all the spheres of radius a, and whose centres are in a straight line.

M. BERTRAND has given in the XIIIth Vol. of LIOUVILLE's *Journal* a very simple and elegant proof of the theorem of M. GAUSS about the product of the radii of curvature.

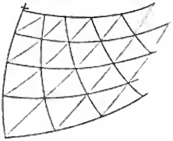
He supposes one extremity of an inextensible thread to be fixed at a point in the surface, and a closed curve to be described on the surface by the other extremity, the thread being stretched all the while. It is evident that the length of such a curve cannot be altered by bending the surface. He then calculates the length of this curve, considering the length of the thread small, and finds that it depends on the product of the principal radii of curvature of the surface at the fixed point. His memoir is followed by a note of M. DIGUET, who deduces the same result from a consideration of the area of the same curve; and by an independent memoir of M. PUISEUX, who seems to give the same proof at somewhat greater length.

NOTE. Since this paper was written, I have seen the Rev. Professor JELLETT's Memoir, *On the Properties of Inextensible Surfaces.* It is to be found in the *Transactions of the Royal Irish Academy*, Vol. XXII. *Science*, &c., and was read May 23, 1853. Professor JELLETT has obtained a system

ON THE PROPERTIES OF A SURFACE CONSIDERED AS THE LIMIT OF THE INSCRIBED POLYHEDRON.

1. ***To inscribe a polyhedron in a given surface, all whose sides shall be triangles, and all whose solid angles shall be hexahedral.***

On the given surface describe a series of curves according to any assumed law. Describe a second series intersecting these in any manner, so as to divide the whole surface into quadrilaterals. Lastly, describe a third series (the dotted lines in the figure), so as to pass through all the intersections of the first and second series, forming the diagonals of the quadrilaterals.



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of three partial differential equations which express the conditions to which the displacements of a continuous inextensible membrane are subject. From these he has deduced the two theorems of GAUSS, relating to the invariability of the product of the radii of curvature at any point, and of the "entire curvature" of a finite portion of the surface.

He has then applied his method to the consideration of cases in which the flexibility of the surface is limited by certain conditions, and he has obtained the following results:—

If the displacements of an inextensible surface be all parallel to the same plane, the surface moves as a rigid body.

Or, more generally,

If the movement of an inextensible surface, parallel to any one line, be that of a rigid body, the entire movement is that of a rigid body.

The following theorems relate to the case in which a curve traced on the surface is rendered rigid:—

If any curve be traced upon an inextensible surface whose principal radii of curvature are finite and of the same sign, and if this curve be rendered immoveable, the entire surface will become immoveable also.

In a developable surface composed of an inextensible membrane, any one of its rectilinear sections may be fixed without destroying the flexibility of the membrane.

In convexo-concave surfaces, there are two directions passing through every point of the surface, such that the curvature of a normal section taken in these directions vanishes. We may therefore conceive the entire surface to be crossed by two series of curves, such that a tangent drawn to either of them at any point shall coincide with one of these directions. These curves Professor JELLETT has denominated *Curves of Flexure,* from the following properties:—

Any curve of flexure may be fixed without destroying the flexibility of the surface.

If an arc of a curve traced upon an inextensible surface be rendered fixed or rigid, the entire of the quadrilateral, formed by drawing the two curves of flexure through each extremity of the curve, becomes fixed or rigid also.

Professor JELLETT has also investigated the properties of partially inextensible surfaces, and of thin material laminæ whose extensibility is small, and in a note he has demonstrated the following theorem:—

If a closed oval surface be perfectly inextensible, it is also perfectly rigid.

A demonstration of one of Professor JELLETT's theorems will be found at the end of this paper.

Aug. 30, 1854. J. C. M.

The surface is now covered with a network of curvilinear triangles. The plane triangles which have the same angular points will form a polyhedron fulfilling the required conditions. By increasing the number of the curves in each series, and diminishing their distance, we may make the polyhedron approximate to the surface without limit. At the same time the polygons formed by the edges of the polyhedron will approximate to the three systems of intersecting curves.

2. *To find the measure of the "entire curvature" of a solid angle of the polyhedron, and of a finite portion of its surface.*

From the centre of a sphere whose radius is unity draw perpendiculars to the planes of the six sides forming the solid angle. These lines will meet the surface in six points on the same side of the centre, which being joined by arcs of great circles will form a hexagon on the surface of the sphere.

The area of this hexagon represents the entire curvature of the solid angle.

It is plain by spherical geometry that the angles of this hexagon are the supplements of the six plane angles which form the solid angle, and that the arcs forming the sides are the supplements of those subtended by the angles of the six edges formed by adjacent sides.

The area of the hexagon is equal to the excess of the sum of its angles above eight right angles, or to the defect of the sum of the six plane angles from four right angles, which is the same thing. Since these angles are invariable, the bending of the polyhedron cannot alter the measure of curvature of each of its solid angles.

If perpendiculars be drawn to the sides of the polyhedron which contain other solid angles, additional points on the sphere will be found, and if these be joined by arcs of great circles, a network of hexagons will be formed on the sphere, each of which corresponds to a solid angle of the polyhedron and represents its "entire curvature."

The entire curvature of any assigned portion of the polyhedron is the sum of the entire curvatures of the solid angles it contains. It is therefore represented by a polygon on the sphere, which is composed of all the hexagons corresponding to its solid angles.

If a polygon composed of the edges of the polyhedron be taken as the boundary of the assigned portion, the sum of its exterior angles will be the same as the sum of the exterior angles of the polygon on the sphere; but the area of a spherical polygon is equal to the defect of the sum of its exterior angles from four right angles, and this is the measure of entire curvature.

Therefore the entire curvature of the portion of the polyhedron enclosed by the polygon is equal to the defect of the sum of its exterior angles from four right angles.

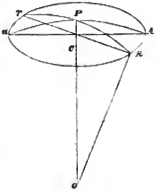
Since the entire curvature of each solid angle is unaltered by bending, that of a finite portion of the surface must be also invariable.

3. *On the "Conic of Contact," and its use in determining the curvature of normal sections of a surface.*

Suppose the plane of one of the triangular facets of the polyhedron to be produced till it cuts the surface. The form of the curve of intersection will depend on the nature of the surface, and when the size of the triangle is indefinitely diminished, it will approximate to the form of a conic section.

For we may suppose a surface of the second order constructed so as to have a contact of the second order with the given surface at a point within the angular points of the triangle. The curve of intersection with this surface will be the conic section to which the other curve of intersection approaches. This curve will be henceforth called the "Conic of Contact," for want of a better name.

To find the radius of curvature of a normal section of the surface.



Let ARa be the conic of contact, C its centre, and CP perpendicular to its plane. rPR a normal section, and O its centre of curvature, then

$$PO = \frac{1}{2}\frac{PR^2}{CP}$$

$$= \frac{1}{2}\frac{CR^2}{CP} \text{ in the limit, when } CR \text{ and } PR \text{ coincide,}$$

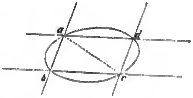
$$= \frac{1}{8}\frac{rR^2}{CP},$$

or calling CP the "sagitta," we have this theorem:

"The radius of curvature of a normal section is equal to the square of the corresponding diameter of the conic of contact divided by eight times the sagitta."

4. *To inscribe a polyhedron in a given surface, all whose sides shall be plane quadrilaterals, and all whose solid angles shall be tetrahedral.*

Suppose the three systems of curves drawn as described in sect. (1), then each of the quadrilaterals formed by the intersection of the first and second systems is divided into two triangles by the third system. If the planes of these two triangles coincide, they form a plane quadrilateral, and if every such pair of triangles coincide, the polyhedron will satisfy the required condition.



Let abc be one of these triangles, and acd the other, which is to be in the same plane with abc. Then if the plane of abc be produced to meet the surface in the conic of contact, the curve will pass through abc and d. Hence $abcd$ must be a quadrilateral inscribed in the conic of contact.

But since ab and dc belong to the same system of curves, they will be ultimately parallel when the size of the facets is diminished, and for a similar reason, ad and bc will be ultimately parallel. Hence $abcd$ will become a parallelogram, but the sides of a parallelogram inscribed in a conic are parallel to conjugate diameters.

Therefore the directions of two curves of the first and second system at their point of intersection must be parallel to two conjugate diameters of the conic of contact at that point in order that such a polyhedron may be inscribed.

Systems of curves intersecting in this manner will be referred to as "conjugate systems."

5. *On the elementary conditions of the applicability of two surfaces.*

It is evident, that if one surface is capable of being applied to another by bending, every point, line, or angle in the first has its corresponding point, line, or angle in the second.

If the transformation of the surface be effected without the extension or contraction of any part, no line drawn on the surface can experience any change in its length, and if this condition be fulfilled, there can be no extension or contraction.

Therefore the condition of bending is, that if any line whatever be drawn on the first surface, the corresponding curve on the second surface is equal to it in length. All other conditions of bending may be deduced from this.

6. *If two curves on the first surface intersect, the corresponding curves on the second surface intersect at the same angle.*

On the first surface draw any curve, so as to form a triangle with the curves already drawn, and let the sides of this triangle be indefinitely diminished, by making the new curve approach to the intersection of the former curves. Let the same thing be done on the second surface. We shall then have two corresponding triangles whose sides are equal each to each, by (5), and since their sides are indefinitely small, we may regard them as straight lines. Therefore by Euclid I. 8, the angle of the first triangle formed by the intersection of the two curves is equal to the corresponding angle of the second.

7. *At any given point of the first surface, two directions can be found, which are conjugate to each other with respect to the conic of contact at that point, and continue to be conjugate to each other when the first surface is transformed into the second.*

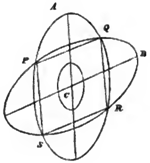
For let the first surface be transferred, without changing its form, to a position such that the given point coincides with the corresponding point of the second surface, and the normal to the first surface coincides with that of the second at the same point. Then let the first surface be turned about the normal as an axis till the tangent of any line through the point coincides with the tangent of the corresponding line in the second surface.

Then by (6) any pair of corresponding lines passing through the point will have a common tangent, and will therefore coincide in direction at that point.

If we now draw the conics of contact belonging to each surface we shall have two conics with the same centre, and the problem is to determine a pair of conjugate diameters of the first which coincide with a pair of conjugate diameters of the second. The analytical solution gives two directions, real, coincident, or impossible, for the diameters required.

In our investigations we can be concerned only with the case in which these directions are real.

When the conics intersect in four points, P, Q, R, S, $PQRS$ is a parallelogram inscribed in both conics, and the axes CA, CB, parallel to the sides, are conjugate in both conics.



If the conics do not intersect, describe, through any point P of the second conic, a conic similar to and concentric with the first. If the conics intersect in four points, we must proceed as before; if they touch in two points, the diameter through those points and its conjugate must be taken. If they intersect in two points only, then the problem is impossible; and if they coincide altogether, the conics are similar and similarly situated, and the problem is indeterminate.

8. *Two surfaces being given as before, one pair of conjugate systems of curves may be drawn on the first surface, which shall correspond to a pair of conjugate systems on the second surface.*

By article (7) we may find at every point of the first surface two directions conjugate to one another, corresponding to two conjugate directions on the second surface. These directions indicate the directions of the two systems of curves which pass through that point.

Knowing the direction which every curve of each system must have at every point of its course, the systems of curves may be either drawn by some direct geometrical method, or constructed from their equations, which may be found by solving their differential equations.

Two systems of curves being drawn on the first surface, the corresponding systems may be drawn on the second surface. These systems being conjugate to each other, fulfil the condition of Art. (4), and may therefore be made the means of constructing a polyhedron with quadrilateral facets, by the bending of which the transformation may be effected.

These systems of curves will be referred to as the "first and second systems of Lines of Bending."

9. *General considerations applicable to Lines of Bending.*

It has been shewn that when two forms of a surface are given, one of which may be transformed into the other by bending, the nature of the lines of bending is completely determined. Supposing the problem reduced to its analytical expression, the equations of these curves would appear under the form of double solutions of differential equations of the first order and second degree, each of which would involve one arbitrary quantity, by the variation of which we should pass from one curve to another of the same system.

Hence the position of any curve of either system depends on the value assumed for the arbitrary constant; to distinguish the systems, let us call one the first system, and the other the second, and let all quantities relating to the second system be denoted by accented letters.

Let the arbitrary constants introduced by integration be u for the first system, and u' for the second.

Then the value of u will determine the position of a curve of the first system, and that of u' a curve of the second system, and therefore u and u' will suffice to determine the point of intersection of these two curves.

Hence we may conceive the position of any point on the surface to be determined by the values of u and u' for the curves of the two systems which intersect at that point.

By taking into account the equation to the surface, we may suppose x, y, and z the co-ordinates of any point, to be determined as functions of the two variables u and u'. This being done, we shall have materials for calculating everything connected with the surface, and its lines of bending. But before entering on such calculations, let us examine the principal properties of these lines which we must take into account.

Suppose a series of values to be given to u and u', and the corresponding curves to be drawn on the surface.

The surface will then be covered with a system of quadrilaterals, the size of which may be diminished indefinitely by interpolating values of u and u' between those already assumed; and in the limit each quadrilateral may be regarded as a parallelogram coinciding with a facet of the inscribed polyhedron.

The *length*, the *breadth*, and the *angle* of these parallelograms will vary at different parts of the surface, and will therefore depend on the values of u and u'.

The *curvature* of a line drawn on a surface may be investigated by considering the curvature of two other lines depending on it.

The first is the projection of the line on a tangent plane to the surface at a given point in the line. The curvature of the projection at the point of contact may be called the *tangential curvature* of the line on the surface. It has also been called the *geodesic* curvature, because it is the measure of its deviation from a geodesic or shortest line on the surface.

The other projection necessary to define the curvature of a line on the surface is on a plane passing through the tangent to the curve and the normal to the surface at the point of contact. The curvature of this projection at that point may be called the *normal curvature* of the line on the surface.

It is easy to show that this normal curvature is the same as the curvature of a normal section of the surface passing through a tangent to the curve at the same point.

10. *General considerations applicable to the inscribed polyhedron.*

When two series of lines of bending belonging to the first and second systems have been described on the surface, we may proceed, as in Art. (1), to describe a third series of curves so as to pass through all their intersections and form the diagonals of the quadrilaterals formed by the first pair of systems.

Plane triangles may then be constituted within the surface, having these points of intersection for angles, and the size of the facets of this polyhedron may be diminished indefinitely by increasing the number of curves in each series.

But by Art. (8) the first and second systems of lines of bending are conjugate to each other, and therefore by Art. (4) the polygon just constructed will have every pair of triangular facets in the same plane, and may therefore be considered as a polyhedron with plane quadrilateral facets all whose solid angles are formed by four of these facets meeting in a point.

When the number of curves in each system is increased and their distance diminished indefinitely, the plane facets of the polyhedron will ultimately coincide with the curved surface, and the polygons formed by the successive edges between the facets, will coincide with the lines of bending.

These quadrilaterals may then be considered as parallelograms, the length of which is determined by the portion of a curve of the second system intercepted between two curves of the first, while the breadth is the distance of two curves of the second system measured along a curve of the first. The expressions for these quantities will be given when we come to the calculation of our results along with the other particulars which we only specify at present.

The angle of the sides of these parallelograms will be ultimately the same as the angle of intersection of the first and second systems, which we may call ϕ; but if we suppose the dimensions of the facets to be small quantities of the first order, the angles of the four facets which meet in a point will differ from the angle of intersection of the curves at that point by small angles of the first order depending on the tangential curvature of the lines of bending. The sum of these four angles will differ from four right angles by a small angle of the second order, the circular measure of which expresses the entire curvature of the solid angle as in Art. (2).

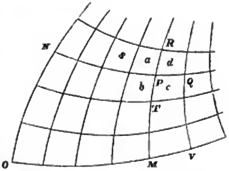
The angle of inclination of two adjacent facets will depend on the normal curvature of the lines of bending, and will be that of the projection of two consecutive sides of the polygon of one system on a plane perpendicular to a side of the other system.

11. *Explanation of the Notation to be employed in calculation.*

Suppose each system of lines of bending to be determined by an equation containing one arbitrary parameter.

Let this parameter be u for the first system, and u' for the second.

Let two curves, one from each system, be selected as curves of reference, and let their parameters be u_0 and u'_0.



Let ON and OM in the figure represent these two curves.

Let PM be any curve of the first system whose parameter is u, and PN any curve of the second whose parameter is u', then their intersection P may be defined as the point (u, u'), and all quantities referring to the point P may be expressed as functions of u and u'.

Let PN, the length of a curve of the second system (u') from $N, (u_0)$ to $P, (u)$, be expressed by s, and PM the length of the curve (u) from (u'_0) to (u'), by s', then s and s' will be functions of u and u'.

Let $(u + \delta u)$ be the parameter of the curve QV of the first system consecutive to PM. Then the length of PQ, the part of the curve of the second system intercepted between the curves (u) and $(u + \delta u)$, will be

$$\frac{ds}{du}\,\delta u.$$

Similarly PR may be expressed by

$$\frac{ds'}{du'}\,\delta u'.$$

These values of PQ and PR will be the ultimate values of the length and breadth of a quadrilateral facet.

The angle between these lines will be ultimately equal to ϕ, the angle of intersection of the system; but when the values of δu and $\delta u'$ are considered as finite though small, the angles a, b, c, d of the facets which form a solid angle will depend on the tangential curvature of the two systems of lines.

Let r be the tangential curvature of a curve of the first system at the given point measured in the direction in which u increases, and let r', that of the second system, be measured in the direction in which u' increases.

Then we shall have for the values of the four plane angles which meet at P,

$$\begin{aligned}
a &= \pi - \phi + \frac{1}{2r}\frac{ds'}{du'}\,\delta u' - \frac{1}{2r'}\frac{ds}{du}\,\delta u,\\
b &= \phi \qquad\; + \frac{1}{2r}\frac{ds'}{du'}\,\delta u' + \frac{1}{2r'}\frac{ds}{du}\,\delta u,\\
c &= \pi - \phi - \frac{1}{2r}\frac{ds'}{du'}\,\delta u' + \frac{1}{2r'}\frac{ds}{du}\,\delta u,\\
d &= \phi \qquad\; - \frac{1}{2r}\frac{ds'}{du'}\,\delta u' - \frac{1}{2r'}\frac{ds}{du}\,\delta u.
\end{aligned}$$

These values are correct as far as the first order of small quantities. Those corrections which depend on the curvature of the surface are of the second order.

Let ρ be the normal curvature of a curve of the first system, and ρ' that of a curve of the second, then the inclination l of the plane facets a and b, separated by a curve of the second system, will be

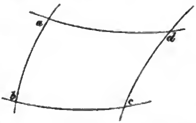
$$l = \frac{1}{\rho \sin\phi}\,\frac{ds'}{du'}\,du',$$

as far as the first order of small angles, and the inclination l' of b and c will be

$$l' = \frac{1}{\rho' \sin\phi}\,\frac{ds}{du}\,\delta u$$

to the same order of exactness.

12. *On the corresponding polygon on the surface of the sphere of reference.*



By the method described in Art. (2) we may find a point on the sphere corresponding to each facet of the polyhedron.

In the annexed figure, let a, b, c, d be the points on the sphere corresponding to the four facets which meet at the solid angle P. Then the area of the spherical quadrilateral a, b, c, d will be the measure of the entire curvature of the solid angle P.

This area is measured by the defect of the sum of the exterior angles from four right angles; but these exterior angles are equal to the four angles a, b, c, d, which form the solid angle P, therefore the entire curvature is measured by

$$k = 2\pi - (a + b + c + d).$$

Since a, b, c, d are invariable, it is evident, as in Art. (2), that the entire curvature at P is not altered by bending.

By the last article it appears that when the facets are small the angles b and d are approximately equal to ϕ, and a and c to $(\pi-\phi)$, and since the sides of the quadrilateral on the sphere are small, we may regard it as approximately a plane parallelogram whose angle $bad = \phi$.

The sides of this parallelogram will be l and l', the supplements of the angles of the edges of the polyhedron, and we may therefore express its area as a plane parallelogram

$$k = l\,l' \sin\phi.$$

By the expressions for l and l' in the last article, we find

$$k = \frac{1}{\rho\rho' \sin\phi}\,\frac{ds}{du}\,\frac{ds'}{du'}\,\delta u\,\delta u'$$

for the entire curvature of one solid angle.

Since the whole number of solid angles is equal to the whole number of facets, we may suppose a quarter of each of the facets of which it is composed to be assigned to each solid angle. The area of these will be the same as that of one whole facet, namely,

$$\sin\phi\,\frac{ds}{du}\,\frac{ds'}{du'}\,\delta u\,\delta u';$$

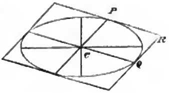
therefore dividing the expression for k by this quantity, we find for the value of the specific curvature at P

$$p = \frac{1}{\rho\rho' \sin^2\phi};$$

which gives the specific curvature in terms of the normal curvatures of the lines of bending and their angle of intersection.

13. *Farther reduction of this expression by means of the "Conic of Contact," as defined in Art.* (3).

Let a and b be the semiaxes of the conic of contact, and h the sagitta or perpendicular to its plane from the centre to the surface.



Let CP, CQ be semidiameters parallel to the lines of bending of the first and second systems, and therefore conjugate to each other.

By (Art. 3,) $\rho = \dfrac{1}{2}\,\dfrac{CP^2}{h}$,

and $\rho' = \dfrac{1}{2}\,\dfrac{CQ^2}{h}$;

and the expression for p in Art. (12), becomes

$$p = \frac{4h^2}{(CP.CQ \sin\phi)^2}.$$

But $CP.CQ \sin\phi$ is the area of the parallelogram $CPRQ$, which is one quarter of the circumscribed parallelogram, and therefore by a well-known theorem

$$CP.CQ \sin\phi = ab,$$

and the expression for p becomes

$$p = \frac{4h^2}{a^2b^2};$$

or if the area of the circumscribing parallelogram be called A,

$$p = \frac{16h^2}{A^2}.$$

The principal radii of curvature of the surface are parallel to the axes of the conic of contact. Let R and R' denote these radii, then

$$R = \frac{1}{2}\frac{a^2}{h} \text{ and } R' = \frac{1}{2}\frac{b^2}{h};$$

and therefore substituting in the expression for p,

$$p = \frac{1}{RR'};$$

or the specific curvature is the reciprocal of the product of the principal radii of curvature.

This remarkable expression was introduced by GAUSS in the memoir referred to in a former part of this paper. His method of investigation, though not so elementary, is more direct than that here given, and will show how this result can be obtained without reference to the geometrical methods necessary to a more extended inquiry into the *modes* of bending.

14. *On the variation of normal curvature of the lines of bending as we pass from one point of the surface to another.*

We have determined the relation between the normal curvatures of the lines of bending of the two systems at their points of intersection; we have now to find the variation of normal curvature when we pass from one line of the first system to another, along a line of the second.

In analytical language we have to find the value of

$$\frac{d}{du}\left(\frac{1}{\rho}\right).$$

Referring to the figure in Art. (11), we shall see that this may be done if we can determine the difference between the angle of inclination of the facets a and b, and that of c and d: for the angle l between a and b is

$$l = \frac{1}{\rho \sin \phi}\frac{ds'}{du'}\delta u',$$

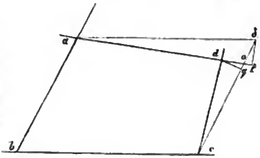
and therefore the difference between the angle of a and b and that of c and d is

$$\delta l = \frac{dl}{du}\delta u = \frac{d}{du}\left(\frac{1}{\rho \sin \phi}\frac{ds'}{du'}\right)\delta u \delta u';$$

whence the differential of ρ with respect to u may be found.

We must therefore find δl, and this is done by means of the quadrilateral on the sphere described in Art. (12).

15. *To find the values of δl and $\delta l'$.*



In the annexed figure let *abcd* represent the small quadrilateral on the surface of the sphere. The exterior angles a, b, c, d are equal to those of the four facets which meet at the point P of the surface, and the sides represent the angles which the planes of those facets make with each other; so that

$$ab = l, \quad bc = l', \quad cd = l + \delta l, \quad da = l' + \delta l',$$

and the problem is to determine δl and $\delta l'$ in terms of the sides l and l' and the angles *abcd*.

On the sides *ba*, *bc* complete the parallelogram *abcd*.

Produce *ad* to *p*, so that $ap = a\delta$. Join δp.

Make $Cq = cd$ and join dq,

$$\begin{aligned} \text{then} \quad \delta l &= cd - ab, \\ &= cq - c\delta, \\ &= -(qo + o\delta). \end{aligned}$$

$$\begin{aligned} \text{Now} \quad qo &= qd \tan qdo \\ &= cd \sin qcd \cot qod, \end{aligned}$$

but $cd = l$ nearly, $\sin qcd = qcd = (c + b + \pi)$ and $qod = \phi$;

$$\therefore \; qo = l(c + b - \pi) \cot \phi.$$

$$\begin{aligned} \text{Also} \quad o\delta &= \frac{p\delta}{\sin \delta op} \\ &= a\delta(\delta ap) \frac{1}{\sin \phi} \\ &= l'(a + b - \pi) \frac{1}{\sin \phi}. \end{aligned}$$

Substituting the values of a, b, c, d from Art. (11),

$$\begin{aligned} \delta l &= -(qo + o\delta) \\ &= -l \frac{1}{r'} \frac{ds}{du} \cot \phi \delta u - l' \frac{1}{r} \frac{ds'}{du'} \frac{1}{\sin \phi} \delta u'. \end{aligned}$$

Finally, substituting the values of l, l', and δl from Art. (14),

$$\frac{d}{du} \left(\frac{1}{\rho \sin \phi} \frac{ds'}{du'} \right) \delta u \delta u' = - \frac{\cot \phi}{\rho \sin \phi} \frac{ds'}{du'} \frac{1}{r'} \frac{ds}{du} \delta u \delta u' - \frac{1}{\rho' \sin^2 \phi} \frac{ds}{du} \frac{1}{r} \frac{ds'}{du'} \delta u \delta u';$$

which may be put under the more convenient form

$$\frac{d}{du} (\log \rho) = \frac{d}{du} \log \left(\frac{1}{\sin \phi} \frac{ds'}{du'} \right) + \frac{1}{r'} \frac{ds}{du} \cot \phi + \frac{\rho}{\rho'} \frac{1}{r} \frac{ds}{du} \frac{1}{\sin \phi};$$

and from the value of $\delta l'$ we may similarly obtain

$$\frac{d}{du'} (\log \rho') = \frac{d}{du'} \log \left(\frac{1}{\sin \phi} \frac{ds}{du} \right) + \frac{1}{r} \frac{ds'}{du'} \cot \phi + \frac{\rho'}{\rho} \frac{1}{r'} \frac{ds'}{du'} \frac{1}{\sin \phi}.$$

We may simplify these equations by putting p for the specific curvature of the surface, and q for the ratio $\frac{\rho}{\rho'}$, which is the only quantity altered by bending.

We have then

$$p = \frac{1}{\rho\rho' \sin^2 \phi}, \quad \text{and } q = \frac{\rho}{\rho'},$$

$$\text{whence} \quad \rho^2 = q\,\frac{1}{p \sin^2 \phi}, \qquad \rho'^2 = \frac{1}{q}\,\frac{1}{p \sin^2 \phi},$$

and the equations become

$$\frac{d}{du}(\log q) = \frac{d}{du}\log\left(p \left.\overline{\frac{ds'}{du'}}\right|^2\right) + \frac{2}{r'}\,\frac{ds}{du}\cot\phi + \frac{2}{r}\,\frac{ds}{du}\,\frac{1}{\sin\phi}\,q,$$

$$\frac{d}{du'}(\log q) = -\frac{d}{du'}\log\left(p \left.\overline{\frac{ds}{du}}\right|^2\right) - \frac{2}{r}\,\frac{ds'}{du'}\cot\phi - \frac{2}{r'}\,\frac{ds'}{du'}\,\frac{1}{\sin\phi}\,\frac{1}{q}.$$

In this way we may reduce the problem of bending a surface to the consideration of one variable q, by means of the lines of bending.

16. *To obtain the condition of Instantaneous lines of bending.*

We have now obtained the values of the differential coefficients of q with respect to each of the variables u, u'.

From the equation

$$\frac{d^2}{du\,du'}(\log q) = \frac{d^2}{du'du}(\log q),$$

we might find an equation which would give certain conditions of lines of bending. These conditions however would be equivalent to those which we have already assumed when we drew the systems of lines so as to be conjugate to each other.

To find the true conditions of bending we must suppose the form of the surface to vary continuously, so as to depend on some variable t which we may call the time.

Of the different quantities which enter into our equations, none are changed by the operation of bending except q, so that in differentiating with respect to t all the rest may be considered constant, q being the only variable.

Differentiating the equations of last article with respect to t, we obtain

$$\frac{d^2}{du\,dt}(\log q) = \frac{2}{r}\,\frac{ds}{du}\,\frac{1}{\sin\phi}\,q\,\frac{d}{dt}(\log q),$$

$$\frac{d^2}{du'dt}(\log q) = \frac{2}{r'}\,\frac{ds'}{du'}\,\frac{1}{\sin\phi}\,\frac{1}{q}\,\frac{d}{dt}(\log q).$$

$$\text{Whence} \quad \frac{d^3}{du\,du'dt}(\log q) = \left\{\frac{d}{du'}\left(\frac{2}{r}\,\frac{ds}{du}\,\frac{1}{\sin\phi}\right) + \frac{2}{r}\,\frac{ds}{du}\,\frac{1}{\sin\phi}\,\frac{d}{du'}(\log q)\right\} q\,\frac{d}{dt}(\log q)$$

$$+ \frac{2}{r}\,\frac{ds}{du}\,\frac{1}{\sin\phi}\,q\,\frac{d}{du'dt}(\log q),$$

and $$\frac{d^3}{du\,du'\,dt}(\log q) = \left\{\frac{d}{du}\left(\frac{2}{r'}\frac{ds'}{du'}\frac{1}{\sin\phi}\right) - \frac{2}{r'}\frac{ds'}{du'}\frac{1}{\sin\phi}\frac{d}{du}\log q\right\}\frac{1}{q}\frac{d}{dt}(\log q)$$
$$+\frac{2}{r'}\frac{ds'}{du'}\frac{1}{\sin\phi}\frac{1}{q}\frac{d}{du\,dt}(\log q),$$

two independent values of the same quantity, whence the required conditions may be obtained.

Substituting in these equations the values of those quantities which occur in the original equations, we obtain

$$q\frac{1}{r}\frac{ds}{du}\left\{\frac{d}{du'}\log\left(pr\frac{ds}{du}\sin\phi\right) + \frac{2}{r}\frac{ds'}{du'}\cot\phi\right\}$$
$$=\frac{1}{q}\frac{1}{r'}\frac{ds'}{du}\left\{\frac{d}{du'}\log\left(pr'\frac{ds'}{du'}\sin\phi\right) + \frac{2}{r'}\frac{ds}{du}\cot\phi\right\},$$

which is the condition which must hold at every instant during the process of bending for the lines about which the bending takes place at that instant. When the bending is such that the position of the lines of bending on the surface alters at every instant, this is the only condition which is required. It is therefore called the condition of Instantaneous lines of bending.

17. *To find the condition of Permanent lines of bending.*

Since q changes with the time, the equation of last article will not be satisfied for any finite time unless both sides are separately equal to zero. In that case we have the two conditions

$$\left.\begin{aligned}\frac{d}{du'}\log\left(pr\frac{ds}{du}\sin\phi\right) + \frac{2}{r}\frac{ds'}{du'}\cot\phi = 0,\\ \text{or}\quad \frac{1}{r}\frac{ds}{du} = 0.\end{aligned}\right\}\quad (1)$$

$$\left.\begin{aligned}\frac{d}{du}\log\left(pr'\frac{ds'}{du'}\sin\phi\right) + \frac{2}{r'}\frac{ds}{du}\cot\phi = 0,\\ \text{or}\quad \frac{1}{r'}\frac{ds'}{du'} = 0.\end{aligned}\right\}\quad (2)$$

If the lines of bending satisfy these conditions, a finite amount of bending may take place without changing the position of the system on the surface. Such lines are therefore called Permanent lines of bending.

The only case in which the phenomena of bending may be exhibited by means of the polyhedron with quadrilateral facets is that in which permanent lines of bending are chosen as the boundaries of the facets. In all other cases the bending takes place about an instantaneous system of lines which is continually in motion with respect to the surface, so that the nature of the polyhedron would need to be altered at every instant.

We are now able to determine whether any system of lines drawn on a given surface is a system of instantaneous or permanent lines of bending.

We are also able, by the method of Article (8), to deduce from two consecutive forms of a surface, the lines of bending about which the transformation must have taken place.

If our analytical methods were sufficiently powerful, we might apply our results to the determination of such systems of lines on any known surface, but the necessary calculations even in the simplest cases are so complicated, that, even if useful results were obtained, they would be out of place in a paper of this kind, which is intended to afford the means of forming distinct conceptions rather than to exhibit the results of mathematical labour.

18. *On the application of the ordinary methods of analytical geometry to the consideration of lines of bending.*

It may be interesting to those who may hesitate to accept results derived from the consideration of a polyhedron, when applied to a curved surface, to inquire whether the same results may not be obtained by some independent method.

As the following method involves only those operations which are most familiar to the analyst, it will be sufficient to give the rough outline, which may be filled up at pleasure.

The proof of the invariability of the specific curvature may be taken from any of the memoirs above referred to, and its value in terms of the equation of the surface will be found in the memoir of GAUSS.

Let the equation to the surface be put under the form

$$z = f(xy),$$

then the value of the specific curvature is

$$p = \frac{\frac{d^2z}{dx^2}\frac{d^2z}{dy^2} - \overline{\frac{d^2z}{dx\,dy}}\Big|^2}{\sqrt{1 + \overline{\frac{dz}{dx}}\Big|^2 + \overline{\frac{dz}{dy}}\Big|^2}}.$$

The definition of conjugate systems of curves may be rendered independent of the reasoning formerly employed by the following modification.

Let a tangent plane move along any line of the first system, then if the line of ultimate intersection of this plane with itself be always a tangent to some line of the second system, the second system is said to be conjugate to the first.

It is easy to show that the first system is also conjugate to the second.

Let the system of curves be projected on the plane of xy, and at the point (x, y) let α be the angle which a projected curve of the first system makes with the axis of x, and β the angle which the projected curve of the second system which intersects it at that point makes with the same axis. Then the condition of the systems being conjugate will be found to be

$$\frac{d^2z}{dx^2}\cos\alpha\cos\beta + \frac{d^2z}{dx\,dy}\sin(\alpha+\beta) + \frac{d^2z}{dy^2}\sin\alpha\sin\beta = 0;$$

α and β being known as functions of x and y, we may determine the nature of the curves projected on the plane of xy.

Supposing the surface to touch that plane at the origin, the length and tangential curvature of the lines on the surface near the point of contact may be taken the same as those of their projections on the plane, and any change of form of the surface due to bending will not alter the form of the projected lines indefinitely near the point of contact. We may therefore consider z as the only variable altered by bending; but in order to apply our analysis with facility, we may assume

$$\frac{d^2z}{dx^2} = PQ\sin^2\alpha + PQ^{-1}\sin^2\beta,$$

$$\frac{d^2z}{dx\,dy} = -PQ\sin\alpha\cos\alpha - PQ^{-1}\sin\beta\cos\beta,$$

$$\frac{d^2z}{dy^2} = PQ\cos^2\alpha + PQ^{-1}\cos^2\beta.$$

It will be seen that these values satisfy the condition last given. Near the origin we have

$$p = \frac{d^2z}{dx^2}\,\frac{d^2z}{dy^2} - \left.\overline{\frac{d^2z}{dx\,dy}}\right|^2 = P^2\sin^2(\alpha-\beta),$$

$$\text{and}\quad q = Q^{-2}.$$

Differentiating these values of $\frac{d^2z}{dx^2}$, &c., we shall obtain two values of $\frac{d^3z}{dx^2dy}$ and of $\frac{d^3z}{dx\,dy^2}$, which being equated will give two equations of condition.

Now if s' be measured along a curve of the first system, and R be any function of x and y, then

$$\frac{dR}{ds'} = \frac{dR}{dx}\cos\alpha + \frac{dR}{dy}\sin\alpha,$$

$$\text{and}\quad \frac{dR}{du'} = \frac{dR}{ds'}\,\frac{ds'}{du'}.$$

We may also shew that
$$\frac{d\alpha}{ds'} = \frac{1}{r}.$$

$$\text{and that}\quad \cos\alpha\frac{d\alpha}{dy} - \sin\alpha\frac{d\alpha}{dx} = \frac{d}{ds}\log\left(\frac{ds'}{du'}\sin\phi\right).$$

By substituting these values in the equations thus obtained, they are reduced to the two equations given at the end of (Art. 15). This method of investigation introduces no difficulty except that of somewhat long equations, and is therefore satisfactory as supplementary to the geometrical method given at length.

As an example of the method given in page (446), we may apply it to the case of the surface whose equation is

$$\left(\frac{x}{c-z}\right)^2 + \left(\frac{y}{c+z}\right)^2 = \left(\frac{a}{c}\right)^2.$$

This surface may be generated by the motion of a straight line whose equation is of the form

$$x = a\cos t\left(1-\frac{z}{c}\right),\qquad y = a\sin t\left(1+\frac{z}{c}\right),$$

t being the variable, by the change of which we pass from one position of the line to another. This line always passes through the circle

$$z = 0, \qquad x^2 + y^2 = a^2,$$

and the straight lines $z = c, \quad x = 0,$

and $z = -c, \quad y = 0,$

which may therefore be taken as the directors of the surface.

Taking two consecutive positions of this line, in which the values of t are t_1, and $t_1 + \delta t$, we may find by the ordinary methods the equation to the shortest line between them, its length, and the co-ordinates of the point in which it intersects the first line.

Calling the length $\delta\zeta$,

$$\delta\zeta = \frac{ac}{\sqrt{a^2 + c^2}} \sin 2t_1 \delta t,$$

and the co-ordinates of the point of intersection are

$$x = 2a \cos^3 t, \qquad y = 2a \sin^3 t, \qquad z = -c \cos 2t.$$

The angle $\delta\theta$ between the consecutive lines is

$$\delta\theta = \frac{a}{\sqrt{a^2 + c^2}} \delta t,$$

The distance $\delta\sigma$ between consecutive shortest lines is

$$\delta\sigma = \frac{3a + 2c}{\sqrt{a^2 + c^2}} \sin 2t \, \delta t,$$

and the angle $\delta\phi$ between these latter lines is

$$\delta\phi = \frac{c}{\sqrt{a^2 + c^2}} \delta t.$$

Hence if we suppose ζ, θ, σ, ϕ, and t to vanish together, we shall have by integration

$$\zeta = \frac{ac}{2\sqrt{a^2 + c^2}} (1 - \cos 2t),$$

$$\theta = \frac{a}{\sqrt{a^2 + c^2}} t,$$

$$\sigma = \frac{3a + 2c}{2\sqrt{a^2 + c^2}} (1 - \cos 2t),$$

$$\phi = \frac{c}{\sqrt{a^2 + c^2}} t.$$

By bending the surface about its generating lines we alter the value of ϕ in any manner without changing ζ, θ, or σ. For instance, making $\phi = 0$, all the generating lines become parallel to the same plane. Let this plane be that of xy, then ζ is the distance of a generating line from that plane. The projections of the generating lines on the plane of xy will, by their ultimate intersections, form a curve, the length of which is measured by σ, and the angle which its tangent makes with the axis of x by θ, θ and σ being connected by the equation

$$\sigma = \frac{3a + 2c}{2\sqrt{a^2 + c^2}} \left(1 - \cos \frac{2\sqrt{a^2 + c^2}}{a} \theta\right),$$

which shows the curve to be an epicycloid.

The generating lines of the surface when bent into this form are therefore tangents to a cylindrical surface on an epicycloidal base, touching that surface along a curve which is always equally inclined to the plane of the base, the tangents themselves being drawn parallel to the base.

We may now consider the bending of the surface of revolution

$$\sqrt[3]{x^2+y^2} + z^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Putting $r = \sqrt{x^2+y^2}$, then the equation of the generating line is

$$r^{\frac{2}{3}} + z^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

This is the well-known hypocycloid of four cusps.

Let s be the length of the curve measured from the cusp in the axis of z, then,

$$s = \frac{3}{2} c^{\frac{1}{3}} r^{\frac{2}{3}},$$

$$\text{wherefore,} \quad r = \left(\frac{2}{3}\right)^{\frac{3}{2}} c^{-\frac{1}{2}} s^{\frac{3}{2}}.$$

Let θ be the angle which the plane of any generating line makes with that of xz, then s and θ determine the position of any point on the surface. The length and breadth of an element of the surface will be δs and $r\delta\theta$.

Now let the surface be bent in the manner formerly described, so that θ becomes θ', and r, r', when

$$\theta' = \mu\theta \text{ and } r' = \frac{1}{\mu} r,$$

$$\text{then} \quad r' = \left(\frac{2}{3}\right)^{\frac{3}{2}} c^{-\frac{1}{2}} \mu^{-1} s^{\frac{3}{2}}$$

$$= \left(\frac{2}{3}\right)^{\frac{3}{2}} c'^{-\frac{1}{2}} s^{\frac{3}{2}},$$

$$\text{provided} \quad c' = \mu^2 c.$$

The equation between r' and s being of the same form as that between r and s shows that the surface when bent is similar to the original surface, its dimensions being multiplied by μ^2.

This, however, is true only for one half of the surface when bent. The other half is precisely symmetrical, but belongs to a surface which is not continuous with the first.

The surface in its original form is divided by the plane of xy into two parts which meet in that plane, forming a kind of cuspidal edge of a circular form which limits the possible value of s and r.

After being bent, the surface still consists of the same two parts, but the edge in which they meet is no longer of the cuspidal form, but has a finite angle $= 2\cos^{-1}\frac{1}{\mu}$, and the two sheets of the surface become parts of two different surfaces which meet but are not continuous.

J. C. MAXWELL.

NOTE.

As an example of the application of the more general theory of "lines of bending," let us consider the problem which has been already solved by Professor **JELLETT**.

To determine the conditions under which one portion of a surface may be rendered rigid, while the remainder is flexible.

Suppose the lines of bending to be traced on the surface, and the corresponding polyhedron to be formed, as in (9) and (10), then if the angle of one of the four edges which meet at any solid angle of the polyhedron be altered by bending, those of the other three must be also altered. These edges terminate in other solid angles, the forms of which will also be changed, and therefore the effect of the alteration of one angle of the polyhedron will be communicated to every other angle within the system of lines of bending which defines the form of the polyhedron.

If any portion of the surface remains unaltered it must lie beyond the limits of the system of lines of bending. We must therefore investigate the conditions of such a system being bounded.

The boundary of any system of lines on a surface is the curve formed by the ultimate intersections of those lines, and therefore at any given point coincides in direction with the curve of the system which passes through that point. In this case there are two systems of lines of bending, which are necessarily coincident in extent, and must therefore have the same boundary. At any point of this boundary therefore the directions of the lines of bending of the first and second systems are coincident.

But, by (7), these two directions must be "conjugate" to each other, that is, must correspond to conjugate diameters of the "Conic of Contact." Now the only case in which conjugate diameters of a conic can coincide, is when the conic is an hyperbola, and both diameters coincide with one of the asymptotes; therefore the boundary of the system of lines of bending must be a curve at every point of which the conic of contact is an hyperbola, one of whose asymptotes lies in the direction of the curve. The radius of "normal curvature" must therefore by (3) be infinite at every point of the curve. This is the geometrical property of what Professor **JELLETT** calls a "Curve of Flexure," so that we may express the result as follows:

If one portion of a surface be fixed, while the remainder is bent, the boundary of the fixed portion is a curve of flexure.

This theorem includes those given at p. (453), relative to a fixed curve on a surface, for in a surface whose curvatures are of the same sign, there can be no "curves of flexure," and in a developable surface, they are the rectilinear sections. Although the cuspidal edge, or *arête de rebroussement*, satisfies the analytical condition of a curve of flexure, yet, since its form determines that of the whole surface, it cannot remain fixed while the form of the surface is changed.

In concavo-convex surfaces, the curves of flexure must either have tangential curvature or be straight lines. Now if we put $\phi = 0$ in the equations of Art. (17), we find that the lines of bending of both systems have no tangential curvature at the point where they touch the curve of flexure. They must therefore lie entirely on the convex side of that curve, and therefore

If a curve of flexure be fixed, the surface on the concave side of the curve is not flexible.

I have not yet been able to determine whether the surface is inflexible on the convex side of the curve. It certainly is so in some cases which I have been able to work out, but I have no general proof.

When a surface has one or more rectilinear sections, the portions of the surface between them may revolve as rigid bodies round those lines as axes in any manner, but no other motion is possible. The case in which the rectilinear sections form an infinite series has been discussed in Sect. (I.).

XX. *On Self-repeating Series.* *By* HENRY WARBURTON, M.A.

[Read *May* 15, 1854.]

SECT. I. *Introduction.*

1. ON reading some years ago the well-known Memoir of Laplace, in which he institutes a comparison between the general terms of the exponential functions $\frac{t}{e^t-1}$ and $\frac{1}{1+e^t}$, I noticed a circumstance not mentioned by the distinguished Author, and, as I conceive, overlooked by him.

The general term of $\frac{1}{1+e^t}$ is $\frac{1}{1.2\ldots x}[0^x - 1^x + 2^x - 3^x + \&c.]$; and we are to substitute for the infinite series which this term contains, its generating fraction. On computing the generating fractions of the series $(1^x - 2^x t + 3^x t^2 - 4^x t^3 + \&c.)$, when the integer values 1, 2, 3, &c. were successively assigned to x, I found that the numerators of these fractions, so far as my computations extended, were all *recurrent* functions of t. Thus, for instance,

$$\begin{aligned} \text{when } x = 1, \text{ the numerator } &= 1, \\ = 2, \ldots\ldots\ldots\ldots\ldots &= 1 - t, \\ = 3, \ldots\ldots\ldots\ldots\ldots &= 1 - 4t + t^2, \\ = 4, \ldots\ldots\ldots\ldots\ldots &= 1 - 11t + 11t^2 - t^3, \\ = 5, \ldots\ldots\ldots\ldots\ldots &= 1 - 26t + 66t^2 - 26t^3 + t^4. \end{aligned}$$

And so on; and if in the infinite series, the terms, retaining the same numerical values, were made all positive, then the terms of the several numerators, retaining the same numerical values, became all positive; and it therefore appeared that the generating fraction of the infinite series $(1^x + 2^x t + 3^x t^2 + \&c.)$ had a *recurrent* numerator.

Dr Brinkley and Sir John Herschel, who have followed up with great success the researches of Laplace, have considered with particular attention the series $(1^x - 2^x t + 3^x t^2 - \&c.)$; but they make no allusion to the fact of the fractions which generate the different series of this form, when different consecutive integer values are given to x, having recurrent numerators

I was therefore led to investigate the question, what are the conditions which the denominator of the generating fraction, and the terms of the recurring series generated must satisfy, in order that the numerator of the generating fraction may be a *recurrent* function of t.

It is the result of that investigation which I now communicate to the Society.

SECT. II. *Self-repeating Series defined.*

2. I call a series self-repeating, if when extended without limit in opposite directions, it admits of separation into two similar arms, each beginning with a finite term; either no zero-term, or one or more zero-terms, lying between these two equal finite terms. The right arm extends from left to right, and increases as the positive index of the series increases; the left arm extends from right to left, and increases as the negative index increases. One arm repeats, and contains, arranged in reverse consecutive order, the terms of the other arm; either all, or none of, the terms so repeated having their signs changed.

3. Let C_0 be the finite term with which the right arm begins; and let C_x be the term, in that arm, corresponding to the positive index x; and let z denote the number of the zero-terms that lie between the two arms. Then $C_{-(z+1)}$ is the finite term with which the left arm begins.

If γ is some integer number or other, constantly odd, or constantly even, for the same series, then between the term C_x in the right arm, and the term $C_{-(z+1+x)}$ in the left arm, there will subsist the following equation:

$$C_x = (-1)^\gamma C_{-(z+1+x)}, \quad . \quad . \quad . \quad . \quad (1)$$

which admits also of the form

$$C_{x-(z+1)} = (-1) C_{-x}. \quad . \quad . \quad . \quad . \quad (1^*)$$

We may call each of these equivalent equations the ***Equation of Repetition***, or the ***Repeating Equation*** of the series; and $(-1)^\gamma$ its sign of repetition.

4. In the sequel, I shall principally have occasion to treat of series, which, besides being self-repeating, are also recurring series.

SECT. III. *Certain elementary properties of Recurring Series, in general.*

5. Let the terms of the numerator, and of the denominator, of a *proper** fraction, and also those of its developement into series, proceed according to the positive integer powers of t; and let this developement, for brevity, be called the right arm of the recurring series; and let the coefficient of the first and of the last term, as well in the numerator as in the denominator, be finite; and, in the denominator, let the coefficient of the first term be 1. Then the coefficient of the first term in the right arm of the series is obviously finite.

Now by means either of the scale of relation, or of any knowledge we may possess what function the general term is of the serial index, let the terms which correspond to negative integer indices be constructed. Then the first term of finite magnitude, taken in that direction, will be the term whose index is the difference between the dimensions of the numerator and of the denominator, taking that difference with a negative sign; and all the terms intermediate between this finite term and the first term in the right arm of the series, will be zero-terms; and, consequently, the number of such zero-terms will be a unit less than the difference between the dimensions of the numerator and of the denominator, taking that difference with a positive sign.

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* By *proper*, I mean that the dimension of t, in the numerator, is lower than it is in the denominator.

For let the generating fraction be,

$$\frac{A_0 + A_1 t + \ldots\ldots + A_a t^a}{1 + B_1 t + \ldots\ldots + B_b t^b}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and let the right arm of the recurring series be,

$$C_0 + C_1 t + \ldots\ldots + C_x t^x + \&\text{c.} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

and let the coefficients of the terms corresponding to negative integer indices be

$$C_{-1},\ C_{-2},\ C_{-3},\ \&\text{c.}$$

Then, supposing x to be less than b, there will subsist between any $(b + 1)$ consecutive terms of the series extended as above directed, in accordance with the scale of relation, the following equation:

$$0 = 1 \times C_x + B_1 \times C_{x-1} + \ldots\ldots + B_x \times C_0 + B_{x+1} \times C_{-1} + B_{x+2} \times C_{-2} + \ldots\ldots + B_b \times C_{-(b-x)}. \quad . \qquad (4)$$

But $$A_x = 1 \times C_x + B_1 \times C_{x-1} + \ldots\ldots + B_x \times C_0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

$$\therefore\ A_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \left[B_{x+1} \times C_{-1} + B_{x+2} \times C_{-2} + \ldots\ldots + B_b \times C_{-(b-x)}\right] \qquad (6)$$

In equation (6), substitute for x, in succession, the values,

$$(b-1),\ (b-2),\ \ldots\ldots,\ (a+1),\ a.$$

Then, first, $A_{b-1} = -B_b \times C_{-1}$; and since B_b is finite, therefore, if $A_{b-1} = 0$, $C_{-1} = 0$.

Secondly, $A_{b-2} = -\left[B_{b-1} \times C_{-1} + B_b \times C_{-2}\right]$; and since B_b is finite, therefore, if A_{b-1} and A_{b-2} are each equal to 0, C_{-1} and C_{-2} will be each equal to 0.

And in like manner it may be shewn, that if A_{b-1}, A_{b-2}, ……, A_{a+1}, are each equal to 0, then C_{-1}, C_{-2}, ……, $C_{-\{b-(a+1)\}}$, will be each equal to 0.

Lastly, $A_a = -B_b \times C_{-(b-a)}$; but A_a and B_b are finite, $\therefore\ C_{-(b-a)}$ is finite.

Hence, if z denotes the number of the zero-terms, intermediate between C_0 and $C_{-(b-a)}$, we have the equation,

$$z = b - (a + 1). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

6. In like manner may the converse proposition be proved; namely, that if a recurring series contains z consecutive terms, each equal to zero, and these zero-terms are immediately preceded by a finite term, $C_{-(z+1)}$, and are immediately followed by another finite term, C_0,

then the fraction which generates the series commencing with C_0, and extending in the direction of the positive indices, will be of higher dimension in the denominator than it is in the numerator, by $(x + 1)$ units.

7. Assuming, either first, that A_0, A_a, B_0, and B_b are finite, and that A_{b-1}, A_{b-2}, ..., A_{a+1}, are each equal to zero; or secondly, that C_0, C_{b-a}, B_0, and B_b are finite, and that C_{-1}, C_{-2}, ..., $C_{-[b-(a+1)]}$, are each equal to zero; on either of those assumptions, Equation (6) gives the following values of A_x and of A_{a-x} in terms of the coefficients B and C.

$$A_x = -\left[B_{(b-a)+x} \times C_{-(b-a)} + B_{(b-a)+1+x} \times C_{-(b+1-a)} + \ldots + B_b \times C_{-[b-x]}\right]; \quad . \qquad (8)$$

$$A_{a-x} = -\left[B_{b-x} \times C_{-(b-a)} + B_{b+1-x} \times C_{-(b+1-a)} + \ldots + B_b \times C_{-[b-a+x]}\right]. \quad . \qquad (9)$$

Both expressions vanish when x is either greater than a, or becomes negative.

SECT. IV. *On the fraction whose developement is the right arm of a* self-repeating *recurring series.*

8. Let $\left(A_0 + A_1 t + \ldots + A_x t^x + \ldots + A_{a-x} t^{a-x} + \ldots + A_{a-1} t^{a-1} + A_a t^a\right)$ . . (10)

be a rational integer function of t, of the finite dimension, a; and let a be some integer number or other, constantly odd, or constantly even, for the same function of the form (10).

Then if there subsists between every two of the coefficients, A, the sum of whose indices is equal to (a), an equation of the form,

$$A_x = (-1)^{a} A_{a-x}, \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

the function itself, and its coefficients, are called *recurrent;* and we may call equation (11) the equation of recurrence, and $(-1)^{a}$ the sign of recurrence, of the function to which they apply.

When (a) is an even positive integer, and a is an odd positive integer, then, since

$$+A_{\frac{a}{2}} = -A_{\frac{a}{2}}, \quad \therefore A_{\frac{a}{2}} = 0.$$

When x, in equation (11), is either negative, or greater than a, the corresponding coefficients will each be equal to zero.

9. *Theorem.* Let the series arising from the developement of a *proper* fraction be the right arm of a self-repeating recurring series. Then,

First, if the denominator of the fraction is recurrent, the numerator also is recurrent; and secondly, if the numerator of the fraction is recurrent, the denominator also is recurrent. And, conversely, if the numerator and the denominator of a *proper* fraction are each recurrent, the developement of the fraction will be the right arm of a self-repeating recurring series.

Case 1, proved. Let the repeating equation of the series be (according to equations 1 and 7),

$$C_x = (-1)^{\gamma} C_{-[x+b-a]} \;; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

and let the equation of recurrence of the denominator be,

$$B_x = (-1)^{\beta} B_{b-x}. \quad . \quad . \qquad \text{(See equation 11)} \quad (13)$$

Substitute in equation (8) for B and C their respective values, as deduced from the two preceding equations. Equation (8) will then become,

$$A_x = (-1)^{\beta+\gamma+1} \left\{ B_{a-x} \times C_0 + B_{a+1-x} \times C_1 + \ldots\ldots + B_0 . C_{a-x} \right\}; \quad . \quad . \qquad (14)$$

that is to say, $$A_x = (-1)^{\beta+\gamma+1} A_{a-x}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

But equation (15) is the equation of recurrence of the numerator of the fraction which has been developed; and from that equation it appears that the sign of recurrence is

$$(-1)^{\alpha} = (-1)^{\beta+\gamma+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Consequently, when β and γ are either both odd, or both even, the numerator will have a negative sign of recurrence; but when one of the said two exponents is odd, and the other even, the numerator will have a positive sign of recurrence.

Case 2, proved. As in the preceding case, let the repeating equation of the series be,

$$C_x = (-1)^{\gamma} C_{-[x+b-a]} \;; \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

and let the equation of recurrence of the numerator be,

$$A_x = (-1)^{\alpha} A_{a-x}. \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Substitute, in equation (9), $(-1)^{\alpha} A_x$ for A_{a-x}; and on the right side of that equation, substitute everywhere for C its values, as deduced from equation (12). Equation (9) will then become,

$$A_x = (-1)^{\alpha+\gamma+1} \left\{ B_{b-x} \times C_0 + B_{b+1-x} \times C_1 + \ldots\ldots + B_b \times C_x \right\}. \quad . \qquad (17)$$

But $$A_x = B_x \times C_0 + B_{x-1} \times C_1 + \ldots\ldots + B_0 \times C_x. \quad . \qquad (5)$$

In equations (17) and (5), let 0, 1, 2, 3, &c. be substituted, in succession, for x.

Then $$B_0 = (-1)^{\alpha+\gamma+1} B_b.$$

Consequently, $$B_1 = (-1)^{\alpha+\gamma+1} B_{b-1}.$$

Consequently, $$B_2 = (-1)^{\alpha+\gamma+1} B_{b-2}.$$

And so on; and since the several antecedent equations of the same form are true,

Consequently, $$B_x = (-1)^{\alpha+\gamma+1} B_{b-x}. \qquad \ldots \ldots \qquad (18)$$

But this is the equation of recurrence of the denominator of the fraction; and the sign of recurrence is,

$$(-1)^{\beta} = (-1)^{\alpha+\gamma+1}. \qquad \ldots \ldots \qquad (19)$$

Therefore, when α and γ are both odd or both even, the denominator will have a negative sign of recurrence; but when one of those two exponents is odd, and the other even, the denominator will have a positive sign of recurrence.

Case 3. The converse Proposition proved.

Let the equations of recurrence of the numerator and of the denominator of the fraction, respectively, be,

$$A_x = (-1)^{\alpha} A_{a-x}, \qquad \ldots \ldots \qquad (11)$$

$$\text{and} \quad B_x = (-1)^{\beta} B_{b-x} \qquad \ldots \ldots \qquad (13)$$

Substitute, in equation (9), $(-1)^{\alpha} A_{a-x}$ for A_x; and on the right side of that equation, substitute everywhere for B its values, as deduced from equation (13). Equation (9) will then become,

$$A_x = (-1)^{\alpha+\beta+1} \left\{ B_x \times C_{-(b-a)} + B_{x-1} \times C_{-(b+1-a)} + \ldots\ldots + B_0 \times C_{-(x+b-a)} \right\} \quad . \qquad (20)$$

But $$A_x = B_x \times C_0 + B_{x-1} \times C_1 + \ldots\ldots + B_0 \times C_x \quad . \qquad (5)$$

In equations (20) and (5), let 0, 1, 2, 3, &c. be substituted, in succession, for x.

Then $$C_0 = (-1)^{\alpha+\beta+1} C_{-(b-a)}.$$

Consequently, $$C_1 = (-1)^{\alpha+\beta+1} C_{-(b+1-a)}.$$

Consequently, $$C_2 = (-1)^{\alpha+\beta+1} C_{-(b+2-a)}.$$

And so on; and since the several antecedent equations of the same form are true,

Consequently, $$C_x = (-1)^{\alpha+\beta+1} C_{-(x+b-a)}. \qquad \ldots \ldots \qquad (21)$$

But this is the equation of repetition of the series arising from the developement of the given fraction, whose numerator and denominator are recurrent; and the sign of repetition of the series is

$$(-1)^\gamma = (-1)^{\alpha+\beta+1}. \qquad (22)$$

Therefore when α and β are either both odd, or both even, the series will have a negative sign of repetition; but when one of the two exponents is odd, and the other even, the series will have a positive sign of repetition.

10. From the first of the last three propositions several obvious and important corollaries immediately follow.

Let the general term of a series be a rational integer function of the index, and also let the series generated be the right arm of a self-repeating series. Then, since the denominator of the generating fraction is a positive integer power of $(1-t)$, the denominator is a recurrent function of t; and therefore the numerator also is a recurrent function of t. Thus, for instance, when the terms of the series are the natural numbers, or their positive integer powers; the odd numbers, or their positive integer powers; the figurate numbers of whatever order, or their positive integer powers, or when the general term is any other rational integer function of the general term of any of those numbers, such as will render the series a self-repeating one, and when the series generated begins with the first term of the right arm, then the numerator of the generating fraction will be a recurrent numerator.

Thus, if we take the triangular of every triangular number; that is to say, if instead of the series 1, 3, 6, 10, 15, 21, &c. we take the series 1, $\frac{3.4}{2}$, $\frac{6\times 7}{2}$, $\frac{10\times 11}{2}$, $\frac{15.16}{2}$, &c., we obtain the self-repeating series 1, 6, 21, 55, 120, &c., whose general term is obviously of four dimensions, and consequently the denominator is $(1-t)^5$; and the corresponding numerator is readily found to be $(1+t+t^2)$; therefore the generating fraction is $\frac{1+t+t^2}{(1-t)^5}$. And in like manner we may take the triangulars of the triangulars of the triangulars; which are the series 1, 21, 231, 1540, 7260, 26796, 82621, &c., in the general term of which the index is of 8 dimensions; and the denominator therefore is $(1-t)^9$; and the corresponding recurrent numerator is

$$1 + 12t + 78t^2 + 133t^3 + 78t^4 + 12t^5 + t^6.$$

We may proceed in like manner with any other figurate series of an *even* order; for in this case the right arm and the left arm of the series have the same sign. But in the figurate series of the odd orders, the right arm of the series consisting of positive terms, but the left arm of negative ones, when we come to take the odd figurate of an odd figurate, we fall, in the left arm, upon numbers differing in numerical value from those that we obtained in the right arm; and consequently the series obtained by a similar process from an odd figurate is not a self-repeating series.

11. Let the general term of the self-repeating series be the product of a rational integer function of the index by the exponential $(-1)^x$; in other words, let the numerical values of the rational integer function, corresponding to the consecutive integer values of the index, 0, 1,

2, 3, 4, &c. be taken alternately with a positive and with a negative sign. Then since the denominator of the generating fraction is some positive integer power of $(1+t)$, and since this power, when expanded, is a recurrent function of t, the numerator also of the generating fraction is a recurrent function of t.

The numerical values of the coefficient of the numerator, other things remaining the same, will be the same as were obtained in the former case, where the denominator, instead of being a power of $(1+t)$, was a power of $(1-t)$. The numerator in this case will differ from the numerator in the former case, in having the signs of all its alternate terms changed. Thus the fraction which generates $(x+1)^7$, to which corresponds the series, $1^7 + 2^7 t + 3^7 t^2 +$ &c., being

$$\frac{1 + 120t + 1191t^2 + 2416t^3 + 1191t^4 + 120t^5 + 1 \,.\, t^6}{(1-t)^8},$$

the fraction which generates $(-1)^x (x+1)^7$, to which corresponds the series

$$1^7 - 2^7 t + 3^7 t^2 - 4^7 t^3 + \text{\&c.}$$

will be

$$\frac{1 - 120t + 1191t^2 - 2416t^3 + 1191t^4 - 120t^5 + 1 \,.\, t^6}{(1+t)^8}.$$

As in the former case, so in the present, the positive integer powers of the natural numbers, of the odd numbers, and of the figurate numbers of the different orders, are elements from which we may form innumerable other self-repeating series. To this case belongs the particular series

$$1^n - 2^n t + 3^n t^2 - 4^n t^3 + \text{\&c.}$$

which first drew my attention to the subject now before us; and I shall proceed to consider this example with some minuteness, for the purpose of shewing that owing to their not being aware of the theorem we are now considering, the distinguished mathematicians who have before directed their attention to this series, have not given to their respective summations all the simplicity which the case admits of.

When the numbers of the natural series are raised to a positive even power, and are taken alternately with opposite signs, the number of the terms in the recurrent numerator being also even, and every two equal recurrent terms having opposite signs, the terms of the numerator, when t is equated to 1, destroy one another, and the value of the generating fraction and of the corresponding infinite series, is zero.

Thus the generating fraction of $1^2 - 2^2 t + 3^2 t^2 -$ &c. being $\dfrac{1-t}{(1+t)^3}$; that of $1^4 - 2^4 t + 3^4 t^2 -$&c. being $\dfrac{1 - 11t + 11t^2 - t^3}{(1+t)^5}$; and so on, all these fractions vanish, when t is equal to 1.

I proceed therefore to the case where the numbers are raised to an odd power; and with a view to brevity, I will consider the particular instance of $2n + 1 = 9$.

Then, if we do not know beforehand that the numerator consists of recurrent terms, the summation, according to the methods in use in the days of Euler, Lagrange, and Laplace, would be made in accordance with the following Diagram, No. I.; but, if we do know beforehand that the numerator is recurrent, then the summation, founded on the same methods, will be made in

DIAGRAM I. p. 478.

$$1^9 - 2^9t + 3^9t^2 - 4^9t^3 + \&c.$$

$$= \frac{A_0 + A_1t + A_2t^2 + A_3t^3 + A_4t^4 + A_5t^5 + A_6t^6 + A_7t^7 + A_8t^8}{(1+t)^{10}}.$$

Determination of the numerator, when it is *not* known beforehand that the coefficients, A, are recurrent.

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	Terms summed horizontally, when $t = 1$.
	$+1^9$	-2^9	$+3^9$	-4^9	$+5^9$	-6^9	$+7^9$	-8^9	$+9^9$	$= + \ 1 \times 285{,}188{,}825$
		$+10\times1^9$	-10×2^9	$+10\times3^9$	-10×4^9	$+10\times5^9$	-10×6^9	$+10\times7^9$	-10×8^9	$= - \ 10 \times 102{,}231{,}664$
			$+45\times1^9$	-45×2^9	$+45\times3^9$	-45×4^9	$+45\times5^9$	-45×6^9	$+45\times7^9$	$= + \ 45 \times 31{,}986{,}064$
				$+120\times1^9$	-120×2^9	$+120\times3^9$	-120×4^9	$+120\times5^9$	-120×6^9	$= - 120 \times 8{,}367{,}543$
					$+210\times1^9$	-210×2^9	$+210\times3^9$	-210×4^9	$+210\times5^9$	$= + 210 \times 1{,}710{,}153$
						$+252\times1^9$	-252×2^9	$+252\times3^9$	-252×4^9	$= - 252 \times 242{,}972$
							$+210\times1^9$	-210×2^9	$+210\times3^9$	$= + 210 \times 19{,}172$
								$+120\times1^9$	-120×2^9	$= - 120 \times 511$
									$+45\times1^9$	$= + \ 45 \times 1$
	$+1$	-502	$+14608$	-88234	$+156190$	-88234	$+14608$	-502	$+1$	$= \ +7936$

DIAGRAM II. p. 482.

Determination of the numerator, when it is known beforehand that the coefficients, A, are recurrent.

	A_0	A_1	A_2	A_3	A_4	A_3	A_2	A_1	A_0	Terms summed horizontally, when $t = 1$.
	$+1^9$	-2^9	$+3^9$	-4^9	$+5^9$	-4^9	$+3^9$	-2^9	$+1^9$	$= + \ 1 \times 1{,}467{,}181$
		$+10\times1^9$	-10×2^9	$+10\times3^9$	-10×4^9	$+10\times3^9$	-10×2^9	$+10\times1^9$		$= - \ 10 \times 223{,}800$
			$+45\times1^9$	-45×2^9	$+45\times3^9$	-45×2^9	$+45\times1^9$			$= + \ 45 \times 18{,}661$
				$+120\times1^9$	-120×2^9	$+120\times1^9$				$= - 120 \times 510$
					$+210\times1^9$					$= + 210 \times 1$
	$+1$	-502	$+14{,}608$	$-88{,}234$	$+156190$	$-88{,}234$	$+14{,}608$	-502	$+1$	$= \ +7936$

DIAGRAM III. p. 482.

$$1^9-2^9t+3^9t^2-4^9t^3+\&c. = \left\{\frac{\Delta^1}{(1+t)^2}\times\frac{1}{1+\frac{\Delta^1 t}{1+t}}\right\}0^9 = \Delta^1 0^9 \times \frac{1}{(1+t)^2} - \Delta^2 0^9\,\frac{t}{(1+t)^3} + \Delta^3 0^9\,\frac{t^2}{(1+t)^4} - \ldots\ldots - \Delta^8 0^9\,\frac{t^7}{(1+t)^9} + \Delta^9 0^9\,\frac{t^8}{(1+t)^{10}}$$

$$= \frac{1}{(1+t)^{10}}\left\{\Delta^1 0^9 (1+t)^8 - \Delta^2 0^9 (1+t)^7 \times t + \Delta^3 0^9 (1+t)^6 \times t^2 = \ldots\ldots\ldots\ldots\ldots\ldots - \Delta^8 0^9 (1+t)^1 \times t^7 + \Delta^9 0^9 \times t^8\right\}.$$

Determination of the numerator, when it is not known beforehand that its coefficients, *A*, are recurrent.

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	The terms summed horizontally, when $t=1$.
$\Delta^1 0^9 \times (1+t)^8 =$	$1\Delta^1$	$+8\Delta^1$	$+28\Delta^1$	$+56\Delta^1$	$+70\Delta^1$	$+56\Delta^1$	$+28\Delta^1$	$+8\Delta^1$	$+1\Delta^1$	$= +256\,\Delta^1 0^9 = +256 \times 1$
$\Delta^2 0^9 \times (1+t)^7 \times t =$		$-1\Delta^2$	$-7\Delta^2$	$-21\Delta^2$	$-35\Delta^2$	$-35\Delta^2$	$-21\Delta^2$	$-7\Delta^2$	$-1\Delta^2$	$= -128\,\Delta^2 0^9 = -128 \times 510$
$\Delta^3 0^9 \times (1+t)^6 \times t^2 =$			$+1\Delta^3$	$+6\Delta^3$	$+15\Delta^3$	$+20\Delta^3$	$+15\Delta^3$	$+6\Delta^3$	$+1\Delta^3$	$= +64\,\Delta^3 0^9 = +64 \times 18{,}150$
$\Delta^4 0^9 \times (1+t)^5 \times t^3 =$				$-1\Delta^4$	$-5\Delta^4$	$-10\Delta^4$	$-10\Delta^4$	$-5\Delta^4$	$-1\Delta^4$	$= -32\,\Delta^4 0^9 = -32 \times 186{,}480$
$\Delta^5 0^9 \times (1+t)^4 \times t^4 =$					$+1\Delta^5$	$+4\Delta^5$	$+6\Delta^5$	$+4\Delta^5$	$+1\Delta^5$	$= +16\,\Delta^5 9^9 = +16 \times 834{,}120$
$\Delta^6 0^9 \times (1+t)^3 \times t^5 =$						$-1\Delta^6$	$-3\Delta^6$	$-3\Delta^6$	$-1\Delta^6$	$= -8\,\Delta^6 0^9 = -8 \times 1{,}905{,}120$
$\Delta^7 0^9 \times (1+t)^2 \times t^6 =$							$+1\Delta^7$	$+2\Delta^7$	$+1\Delta^7$	$= +4\,\Delta^7 0^9 = +4 \times 2{,}328{,}480$
$\Delta^8 0^9 \times (1+t) \times t^7 =$								$-1\Delta^8$	$-1\Delta^8$	$= -2\,\Delta^8 0^9 = -2 \times 1{,}451{,}520$
$\Delta^9 0^9 \times t^8 =$									$+1\Delta^9$	$= +1\,\Delta^9 0^9 = +1 \times 362{,}880$
Numerator =	1	$-502t$	$+14608t^2$	$-88234t^3$	$+156190t^4$	$-88234t^5$	$+14608t^6$	$-502t^7$	$+1t^8$	$= +7936$

DIAGRAM IV. p. 482.

Determination of the numerator, when it is known beforehand that its coefficients, A, are recurrent.

	A_0	A_1	A_2	A_3	A_4	A_3	A_2	A_1	A_0	The terms summed horizontally, when $t=1$.	
	$+1\Delta^1$	$+8\Delta^1$	$+28\Delta^1$	$+56\Delta^1$	$+70\Delta^1$	$+56\Delta^1$	$+28\Delta^1$	$+8\Delta^1$	$+1\Delta^1$	$=+256\Delta^1 0^9 \quad =+256\Delta^1 0^9 = +256 \times \quad 1$	$256 = 70 + 35\times 2 + 15\times 2^2 + 5\times 2^3 + 1\times 2^4$
		$-1\Delta^2$	$-7\Delta^2$	$-21\Delta^2$	$-35\Delta^2$	$-21\Delta^2$	$-7\Delta^2$	$-1\Delta^2$		$=-\frac{256-70}{2}\Delta^2 0^9 = -93\Delta^2 0^9 = -93\times \quad 510$	$93 = 35\times 1 + 15\times 2 + 5\times 2^2 + 1\times 2^3$
			$+1\Delta^3$	$+6\Delta^3$	$+15\Delta^3$	$+6\Delta^3$	$+1\Delta^3$			$=+\frac{93-35}{2}\Delta^3 0^9 = +29\Delta^3 0^9 = +29\times \; 18{,}150$	$29 = 15\times 1 + 5\times 2 + 1\times 2^2$
				$-1\Delta^4$	$-5\Delta^4$	$-1\Delta^4$				$=-\frac{29-15}{2}\Delta^4 0^9 = -7\Delta^4 0^9 = -7\times 186{,}480$	$7 = 5\times 1 + 1\times 2$
					$+1\Delta^5$					$=+\frac{7-5}{2}\Delta^5 0^9 = +1\Delta^5 0^9 = +1\times 834{,}120$	$1 = 1\times 1$
Numerator =	1	$-502t$	$+14608t^2$	$-88234t^3$	$+156190t^4$	$-88234t^5$	$+14608t^6$	$-502t^7$	$+1t^8$	$=+7936$	

The equality that subsists between the terms in the columns equidistant from the middle column of Diagram III., and on opposite sides of it, leads to the following general property of the differences of the powers of zero; namely,

$$\frac{\Delta^v}{\left(1+\frac{1}{\Delta}\right)^{n+1-v}} 0^n = \frac{\Delta^{n+1-v}}{\left(1+\frac{1}{\Delta}\right)^v} 0^n,$$

the developements ceasing with the term that contains $\Delta^1 0^n$.

accordance with the Diagram No. II. In this second Diagram the numerator being recurrent, the respective columns equidistant from the middle column on either side of it must be the exact counterparts, one of the other. Thus the high numbers 6^9, 7^9, 8^9, and 9^9, are at once excluded from the process of summation; we have only 5 lines, that is $(n+1)$, instead of 9, that is $(2n+1)$, to sum; and the greatest Number in any line, instead of being, as in Diagram I., 285,188,825, is in Diagram II. only 1,467,181; and the number of the multiplications to be made of digit by digit, is reduced from 129 to 41.

But let us now proceed to the improved mode of summation, depending on the employment of the differences of the powers of zero, for the introduction of which into analysis the mathematical world are mainly indebted to Dr Brinkley and Sir John Herschel, members of this university.

Then, according to the formula given by Sir John Herschel, in Jameson's *Philosophical Journal* for January, 1820, we have,

$$1^9 - 2^9 t + 3^9 t^2 - \&c. = \left[\frac{\Delta^1}{(1+t)^2} \times \frac{1}{1 + \dfrac{\Delta^1 t}{1+t}}\right] 0^9, \quad . \quad . \quad . \quad . \quad (23)$$

$$= \Delta^1 0^9 \times \frac{1}{(1+t)^2} - \Delta^2 0^9 \frac{t}{(1+t)^3} + \Delta^3 0^9 \frac{t^2}{(1+t)^4} - \ldots\ldots\ldots - \Delta^8 0^9 \frac{t^7}{(1+t)^9} + \Delta^9 0^9 \frac{t^8}{(1+t)^{10}} \quad . \quad (24)$$

and reducing these terms to the common denominator $(1+t)^{10}$, and expanding the powers of $(1+t)$ in the numerator according to the powers of t, we obtain the terms shewn in the Diagram No. III.; which when t is equated to 1, give for the numerator the expression,

$$256\Delta^1 0^9 - 128\Delta^2 0^9 + 64\Delta^3 0^9 - 32\Delta^4 0^9 + 16\Delta^5 0^9 - 8\Delta^6 0^9 + 4\Delta^7 0^9 - 2\Delta^8 0^9 + \Delta^9 0^9;$$

with a denominator 1024. . . (25)

But on the assumption that we know beforehand that the numerator is recurrent, then the columns equidistant from the middle one, on either side of it, become the counterparts of each other; and the numerator acquires the form exhibited in Diagram IV.

But then, this farther question arises. When t is equated to 1, can a simple rule be given for the horizontal summation of the numerical coefficients which stand on the several lines, those on the same line having all the same sign. The answer to this question is very satisfactory.

I proceed to shew that, in a Table constructed on the principle of Diagram IV, the half of the sum of the coefficients in any line, less the half of the coefficient of the middle term in that line, is equal to the sum of the coefficients in the following line.

For in the developement of any two consecutive powers of $(1+t)$; say $(1+t)^p$ and $(1+t)^{p-1}$, it is a known property, that the sum of all the terms in $(1+t)^p$ from the coefficient of t^v to that of t^{p-v}, inclusive of the two extremes, is twice the sum of all the terms in $(1+t)^{p-1}$, from the coefficient of t^v to that of t^{p-v}, inclusive of the two extremes. That is to say, adopting the factorial notation,

$$\frac{p^{v|-1}}{1^{v|1}} + \frac{p^{v+1|-1}}{1^{v+1|1}} + \ldots + \frac{p^{p-v|-1}}{1^{p-v|1}} = 2 \times \left\{\frac{(p-1)^{v|-1}}{1^{v|1}} + \frac{(p-1)^{v+1|-1}}{1^{v+1|1}} + \ldots + \frac{(p-1)^{p-(v+1)|-1}}{1^{p-(v+1)|1}} + \frac{(p-1)^{p-v|-1}}{1^{p-v|1}}\right\}.$$

Change the place of $\frac{p^{v|-1}}{1^{v|1}}$, and write the preceding equation,

$$\frac{p^{v+1|-1}}{1^{v+1|1}} + \ldots + \frac{p^{p-v|-1}}{1^{p-v|1}} + \frac{p^{v|-1}}{1^{v|1}} = 2 \times \left\{\frac{(p-1)^{v|-1}}{1^{v|1}} + \frac{(p-1)^{v+1|-1}}{1^{v+1|1}} + \ldots + \frac{(p-1)^{p-(v+1)|-1}}{1^{p-(v+1)|1}} + \frac{(p-1)^{p-v|-1}}{1^{p-v|1}}\right\}.$$

To give greater symmetry to the expression, change the factorials with diminishing factors, into their respectively equivalent factorials with increasing factors; and the expression will become,

$$\frac{[p-v]^{v+1|+1}}{1^{v+1|1}} + \ldots + \frac{[v+1]^{p-v|+1}}{1^{p-v|1}} + \frac{[p+1-v]^{v|+1}}{1^{v|1}} = 2\left\{\frac{[p-v]^{v|+1}}{1^{v|1}} + \ldots + \frac{[v+1]^{p-(v+1)|+1}}{1^{p-(v+1)|1}} + \frac{v^{p-v|+1}}{1^{p-v|1}}\right\}.$$

Now in order that the term $\frac{[p+1-v]^{v|+1}}{1^{v|1}}$ may coincide with the term of the middle column, $\frac{[n+1]^{v|+1}}{1^{v|1}}$, take $(p+1-v) = (n+1)$; that is, take $p - v = n$. Then the expression will become,

$$\frac{n^{v+1|+1}}{1^{v+1|1}} + \ldots + \frac{[v+1]^{n|+1}}{1^{n|1}} + \frac{[n+1]^{v|+1}}{1^{v|1}} = 2\left\{\frac{n^{v|+1}}{1^{v|1}} + \ldots + \frac{(v+1)^{n-1|+1}}{1^{n-1|1}} + \frac{v^{n|1}}{1^{n|1}}\right\}. \quad (26)$$

And since $p = n + v$, it is plain that

$$\left\{\frac{n^{v+1|+1}}{1^{v+1|1}} + \ldots\ldots + \frac{(v+1)^{n|+1}}{1^{n|1}}\right\}$$

is the sum of the terms which in $(1+t)^{n+v}$ are omitted, in order to produce Diagram IV. from Diagram III.; and that

$$\left\{\frac{n^{v|+1}}{1^{v|1}} + \ldots\ldots + \frac{(v+1)^{n-1|+1}}{1^{n-1|1}} + \frac{v^{n|1}}{1^{n|1}}\right\}$$

is the sum of the terms which in $(1+t)^{n+v-1}$ are omitted, in order to produce Diagram IV. from Diagram III.

And in Diagram III., $(1+t)^{n+v}$ is the factor of $\Delta^{n+1-v}0^{2n+1}$; and $(1+t)^{n+v-1}$ is the factor of $\Delta^{n+2-v}0^{2n+1}$.

Therefore, in Diagram IV.,

$$\left[2^{n+v} - \left\{\frac{n^{v+1|+1}}{1^{v+1|1}} + \ldots\ldots + \frac{(v+1)^{n|+1}}{1^{n|1}}\right\}\right]$$

is the coefficient of $\Delta^{n+1-v}0^{2n+1}$; and

$$\left[2^{n+v-1} - \left\{\frac{n^{v|+1}}{1^{v|1}} + \ldots\ldots + \frac{(v+1)^{n-1|+1}}{1^{n-1|1}} + \frac{v^{n|1}}{1^{v|1}}\right\}\right]$$

is the coefficient of $\Delta^{n+2}0^{2n+1}$.

Hence if from 2^{n+v} we subtract the equals of the equation (26), and divide the left side of the resulting equation by 2, we shall obtain the equation,

$$\tfrac{1}{2}\left[\text{the coefficient of } \overset{n+1-v}{\Delta}0^{2n+1} - \text{the middle term of that coefficient}\right] = \text{the coefficient of } \overset{n+2-v}{\Delta}0^{2n+1}; \quad (27)$$

and the middle term of the coefficient of $\overset{n+1-v}{\Delta}0^{2n+1}$ is the term $\dfrac{[n+1]^{v|+1}}{1^{v|1}}$, or the $(v+1)^{\text{th}}$ term of the figurate series of the n^{th} order.

But in Diagram IV., when $v=0$, the coefficient of $\overset{n+2}{\Delta}0^{2n+1}=0$; and when $v=0$, $\dfrac{(n+1)^{v|+1}}{1^{v|1}}=1$; and therefore the coefficient of $\overset{n+1}{\Delta}0^{2n+1}=1$. Consequently,

the coefficient of $\overset{n}{\Delta}0^{2n+1} = (n+1) + 1 \times 2.$

.... $\overset{n-1}{\Delta}0^{2n+1} = \dfrac{[n+1]^{2|+1}}{1^{2|1}} + (n+1) \times 2 + 1 \times 2^2.$

.... $\overset{n-2}{\Delta}0^{2n+1} = \dfrac{(n+1)^{3|+1}}{1^{3|1}} + \dfrac{(n+1)^{2|+1}}{1^{2|1}} \times 2 + (n+1) \times 2^2 + 1 \times 2^3$ . . (28)

....

.... $\overset{1}{\Delta}0^{2n+1} = \dfrac{[n+1]^{n|+1}}{1^{n|1}} + \dfrac{[n+1]^{n-1|+1}}{1^{n-1|1}} \times 2 + \ldots\ldots\ldots + (n+1)\,2^{n-1} + 1 \times 2^n = 2^{2n}.$

And thus, when $2n+1=9$,

the coefficient of $\Delta^1 0^9 = 70 \times 1 + 35 \times 2 + 15 \times 2^2 + 5 \times 2^3 + 1 \times 2^4 = 2^8.$

.... $\Delta^2 0^9 = 35 \times 1 + 15 \times 2 + 5 \times 2^2 + 1 \times 2^3$

.... $\Delta^3 0^9 = 15 + 5 \times 2 + 1 \times 2^2$

.... $\Delta^4 0^9 = 5 \times 1 + 1 \times 2$

.... $\Delta^5 0^9 = 1 \times 1.$

And hence the following ready mode of determining these coefficients:

the coefficient of $\Delta^1 0^9 = 256.$

.... $\Delta^2 0^9 = \dfrac{256-70}{2} = 93.$

.... $\Delta^3 0^9 = \dfrac{93-35}{2} = 29.$

.... $\Delta^4 0^9 = \dfrac{29-15}{2} = 7.$

.... $\Delta^5 0^9 = \dfrac{7-5}{2} = 1$

Which coefficients, 256 being positive, are to be with signs alternately positive and negative; so that when $t = 1$, the fraction becomes,

$$\frac{1}{1024}\left[256\Delta^1 0^9 - 93\Delta^2 0^9 + 29\Delta^3 0^9 - 7\Delta^4 0^9 + 1\Delta^5 0^9\right], \quad . \quad . \quad . \quad . \qquad (29)$$

instead of the elegant, but more complex expression,

$$\frac{1}{1024}\left[256\Delta^1 0^9 - 128\Delta^2 0^7 + \ldots\ldots\ldots + 2^4\Delta^5 0^9 - 2^3\Delta^6 0^9 + 2^2\Delta^7 0^9 - 2\Delta^8 0^9 + 1\Delta^9 0^9\right].$$

In the former expression, the labour of computation is less, by a half, than it is in the latter; and the proportion of the labour saved becomes greater and greater, as the numbers are raised to a higher power.

12. I should enter upon a vast field of enquiry, were I to enlarge upon the application of this theorem to the different cases where the denominator of the generating fraction is represented by different recurrent functions of t. It will be sufficient to point out as a fertile soil for new theorems, the cases where the recurrent denominator is a power, or rational integer function of $\left\{1 \pm \cos(\theta) \times t + t^2\right\}$.

13. The third, or converse proposition suggests several important corollaries. It is too obvious to require proof, that if two or more rational integer functions of t are severally recurrent, their product will be a recurrent function of t; and that in the product of any two such functions, the sign of recurrence will be positive or negative, according as the two functions agree or differ in their respective signs of recurrence.

Hence it follows that if two or more proper, or improper, fractions have for their respective numerators and denominators recurrent functions of t, then, if the highest dimension of t, in the product of all the numerators, is lower than the highest dimension of t in the product of all the denominators, the series arising from the developement of this fractional product, will be the right arm of a self-repeating recurring series. For the product will then be a proper fraction, having a recurrent numerator and a recurrent denominator. This obviously includes the case of one of the factors being a recurrent *integer* function of t, since it may be considered as a fraction, whose denominator is 1, the limit of all recurrent integer functions.

Let a and b be the respective dimensions, as to t, of the recurrent numerator, and of the recurrent denominator, of a proper fraction; and let n be any integer not greater than $b - (a + 1)$. Then the series generated by this fraction will be the right arm of a self-repeating series; and if a new series is constructed by adding together every $(n + 1)$ consecutive terms of the original series, the new series will be self-repeating. For the process directed to be performed is equivalent to multiplying the numerator of the fraction by the recurrent factor, $(1 + t + t^2 + \ldots\ldots\ldots + t^n)$; and, by supposition, the dimension of the product, as to t, is less than b, the dimension, as to t, of the denominator. And, in like manner, with the like limitation, may the recurrent factor be $(1 - t + t^2 - t^3 \ldots\ldots\ldots \pm t^n)$.

Thus if we add together every n^1 consecutive terms of a figurate series of the n^{th} order, n^1 being any integer not greater than n, we shall obtain a self-repeating series, between the arms of which $(n - n^1)$ zero terms will intervene. For instance, if we add together every two consecutive terms of the triangular series 1, 3, 6, 10, &c., whose generating fraction is $\frac{1}{(1-t)^3}$, we have the self-repeating series of the square numbers, the zero-terms being reduced from two to a single term.

The generating fraction is then, $\frac{1+t}{(1-t)^3}$.

If we add together every three consecutive terms, we have the series, 1, 4, 10, 19, 31, &c., from which all the zero-terms are obliterated; the generating fraction being in this case

$$\frac{1+t+t^2}{(1-t)^3}.$$

14. But what, it may be asked, would be the result, if having by previous addition obliterated all the zero-terms included between the right and the left arm, we were to add together every $(n+1)$, or $(n+2)$, or more, consecutive terms of the series. My answer is, that we should obtain an improper fraction whose numerator and denominator would be recurrent; and we might then possibly have to consider the case of a self-repeating series, differing from those of which I have been treating, in its containing between the two arms a finite middle term.

A theorem respecting this second class of self-repeating series, analogous to that which applies to the first class of which we have been treating, I shall make the subject of a future communication.

HENRY WARBURTON.

May 15, 1854.

XXI. *On the Determination of the Longitude of the Observatory of Cambridge by Galvanic Signals. By the* REV. J. CHALLIS, M.A., F.R.S., F.R.A.S., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *May* 15, 1854.]

ON March 10, 1853, I received a proposal from the Astronomer Royal to make an exchange of galvanic signals between Greenwich and Cambridge, for the determination of the Longitude of the Cambridge Observatory, to which proposal I immediately signified my assent. The Greenwich Galvanic Signal Apparatus had already been tested by giving and receiving signals between Greenwich and Dover, but had not yet been applied to the determination of Longitude. Various arrangements were to be made, which necessarily occupied time. In the first place, I had to apply to the Electric Telegraph Company for permission to make use of the telegraphic lines and apparatus on the Eastern Counties' Railway. I am bound to say that my application was responded to by the Directors in the most liberal manner, and I was immediately put in communication with the Company's Engineer, Mr Edwin Clark, for the purpose of facilitating the requisite arrangements. The completion of the circuit from Greenwich to Lothbury in the Strand, London, having been undertaken at Mr Airy's request by Mr Walker, Engineer and Superintendent of Telegraphs of the South-Eastern Railway, Mr Clark provided carefully for the junctions at Lothbury and the Eastern Counties' Terminus, and for the exclusive use of a line for the transmission of signals to and from the Cambridge Station. I was also furnished with a galvanic battery, a separate signal-needle, and all the personal aid that I required. I did not propose the carrying a temporary wire from the Cambridge Station to the Observatory, (a distance of about 2½ miles), thinking that great expense and trouble would be thereby incurred, but had I considered this arrangement necessary for the success of the experiment, I have reason to say that the authorities of the Electric Telegraph Company would have lent their aid in effecting it.

All necessary preparations having been made, the exchange of signals eventually took place on the nights of May 17 and 18. I cannot better commence my account of the operation than by citing the scheme for conducting it communicated to me by Mr Airy, which was closely followed. It should first be premised that this method of determining longitude is in principle the ordinary method of signals, the instant of the same signal being recorded in the mean or sidereal times of two observatories, so as to give by the difference of times the difference of the longitudes of the observatories. On the supposition of the instantaneous transmission of galvanic action, two needles, however distant from each other, would start at the same instant by the completion of a galvanic circuit, and afford the means of recording the times at two positions of the same event. And if the galvanic action takes time to pass along the wire, the effect of the retardation on the determination of longitude is eliminated by transmitting the signals in nearly equal numbers from the two positions, and taking the mean of the results given by the two sets. The following is Mr Airy's "Scheme for the making and observing of Galvanic Signals for the Comparison of Distant Clocks," drawn out subsequently to the trial of the transmission of signals between Greenwich and Dover.

1. Suppose an hour to be granted for the use of the telegraph wire. The hour is divided into four quarters, and during the first and third quarters signals are sent from Greenwich, and during the second and fourth quarters signals are received at Greenwich.

2. Consequently the fittings at both ends of the line must be generally similar. There must be a battery (one of 72 pairs of plates was very vigorous from Greenwich to Dover), and means of putting the battery out of circuit for receiving signals, and of putting it in circuit by starts for giving signals. The mechanism for this purpose of the common needle telegraph answers perfectly well. There must also be a signal-needle *in another room*, or *in a distant corner of the same room*, at which the time-observations of signals going both ways are made. The separation of the signal-giving place from the time-observing place has been found to be indispensable for accuracy.

3. The signal is rather startling and requires an observer in a good state of nerve. Considering that a half-second beat comes so quick as to endanger the requisite coolness, it would be best, if the means are at hand, to erect a clock near to the signal-needle. But if not, a chronometer may be used: at Greenwich it was done every time with fair success.

4. The signals are made in the following way. The signal-giver fixes in his mind on an arbitrary number of signals to be given in the next batch: not fewer than three and not more than nine. Suppose the number to be five. Then he gives five warning signals at intervals of about $1\frac{1}{2}$ or 2 seconds (to prepare the mind of the signal-receiver), and follows them with five time-signals at arbitrary intervals, not less than 10^s nor greater than 15^s. Graphically a batch is represented .... Then he waits 20 or 30 seconds before making another batch.

5. At each end there must be three persons. The signal-giver A will have nothing to do but to give signals, as above described, for half the time. The signal-watcher B will take seconds from the clock, count beats and observe the start of the needle, and enter the times in his book during the *whole* time. The assistant C (who is to be a very quiet person), must look over the shoulder of B, and count the warning-signals and say "five signals coming." He must quietly count them as they come, and after the entry of the fifth must say "batch finished." This assistance is a great relief to B.

6. In the successive quarters of hour, the needle has been thrown *right* in the first and second, and *left* in the third and fourth.

7. It is thought that accuracy is gained by covering the needle-dial leaving a slit through which the needle is seen. Thus the start only of the needle is seen, not the end of the swing.

All the above rules were attended to in the transmission of the signals from the Cambridge Telegraph Station, excepting the sixth and seventh. The former was neglected from oversight, and the latter was found to be inconvenient, because, for reasons stated further on, the signal-times were noted by two observers at the same time.

After receiving the above scheme of operations, I waited on the Astronomer Royal for the purpose of seeing the Greenwich arrangements, and I took the opportunity of proposing to him that signals should be made on two successive nights, and that there should be an interchange of observers, the Greenwich observer both of transits and signals going to Cambridge, and the Cambridge observer of transits and signals going to Greenwich, after the first night's observations. This proposal, which was made with the view of eliminating the effect of any difference between the personal equations of the two observers, was assented to by Mr Airy. I proposed also that in addition to the stars ordinarily used at the two Observatories for the determination of clock-errors, a selected list of other stars should also be observed, and in the instances in which the same star was observed at both Observatories, that clock-errors should be deduced from the same values of the Apparent Right Ascensions of those stars. It is clear that by this process, in which exact values of the Right Ascensions of the stars are not essential, a larger number of stars are available for clock-errors, in case the state of the sky be capricious. Mr Airy agreed to this arrangement, stipulating only that the longitude deduced from the ordinary clock-stars should be calculated apart from that deduced from corresponding stars. I consequently selected from the Greenwich Twelve-year Catalogue a list of stars, whose meridian transits would occur shortly before and after the time occupied at Cambridge in the observations of the signals, and in the conveyance of the chronometers to and from the Station.

At Greenwich, on May 17, the observations throughout the series, both of selected stars and signals, were made by Mr Dunkin. The signals were observed by noting the time of deflection of the needle at the galvanometer attached to the case of the transit-clock, the time of each signal being the instant of the first start of the needle. The contact was made without exception at the Transit-circle, by the apparatus intended for use in the American method of recording transits. In four batches the signals were made at the instants when a star passed the Transit wires. The signals were given throughout the series by Mr Henderson. The warning-signals were counted and a register kept of the number by Mr Criswick.

At Cambridge, on the same day, Mr C. Todd observed the transits of stars for clock-errors, and the signal-times. Mr Breen also observed signal-times. As at Greenwich, the instant of the first start of the needle was noted. The Cambridge signals were all given by myself, stationed at one corner of the room in which the observations were made, as far from the observers (whose backs were turned towards me) as the limited dimensions of the room admitted. Mr H. Todd counted the warnings and signals. Mr Sach, local superintendent of the Eastern Counties' Telegraph lines, was present to arrange the battery and wires, and to perform the necessary changes of connexion during the observations. The greatest care was taken by all parties engaged in the Telegraph Office to avoid making noises that might disturb the experiment, but some inconvenience was unavoidably experienced from the occasional passage of railway carriages.

On May 18, the operators and the distribution of their duties, both at Greenwich and Cambridge, were the same as on the preceding night, excepting that Mr Dunkin and Mr C. Todd changed places. My objects in having the signals observed simultaneously by two observers, were, to check mistakes in writing down the seconds, and to have the opportunity of deducing difference of personal equations in a mode of observing in which the fraction of a second is estimated in a different manner from that in which it is estimated in a transit observation. In order that the kind of observation might not be entirely strange to the observers, both Mr Breen and Mr Todd practised, a short time previous, taking notes of time of sudden jerks which I gave at arbitrary intervals to a temporary substitute for the galvanic needle.

The Cambridge Observatory time was transferred to the Telegraph Station by means of three chronometers, beating half-seconds, one going mean solar time, and the other two sidereal time*. The two sidereal chronometers were compared with the transit-clock by the intervention of the solar chronometer just before we left the Observatory for the Station, they were compared with the solar chronometer at the Station both before the signals commenced and when they were finished, and lastly they were compared again with the transit-clock by means of the solar chronometer on our return to the Observatory. All the comparisons were made by coincidences of beats. The signal-times were noted by one of the sidereal chronometers.

Exactly at 11^h Greenwich Mean Time of May 17, the first signal reached the Cambridge Station. Four others followed at intervals of about two seconds, and then five signals were similarly sent from Cambridge. These by previous understanding signified that all was right. The batches of signals then commenced, and were continued to 12^h, Greenwich giving signals during the first and third quarters of the hour, and Cambridge during the second and fourth. The following is a list of the batches of signal-times recorded at Greenwich and and Cambridge on May 17, inclusive of those recorded by Mr Breen:—

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* A conveyance was hired for the transport of the chronometers and the party of observers, each chronometer was held by hand, and when it was necessary to pass over the street-pavement, we alighted from the conveyance and carried the chronometers walking.

1853. May 17.

First Set.—Greenwich Signals.

Batch.	No. of the series.	Signal-time noted by Mr Dunkin at Greenwich.	Signal-time noted by Mr Todd at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
I.	1*	14.43.45,9	14.46.	
	2	43.57,3	46.26,0	26,0
	3	44.10,0	46,37,9	38,0
	4	44.19,9	46.48,8	49,1
	5*	44.30,1	46.58,0	58,9
II.	6*	14.45.19,9	14.48.	
	7	45.32,0	48. 0,1	0,2
	8	45.42,9	48.10,7	10,9
	9	45.53,0	48.21,0	21,1
	10	46. 3,9	48.32,0	32,2
	11	46.14,9	48.42,8	43,2
III.	12	14.48.45,0	14.51.13,0	13,2
	13	48.58,0	51.26,0	26,1
	14	49.10,8	51.39,0	39,1
	15	49.22,1	51.50,8	50,9
	16	49.33,1	52. 1,7	1,9
	17	49.46,1	52.14,4	14,2
	18	50. 0,0	52.28,0	28,0
IV*.	19*	14.50.55,0	14.53.	23,1
	20	51. 8,8	53.37,1	37,0
	21	51.23,7	53.51,8	52,0
	22	51.38,0	54. 6,0	6,1
V*.	23	14.52.53,1	14.55.20,3	20,1
	24	53. 7,1	55.35,2	35,2
	25	53.21,0	55.49,2	49,1
	26	53.34,9	56. 3,0	3,0
	27	54.48,9	56.17,0	17,1
	28	55. 2,8	56.31,1	31,0
	29	55.16,7	56.45,0	45,0

Second Set.—Cambridge Signals.

VI.	30*	14.57.42,0	15. 0.10,2	11,0
	31	57.55,8	0.24,3	24,1
	32	58. 8,0	0.36,0	36,1
	33	58.21,2	0.49,7	49,3
	34	58.34,3	1. 2,9	2,9

Second Set.—Cambridge Signals—(continued).

Batch.	No. of the series.	Signal-time noted by Mr Dunkin at Greenwich.	Signal-time noted by Mr Todd at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
VII.	35*	14.59.20,5	15. 1.49,0	
	36	59.32,7	2. 0,9	0,9
	37	59.46,6	2.15,0	14,8
VIII.	38*	15. 0.37,9	15. 3.16,2	6,1
	39	0.51,9	3.20,2	20,3
	40	1. 5,2	3.33,3	33,1
	41	1.17,2	3.45,8	45,8
	42	1.32,5	4. 0,8	0,6
	43*	1.45,2		
IX.	44	15. 2.29,9	15. 4.58,0	58,0
	45	2.43,0	5.11,2	11,1
	46	2.54,1	5.22,1	22,3
	47	3. 7,6	5.35,9	35,4
X*.	48	15. 3.58,2	15. 6.26,8	26,3
	49	4.12,4	6.40,8	40,7
	50	4.26,0	6.53,7	54,1
	51	4.38,9	7. 7,0	7,2
	52	4.52,8	7.21,0	21,2
	53	5. 5,0	7.32,7	33,3
	54	5.17,2	7.46,0	46,0
XI.	55	15. 6. 8,0	15. 8.36,0	36,1
	56	6.20,8	8.49,0	48,9
	57	6.33,1	9. 1,5	1,2
	58	6.46,0	9.14,0	14,0
	59	6.58,9	9.27,0	27,0
	60	7.13,0	9.41,5	41,2
XII*.	61	15. 8. 8,0	15.10.36,6	36,7
	62	8.22,0	10.51,3	51,3
	63	8.36,0	11. 5,3	5,0
	64	8.48.9	11.17,4	17,8
	65	9. 2,1	11.31,3	31,2
XIII.	66	15. 9.57,2	15.12.25,6	25,2
	67	10. 7,0	12.35,4	35,1
	68	10.21,2	12.49,4	49,4
	69	10.32,0	13. 0,5	0,6
	70	10.46,2	13.14,2	14,1
	71	10.58,2	13.26,9	27,2

Third Set.—Greenwich Signals.

Batch.	No. of the series.	Signal-time noted by Mr Dunkin at Greenwich.	Signal-time noted by Mr Todd at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
XIV.	72	15.12.54,0	15.15.22,0	22,3
	73	13. 7,9	15.36,2	36,1
	74	13.19,1	15.47,3	47,2
	75	13.30,0	15.58,1	58,1
XV*.	76	15.14.36,2	15.17. 5,0	4,9
	77	14.50,2	17.19,0	18,9
	78	15. 4,3	17.32,8	33,2
	79	15.19,2	17.47,0	47,1
	80	15.33,0	18. 1,2	1,2
	81	15.47,1	18.15,7	15,4
	82*	16. 1,2		
XVI.	83	15.17.13,9	15.19.42,8	43,1
	84	17.28,1	19.56,7	56,8
	85	17.40,3	20. 8,8	9,1
	86	17.54,0	20.22,0	22,0
	87	18. 8,0	20.36,3	36,8
	88*	18.23,0	20.51,1	51,6
XVII*.	89	15.20.24,0	15.22.52,8	52,6
	90	20.38,2	23. 6,7	6,9
	91*	20.52,1	23.20,7	19,6
	92	21. 6,1	23.34,5	34,0
	93	21.19,9	23.47,7	48,2
	94	21.33,7	24. 2,0	2,0
	95	21.47,4	24.15,8	15,2
XVIII*.	96	15.22.47,0	15.25.15,1	15,1
	97	23. 0,0	25.28,2	28,3
	98	23.13,8	25.42,1	42,2
	99	23.26,1	25.54,2	54,3
XIX*.	100	15.24.35,2	15.27. 3,7	3,9
	101	24.49,2	27.17,6	17,3
	102	25. 3,2	27.32,0	31,2
	103	25.17,0	27.45,2	45,2
	104	25.30,8	27.59,1	59,2
	105	25.44,8	28.12,9	13,0
	106	25.58,5	28.26,7	26,6

Fourth Set.—Cambridge Signals.

XX.	107	15.27.37,8	15.30. 5,9	6,0
	108	27.50,8	30.18,8	18,9

Fourth Set.—Cambridge Signals—(continued).

Batch.	No. of the series.	Signal-time noted by Mr Dunkin at Greenwich.	Signal-time noted by Mr Todd at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
XX.	109	15.28. 4,2	15.30.32,6	32,3
	110*		30.44,5	44,7
XXI.	111*	15.29. 1,5	15.31.30,8	30,1
	112	29.15,5	31.43,7	43,7
	113*	29.28,5	31.55,6	56,3
	114*		32. 9,7	9,8
	115	29.55,0	32.23,0	23,2
	116	30. 6,2	32.34,7	34,4
	117	30 21,2	32.49,3	49,2
XXII.	118	15.31. 1,5	15.33.29,6	29,3
	119	31.14,2	33.42,3	42,3
	120	31.26,1	33.54,6	55.0
	121	31.40,4	34. 8,7	8,7
	122	31.54,3	34.22,7	22,2
XXIII.	123	15.32.37,8	15.35. 6,2	6,1
	124	32.50,5	35.19,0	19,0
	125	33. 3,5	35.31,7	32,1
	126	33.16,0	35.44,3	44.2
	127	33.28,2	35.57,1	57,0
	128	33.43,0	36.10,8	11,1
XXIV.	129	15 34.26,2	15.36.54,7	54,4
	130	34.40,2	37. 8,7	9,2
	131	34.53,7	37.21,7	21,7
	132	35. 6,2	37.35,0	35,1
XXV.	133*	15.35.50,4	15.38.19.0	18,2
	134	36. 3,5	38.31,8	31,9
	135	36.16,3	38.44,4	44,4
	136	36.30,0	38.58,3	58,2
	137	36.43,3	39.11,4	11,4
	138	36.56,2	39.24,3	24,3
	139		39.39,9	39,8
XXVI.	140	15.37.54,0	15.40.22,1	22,1
	141	38. 7,2	40.35,2	35,3
	142	38.24,4	40.50,0	50,2
	143	38.34,0	41. 2,0	1,8
	144	38.46,3	41.14,7	14,9
XXVII*.	145	15.39.28,1	15.41.56,0	56,2
	146	39.39,2	42. 7,7	8,1

Fourth Set.—Cambridge Signals—(continued.)

Batch.	No. of the series.	Signal-time noted by Mr Dunkin at Greenwich.	Signal-time noted by Mr Todd at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
	147	39.56,9	42.24,7	25,3
XXVIII.	148	15.40.42,1	15.43.10,4	10,2
	149	40.56,8	43.24,9	25,1
	150	41. 8,2	43.36,4	36,3
	151	41.21,2	43.49,3	49,7
	152	41.36,2	44. 4,6	4,2
	153	41.48,3	44.16,8	16,4
	154	42. 1,3	44.29,8	29,8

Notes to the foregoing Observations.

⁂ The letters D, T, and B are put respectively for Mr Dunkin, Mr Todd, and Mr Breen. The Batches and Nos. of the series to which the notes apply have asterisks attached to them.

No. 5, 58,0 in T's observation is probably a mistake for 59,0.

Batch IV., Note by B, 'bad,—too quick.'

No. 30, B's observation appears to be 1^s in excess.

No. 38, T's observation is 10^s in excess, as shewn by B's.

Batches X. and XII., 'Noise.' (T).

Batch XII., 'At the end of this batch I found myself counting the clock a second in error.' (D).

No. 88, B's observation was marked 'bad.'

No. 91, B's observation is probably 1^s in defect.

Batch XVIII., 'good' (T); 'very good' (B).

Nos. 111 and 113, by comparison of T's times with B's it appears probable that 30,8 is a mistake for 29,8, and 55,6 for 56,6.

No. 133, T's signal-time was doubtful.

Batch XXVII., 'bad,—confused.' (T).

The noted times in batches V., XV., XVII., and XIX., are respectively the times of transit of B.A.C. 4941, 7 Serpentis, 10 Serpentis, and 11 Serpentis, across the wires of the Greenwich Transit Circle.

The omissions in Nos. 1, 6, 19, 35, 43, 82, 110 and 114 appear in most cases to be due to the observers being unprepared, which might arise either from a miscounting of the warning signals, or from the small intervals allowed between the batches. In the observations of May 18, the batches were separated by larger intervals, and in consequence a smaller number of signals were given in the course of the hour.

1853, May 18.

First Set.—Greenwich Signals.

Batch.	No. of the series.	Signal-time noted by Mr Todd at Greenwich.	Signal-time noted by Mr Dunkin at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
I.*	1	14.48.40,7	14.51.14,0	14,1
	2	48.55,2	51.27,9	28,0
	3	49. 9,0	51.42,0	41,9
	4	49.23,0	51.55,9	56,0
	5	49.36,9	52. 9,5	9,8
	6	49.50,7	52.23,0	23,5
	7	50. 4,6	52.37,0	37,1
II.	8*	14.51.	14.54. 2,3	2,1
	9	51.39,8	54.12,0	12,2
	10	51.50,7	54.23,0	23,3
	11*	52. 0,6	54.32,1	33,0
	12	52. 8,9	54.41,2	41,3
III.*	13	14.52.52,7	14.55.25,0	25,1
	14	53. 6,6	55.39,2	39,1
	15	53.20,3	55.52,9	53,0
	16	53.34,2	56. 7,0	7,0
	17	53.48,1	56.20,8	20,9
	18	54. 2,0	56.34,5	34,7
	19	54.15,7	56.48,1	48,3
IV.	20	14.55.35,2	14.58. 8,0	8,1
	21	55.47,3	58.20,1	20,3
	22	55.57,1	58.30,0	30,0
	23	56.10,4	58.43,1	43,2
V.	24	14.57.47,9	15. 0.20,3	20,3
	25	58. 2,7	0.35,1	35,7
	26	58.17,9	0.50,4	50,7
	27	58.28,9	1. 1,2	1,8
	28	58.43,0	1.15,4	15,3
	29	58.56,9	1.29,2	29,8
	30	59.12,0	1.44,2	44,4
	31	59.24,9	1.57,4	57,8
	32	59.37,0	2. 9,4	9,7

Second Set.—Cambridge Signals.

VI.	33	15. 1.52,1	15. 4.25,0	24,8
	34	2. 3,3	4.36,0	36,0
	35	2.17,0	4.49,9	49,9

Second Set.—Cambridge Signals—(continued).

Batch.	No. of the series.	Signal-time noted by Mr Todd at Greenwich.	Signal-time noted by Mr Dunkin at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
	36	15. 2.29,5	15. 5. 2.0	2,1
	37	2.43,7	5.16,1	16,0
VII.	38	15. 3.46,7	15. 6.19,0	19,1
	39	3.59,8	6.32,1	32,4
	40*	4.11,7	6.43,1	44,2
	41	4.24,3	6.57,1	57,1
	42	4.38,0	7.10,3	10,3
	43	4.50,7	7.23,3	23,3
	44	5. 4,4	7.37,0	37,0
VIII.	45	15. 6.10,2	15. 8.42,2	42,4
	46	6.21,7	8.54,1	54,0
	47	6.33,6	9. 6,1	6,1
	48*	6.46,6	9.20,2	19.9
IX.	49	15. 7.51,1	15.10.24,0	23,8
	50	8. 4,0	10.36,8	37,0
	51	8.17,9	10.50,5	50,4
	52	8.30,2	11. 3,0	2,9
	53	8.44,0	11.16,4	16,7
X.	54	15. 9.48,0	15.12.20,5	20,7
	55	10. 1,3	12.33,7	33,9
	56	10.14,2	12 46,9	46,9
	57	10.27,0	12.59,5	59,4
	58	10.39,9	13.12,4	12,8
	59	10.54,5	13.27,1	27,1
XI.	60	15.11.55,1	15.14.27,9	27,8
	61	12. 8,4	14.41,0	40,9
	62	12.20,0	14.52,5	52,7
	63*		15. 6,4	6,2
XII.	64	15.13.40,0	15.16.12,5	12,3
	65	13.52,2	16.24,9	24,8
	66	14. 5,4	16.37,8	37,9
	67	14.19,5	16.52,0	51,9
	68	14.32,4	17. 5,0	4.9
	69	14.44,8	17.17,1	17,6
	70	14.58,9	17.31,5	31,8
	71*		17.44,5	44,7

Third Set.—Greenwich Signals.

Batch.	No. of the series.	Signal-time noted by Mr Todd at Greenwich.	Signal-time noted by Mr Dunkin at Cambridge.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
XIII.	72*	15.17.25,2	15.19.57,0	57,8
	73	17.36,9	20. 9,4	9,3
	74	17.48,7	20.21,2	21,0
	75	18. 0,6	20.33,0	33,0
	76	18.13,6	20.46,1	46,4
	77	18.25,6	20.58,0	58,0
XIV.*	78*	15.20.23,8	15.22.56,3	56,9
	79	20.37,3	23.10,2	10,1
	80	20.51,6	23.24,1	24,0
	81	21. 5,3	23.38,1	38,1
	82	21.19,2	23.52,1	51,7
	83	21.32.9	24. 5,9	5,9
	84	21.46,7	24.19,2	19,6
XV.*	85	15.24.34,7	15.27. 7,5	7,3
	86	24.48,4	27.21,4	21,1
	87	25. 2,3	27.35,2	34,9
	88	25.16,4	27.49,1	48,9
	89	25.30,1	28. 2,8	2,8
	90	25.43,6	28,16.5	16,4
	91	25.57,5	28.30,5	30,3
XVI.*	92	15.28.12,2	15.30.44,9	45,1
	93	28.26,1	30.58,6	59,0
	94	28.39,7	31.12,5	12,4
	95	28.53,8	31.26,5	26,3
	96	29. 7,5	31.40,5	40,2
	97*	29.21,3	31.52,2	54,0
	98	29.35,2	32. 8,0	8,0

Fourth Set.—Cambridge Signals.

XVII.	99	15.31.37,2	15.34.10,0	9,9
	100	31.51,0	34.23,8	23,4
	101	32. 3,6	34.36,0	36,1
	102	32.15,3	34.47,9	48,0
	103	32.30,0	35. 2,5	2,7
XVIII.	104	15.33.31,5	15.36. 4,0	4,1
	105	33.43,9	36.16.7	16,7
	106	33.57,4	36.29,9	29,9
	107	34.11,4	36.44,0	43,8
	108	34.23,8	36.56,2	56,7

Fourth Set.—Cambridge Signals—(continued).

Batch.	No. of the series.	Signal-time noted by Mr Todd at Greenwich.	Signal-time noted by Mr Dunkin at Greenwich.	Seconds of signal-time noted by Mr Breen.
		h. m. s.	h. m. s.	s.
	109	15.34.37,0	15.37. 9,5	9,4
	110	34.50,7	37.23,5	23,3
XIX.	111	15.36.17,9	15.38.50,4	50,2
	112	36.31,4	39. 4,0	3,8
	113	36.44,4	39.17,0	17,0
	114	36.57,4	39.30,1	30,0
	115	37.10,7	39.43,2	43,1
	116	37.23,6	39.56,3	56,3
	117	37.36,5	40. 9,0	9,0
	118	37.47,9	40.20,4	20,3
	119	38. 0,1	40.33,0	33,2
XX.	120	15.38.59,5	15.41.32,1	32,2
	121	39.12,9	41.45,3	45,3
	122	39.25,6	41.58,2	58,2
	123	39.40,0	42.12,5	12,4
	124	39.53,5	42.26,3	26,2
XXI.	125	15.41. 0,9	15.43.33,5	33,7
	126	41.14,6	43.47,0	47,2
	127	41.26,7	43.59,2	59,4
	128	41.39,9	44.12,1	12,7
	129	41.53,7	44.26,5	26,4
	130	42. 6,7	44.39,3	39,2
	131*		44.53,0	53,0
XXII.	132	15.43.39,8	15.46.12,4	12,2
	133	43.53,8	46.26,4	26,3
	134	44. 7,4	46.40,0	40,0
	135	44.21,7	46.54,0	54,3
	136	44.35,6	47. 7,9	7,8
	137	44.49,7	47.22,2	22,3
	138	45. 3,8	47.36,5	36,5
	139	45.16,6	47.49,2	49,0

Notes to the foregoing Observations.

No. 11, in D's observation, 32,1 appears to be a mistake for 33,1.

No. 40, D's observation is 1ˢ in defect, as shewn by B's.

No. 48, the seconds of T's observation should probably be 47,6.

No. 72, in D's observation, 57,0 appears to be a mistake for 58,0.

No. 78, B's observation was doubtful on account of noise.

No. 97, D's signal-time is 2^s in defect, as appears by T's and B's.

Batches I., III., XIV., XV. and XVI. are respectively transits of 16 Libræ, B.A.C. 4941, 10 Serpentis, 11 Serpentis and A′ Serpentis across the wires of the Greenwich Transit-Circle.

The omissions in Nos. 8, 63, 71 and 131 are due apparently to the same accidents as those which occurred on May 17.

Some inferences may be drawn from the noted signal-times before reducing them to the sidereal times of the Observatories. In the first place, we may compare the times of the same signals as recorded by two observers. By 147 comparisons of the signal-times noted at Cambridge on May 17, it appears that Mr Breen observed *later* than Mr Todd by the mean interval $0^s,016$. On May 18, some of Mr Breen's signal-times in Batch V. appear to be anomalous, possibly by mistakes of half a second. If this batch be excluded, the mean result of 130 comparisons is, that Mr Breen observed *later* than Mr Dunkin by $0^s,020$. It seems, therefore, that Mr Dunkin's and Mr Todd's observations of signals agree very closely, and that little difference exists between the three observers in that kind of observation. This result is the more to be remarked because a decided difference of personal equation exists between Mr Breen's transits and those of Mr Todd. In the year 1848 Mr Breen observed *earlier* than Mr Todd by $0^s,35$. (See the *Cambridge Observations*, Vol. XVII. p. 55). By transits taken in 1853, on May 17 and May 19, the differences of the personal equations of these two observers were $0^s,17$ and $0^s,19$, and by transits taken on May 18 and May 19, the differences of the personal equations of Mr Dunkin and Mr Breen were $0^s,14$ and $0^s,18$, Mr Breen's observations being the earlier in each instance. It follows, therefore, that there is not necessarily the same personal equation in observing signals as in observing transits. If such were the case, the determination of longitude by signals would plainly not be affected by personal equation, provided the transits for clock-errors and the signals were observed by the same observer.

Again, from the unreduced signal-times, the interval occupied by the transmission of the galvanic action between Greenwich and Cambridge may be ascertained, if it be of sensible amount. In order to get a result independent of the rates of the time-pieces used for recording the signal-times, and of the comparisons of chronometers, the mean excess of the Cambridge signal-times above the Greenwich signal-times, as derived from the first and third sets, will be compared with the mean excess derived from the second set, and the mean excess of the Cambridge above the Greenwich signal-times given by the second and fourth sets will be compared with the excess given by the third set. It will only be necessary to indicate the fractional parts of these excesses. Let G_1, G_3 represent the fractional parts of the mean excess of the Cambridge above the Greenwich signal-times for the first and third sets of signals, or those given at *Greenwich*, and C_2, C_4 represent the like quantities for the second and fourth sets, or those given at *Cambridge*. Then the following results may be obtained.

By D's and T's signal-times, May 17.

No. of comparisons.

$G_1 = \overset{s.}{0},185$ 26

$G_3 = 0,344$ 27

$\dfrac{G_1 + G_3}{2} = 0,264$ 53

$C_2 = 0,290$ 41

$C_4 = 0,264$ 45

$\dfrac{C_2 + C_4}{2} = 0,277$ 86

$\therefore \dfrac{G_1 + G_3}{2} - C_2 = - \overset{s.}{0},026,$

and $G_3 - \dfrac{C_2 + C_4}{2} = + 0,067.$

By D's and B's signal-times, May 17.

No. of comparisons.

$G_1 = \overset{s.}{0},281$ 27

$G_3 = 0,362$ 27

$\dfrac{G_1 + G_3}{2} = 0,322$ 54

$C_2 = 0,320$ 40

$C_4 = 0,269$ 45

$\dfrac{C_2 + C_4}{2} = 0,295$ 85

$\therefore \dfrac{G_1 + G_3}{2} - C_2 = + \overset{s.}{0},002,$

and $G_3 - \dfrac{C_2 + C_4}{2} = + 0,067.$

By T's and D's signal-times, May 18.

No. of comparisons.

$G_1 = \overset{s.}{0},548$ 31

$G_3 = 0,726$ 27

$\dfrac{G_1 + G_3}{2} = 0,637$ 58

$C_2 = 0,557$ 37

$C_4 = 0,568$ 40

$\dfrac{C_2 + C_4}{2} = 0,563$ 77

$\therefore \dfrac{G_1 + G_3}{2} - C_2 = + \overset{s.}{0},080,$

and $G_3 - \dfrac{C_2 + C_4}{2} = + 0,163.$

By T's and B's signal-times, May 18.

No. of comparisons.

$G_1 = \overset{s.}{0},726$ 31

$G_3 = 0,678$ 27

$\dfrac{G_1 + G_3}{2} = 0,702$ 58

$C_2 = 0,589$ 37

$C_4 = 0,578$ 40

$\dfrac{C_2 + C_4}{2} = 0,584$ 77

$\therefore \dfrac{G_1 + G_3}{2} - C_2 = + \overset{s.}{0},113,$

and $G_3 - \dfrac{C_2 + C_4}{2} = + 0,094.$

If we consider all the eight results to be of equal weight, the mean excess of the Cambridge signal-times above the Greenwich signal-times (the time-pieces having virtually the same rates), is greater for the signals given at Greenwich than for those given at Cambridge by the mean quantity $0^s,070$. The interval occupied by the transmission of the galvanic action is therefore apparently $0^s,035$. This quantity can only be regarded as a limit which the interval of transmission did not exceed. The whole of it may with much probability be accounted for by supposing the start of the needle to catch the eye of the observer *sooner* at the signal-giving place than at the signal-receiving place, the galvanic action being of greater intensity at the former. During the whole of the experiment the air was dry and the insulation good. The number of battery-cells used at each station was 72.

I proceed now to state by what means the signal-times were converted into sidereal times of the respective Observatories. Two methods were employed, which for distinction I will call method (A) and method (B). In the former, the errors of the transit-clocks were determined by the ordinary processes employed at the two Observatories, excepting that the assumed apparent Right Ascensions of the stars observed for clock-errors at Cambridge, after being taken from the Nautical Almanac, were corrected by small quantities derived from the results of *Greenwich* observations, so that their assumed mean R. A. 1853,0 are the same as those adopted for the year at Greenwich. The following are the stars which were observed for clock-errors on the nights of May 17 and May 18, with their assumed apparent Right Ascensions.

At Greenwich, May 17.

Name of Star.	Apparent R. A.
	h. m. s.
Aldebaran............	4.27.28,11
δ Virginis............	12.48.12,98
η Bootis	13.47.42,40
ζ Herculis	16.35.46,28

At Cambridge, May 17.

	h. m. s.
Aldebaran............	4.27.28,11
α Orionis	5.47.11,75
δ Crateris............	11.12. 0,12
β Leonis	11.41.34,14
β Corvi	12.26.41,28
Spica	13.17.28,32

At Greenwich, May 18.

Name of Star.	Apparent R. A.
	h. m. s.
δ Leonis	11. 6.17,59
δ Crateris	11.12. 0,11
β Leonis	11.41.34,13
β Corvi	12.26.41,27
δ Virginis............	12.48.12,98
η Bootis	13.47.42,39
κ Virginis............	14. 5. 4,78
ζ Herculis	16.35.46,30
η Ophiuchi	17. 1.58,47
α Herculis	17. 7.58,10
α Ophiuchi	17.28. 8,03

At Cambridge, May 18.

	h. m. s.
β Leonis	11.41.34,13
Spica	13.17.28,32
α Ophiuchi	17.28. 8,03

In method (B), a list of stars having been previously prepared, clock-errors were deduced from the observations of those that were observed at both Observatories on the same night, the same values of their apparent Right Ascensions being employed for that purpose. The following are the stars so used. Their apparent R. A., which it is unnecessary to subjoin, were derived from the Greenwich Twelve-year Catalogue.

Stars observed at Greenwich
and Cambridge, May 17.

Aldebaran
38 Virginis
δ Virginis
g Virginis
53 Virginis
61 Virginis
l^2 Virginis

Stars observed at Greenwich
and Cambridge, May 18.

β Leonis
γ Virginis
38 Virginis
δ Virginis
g Virginis
53 Virginis
61 Virginis
l^2 Virginis
32 Ophiuchi
η Ophiuchi
λ Herculis
α Ophiuchi

It is needless to say that the determination of the instrumental errors of the Transits was carefully attended to at both Observatories. The following reduced results of the Greenwich transit observations, communicated to me by Mr Airy, suffice for converting the signal-times noted at Greenwich into sidereal times.

Method (A).

	h. m.		s.
May 17.	11.0	Mean Time, Transit-Clock slow	+ 8,77
	12.0	....	+ 8,79
May 18.	11.0	....	+ 9,48
	12.0	....	+ 9,51

Method (B).

	h. m.		s.
May 17.	11.0	Mean Time, Transit-Clock slow	+ 8,87
	12.0	....	+ 8,89
May 18.	11.0	....	+ 9,54
	12.0	....	+ 9,57

In the reduction of the Cambridge transit observations, the level error was the mean of two levellings, taken on May 17 and May 21; the collimation error was the mean of two results obtained by the collimating eye-piece on May 24; and the azimuth error was deduced from transits of Polaris SP and Polaris on May 16. The three errors were calculated on the supposition that the pivots of the transit-instrument are cylindrical and equal, (which, however, is not strictly the case), and the errors of the transit-clock were determined accordingly. The longitude will first be calculated from clock-errors founded on the above supposition, and I shall afterwards correct the result by taking account of the true forms of the pivots. The following are the errors of the transit-clock (Hardy), deduced immediately from the observations.

Method (A).

	h. m.		m. s.
May 17, at	9.49	sidereal time, Hardy slow	1.20,66
18, at	14. 9	....	1.22,55

Method (B).

	h. m.		m. s.
May 17, at	11.49	sidereal time, Hardy slow	1.20,84
18, at	14.17	....	1.22,72

The daily losing rate of Hardy was found to be 1ˢ,58 by observations of Mr Todd on May 16 and 17, 1ˢ,58 and 1ˢ,59 by observations of Mr Breen on May 17, 18 and 19, and 1ˢ,57 by observations of Mr Dunkin on May 18 and 19. The adopted losing rate for May 17 and May 18 is 1ˢ,58.

It is now required to state what comparisons were made between the transit-clock and the three chronometers, for the purpose of converting the noted signal-times into sidereal times of the Cambridge Observatory. The transit-clock will be designated by the letter H, the mean-time chronometer by the letter W, and the sidereal-time chronometers by X and U.

The chronometer X is that which was used for noting the signal-times. The comparisons were made in two sets to serve as checks to each other: it will only be necessary to give here the comparisons which were made use of, which are generally means between two sets.

May 16.

h. m. s. h. m. s.
12.46.55,5 W = 16.20.51,0 U
12.49.31,0 W = 16.28.57,5 X
12.53. 2,0 W = 16.29.14,0 H
21.22.14,0 W = 0.57.29,0 U
21.23.22,5 W = 1. 4.11,5 X
21.29.27,5 W = 1. 7. 1,5 H
21.32.25,5 W = 1.13.16,0 X

May 17.

h. m. s. h. m. s.
2.10. 0,0 W = 5.51.35,0 X
2.13.20,0 W = 5.51.39,0 H
2.17.44,0 W = 5.53.44,5 U
7.35. 0,0 W = 11.14.10,0 H
7.37. 4,5 W = 11.19.32,0 X
7.40.11,0 W = 11.17. 1,0 U
9.46.43,5 W = 13.23.53,0 U
9.47.39,0 W = 13.30.27,5 X
9.50. 5,5 W = 13.29.37,0 H
10.31. 5,0 W = 14.14. 0,5 X
10.46.30,0 W = 14.29.28,0 X
10.48.46,0 W = 14.26. 5,0 U
10.52.38,0 W = 14.35.37,0 X
12. 7.22,5 W = 15.50.33,5 X
12. 9.33,5 W = 15.47. 5,0 U
12.57.41,5 W = 16.37.43,0 H

May 17.

h. m. s. h. m. s.
12.59.34,0 W = 16.42.55,5 X
13.11.25,0 W = 16.49. 6,0 U
13.18.10,5 W = 17. 1.35,0 X
13.22.49,5 W = 17. 2.55,0 H
13.29.25,5 W = 17. 9.32,0 H
13.36.35,0 W = 17.20. 2,5 X

May 18.

h. m. s. h. m. s.
9.24.51,5 W = 13. 8. 8,0 H
9.38. 2,0 W = 13.24.42,5 X
9.41. 9,0 W = 13.27.50,0 X
9.44.44,5 W = 13.25.32,0 U
9.49.25,5 W = 13.32.46,0 H
10.34.24,5 W = 14.15.19.5 U
10.40.57,0 W = 14.21.53,0 U
10.45.19,0 W = 14.32.11,0 X
10.48.27,5 W = 14.35.20,0 X
12. 6. 8,0 W = 15.53.13,0 X
12. 9.24,0 W = 15.50.33,5 U
13. 4. 4,5 W = 16.47.56,0 H
13. 4.39,5 W = 16.48.31,0 H
13. 8. 0,0 W = 16.55.16,0 X
13. 8.37,0 W = 16.55.53,0 X
13.11.26,0 W = 16.52.45,0 U
13.11.56,0 W = 16.53.15,0 U

From the above comparisons, and the errors of Hardy given by the method (A), the following errors of W on Cambridge mean time, and of X and U on Cambridge sidereal time were calculated.

	Mean Time. h. m.		m. s.
May 16.	12.52......	W fast	1. 1,70
	21.28......	——	1. 3,98
17.	2.12......	——	1. 5,29
	7.34......	——	1. 6,78
	9.49......	——	1. 7,31
	12.57......	——	1. 7,93

	Mean Time. h. m.		m. s.
May 17.	13.22......	W fast	1. 8,03
	13.28......	——	1. 8,10
18.	9.24......	——	1.13.11
	9.48......	——	1.13,13
	13. 3......	——	1.13,89
	13. 3......	——	1.13,98

	Mean Time. h. m.		m. s.
May 16.	12.52......	X fast	1.55,53
	21.28......	——	1.55,89
	21.28......	——	1.55,94
17.	2.12......	——	1.56,12
	7.34......	——	1.56,42
	9.49......	——	1.56,49
	12.56......	——	1.58,61
May 16.	12.46......	U slow	3.35,04
	21.21......	——	3.37,92
17.	2.16......	——	3.39,60
	7.39......	——	3.41,58
	9.45......	——	3.42,37

	Mean Time. h. m.		m. s.
May 17.	13.22......	X fast	1.58,61
	13.28......	——	1.58,72
18.	9.37......	——	1.59,32
	9.40......	——	1.59,38
	13. 3......	——	2. 1,15
	13. 3......	——	2. 1,14
May 17.	13.10......	U slow	3.43,83
18.	9.43......	——	3.54,76
	13.10......	——	3.56,42
	13.10......	——	3.56,40

These results shew that on both days from 10^h to 13^h, that is, during the intervals occupied in transporting the chronometers and taking the observations, W and U maintained rates not materially differing from their general rates inferred from the comparisons on May 16, 17, and 18, but that this was not the case with the Chronometer X. This chronometer was selected for recording the signal-times because its rate was known to be much smaller and generally more steady than the rates of the other two. The appearance of its errors leads to the suspicion that half-seconds were gained during the transports by *tripping*, and that the rates were in other respects good. I made a calculation of the longitude on this supposition, assuming four half-seconds to have been gained in this way on May 17 and three half-seconds on May 18, and I found that the longitude resulting from the observations of May 17 would thus be diminished by $0^s,020$, and that resulting from the observations of May 18 be increased by $0^s,020$, equal weights being given to the three chronometers. As therefore the final result would not be affected, and as the above supposition may be somewhat precarious, I have preferred trusting exclusively to the Chronometers W and U. The calculations will therefore be made on the supposition that these chronometers maintained uniform rates from the times of comparison with Hardy before leaving the Observatory, to the times of comparison on returning to the Observatory, and errors of X will be deduced by means of the comparisons made at the station, first from W's errors, and then from U's errors. On May 17 from $9^h. 49^m$ to $12^h. 57^m$, W gained $0^s,62$, and on May 18 from $9^h. 48^m$ to $13^h. 3^m$, the gain was $0^s,76$. On the first day U lost $1^s,46$ from $9^h. 45^m$ to $13^h. 10^m$, and on the second day the loss was $1^s,66$ from $9^h. 43^m$ to $13^h. 10^m$. From these data and the comparisons of X and U with W at the station the following errors of X may be obtained.

	Mean Time. h. m.		m. s.		m. s.
May 17.	10.30...	X fast by W...	1.56,50...	X fast by U...	1.56,53
	10.45...	——————	...1.56.52...	——————	...1.56,55
	10.51...	——————	...1.56.53...	——————	...1.56,58
	12. 6...	——————	...1.56,50...	——————	...1.56,47

	Mean Time.		
	h. m.	m. s.	m. s.
May 18.	10.44	...X fast by W...2. 0,03	...X fast by U...2. 0,10
	10.47	...2. 0,04	...2. 0,10
	12. 5	...2. 0,05	...2. 0,11

Some of the comparisons were made under disturbance by railway noises. Giving double weight to the results by Chronometer W, as depending on direct comparison with X, we obtain as follows:—

Method (A).

	h.	m. s.
May 17.	11 mean time, X fast......	1.56,52
	12	1.56,49
May 18.	11	2. 0,06
	12	2. 0,07

By these errors of X the signal-times were converted into Cambridge sidereal times, each of which, compared with the corresponding Greenwich sidereal time, gives a separate determination of the longitude. Subjoined are the individual results classed in groups, according as the signals were made at Greenwich or Cambridge. The reference numbers correspond to the series-numbers in the lists of signal-times. In this calculation Mr Breen's observations are not made use of.

DETERMINATIONS OF THE LONGITUDE BY THE SIGNALS OF MAY 17.

First Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
2	14.44. 6,07	29,48	23,41
3	44.18,77	41,38	22,61
4	44.28,67	52,28	23,61
5	44.38,67	61,48	22,61
7	45.40,77	63,58	22,81
8	45.51,67	74,18	22,51
9	46. 1,77	24,48	22,71
10	46.12,67	35,48	22,81
11	46.23,67	46,28	22,61
12	48.53,77	76,48	22,71
13	49. 6,77	29,48	22,71
14	49.19,57	42,48	22,91
15	49.30,87	54,28	23,41
16	49.41,87	65,18	23,31
17	49.54,87	77,88	23,01
18	50. 8,77	31,48	22,71
20	51.17,57	40,58	23,01
21	51.32,47	55,28	22,81

First Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
22	14.51.46,77	69,48	22,71
23	53. 0,87	23,78	22,91
24	53.15,87	38,68	22,81
25	53.29,77	52,68	22,91
26	53.43,67	66,48	22,81
27	53.57,67	80,48	22,81
28	54.11,57	34,58	23,01
29	54.25,47	48,48	23,01

Second Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
30	14.57.50,77	73,69	22,92
31	58. 4,57	27,79	23,22
32	58.16,77	39,49	22,72
33	58.29,97	53,19	23,22
34	58.43,07	66,39	23,32
35	59.29,27	52,49	23,22
36	59.41,47	64,39	22,92

Second Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
37	14.59.55,37	78,49	23,12
38	15. 0.46,68	69,69	23,01
39	1. 0,68	23,69	23,01
40	1.13,98	36,79	22,81
41	1.25,98	49,29	23,31
42	1.41,28	64,29	23,01
44	2.38,68	61,49	22,81
45	2.51,78	74,69	22,91
46	3. 2,88	25,59	22,71
47	3.16,38	39,39	23,01
48	4. 6,98	30,29	23,31
49	4.21,18	44,29	23,11
50	4.34,78	57,19	22,41
51	4.47,68	70,49	22,81
52	5. 1,58	24,49	22,91
53	5.13,78	36,19	22,41
54	5.25,98	49,49	23,51
55	6.16,78	39,49	22,71
56	6.29,58	52,49	22,91
57	6.41,88	64,99	23,11
58	6.54,78	77,49	22,71
59	7. 7,68	30,49	22,81
60	7.21,78	44,99	23,21
61	8.16,78	40,09	23,31
62	8.31,78	54,79	23,01
63	8.45,78	68,79	23,01
64	8.57,68	80,89	23,17
65	9.11,88	34,79	22,91
66	10. 5,98	29,09	23,11
67	10.15,78	38,89	23,11
68	10.29,98	52,89	22,91
69	10.40,78	63,99	23,21
70	10.54,98	77,69	23,71
71	11. 6,98	30,39	22,41

Third Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
72	15.13. 2,78	25,50	22,72
73	13.16,68	39,70	23,02
74	13.27,88	50,80	22,92
75	13.38,78	61,60	22,82
76	14.44,98	68,50	23,52
77	14.58,98	82,50	23,52
78	15.13,08	36,30	23,22

Third Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
79	15.15.27,98	50,50	22,52
80	15.41,78	64,70	22,92
81	15.55,88	79,20	23,32
83	17.22,68	46,30	23,62
84	17.36,88	60,20	23,32
85	17.49,08	72,30	23,22
86	18. 2,78	25,50	22,72
87	18.16,78	39,80	23,02
88	18.31,78	54,60	22,82
89	20.32,78	56,30	23,52
90	20.46,98	70,20	23,22
91	21. 0,88	24,20	23,32
92	21.14,88	38,00	23,12
93	21.28,68	51,20	22,32
94	21.42,48	65,50	23,02
95	21.56,18	79,30	23,12
96	22.55,78	78,60	22,82
97	23. 8,78	31,70	22,92
98	23.22,58	45,60	23,02
99	23.34,88	57,70	22,82
100	24.43,99	67,20	23,21
101	24.57,99	81,10	23,11
102	25.11,99	35,50	23,51
103	25.25,79	48,70	22,91
104	25.39,59	62,60	23,01
105	25.53,59	76,30	22,81
106	26. 7,21	30,20	22,91

Fourth Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
107	15.27.46,59	69,41	22,82
108	27.59,59	82,31	22,72
109	28.12,99	36,11	23,12
111	29.10,29	33,31	23,02
112	29.24,29	47,21	22,92
113	29.37,29	60,11	22,82
115	30. 3,79	26,51	22,72
116	30.14,99	38,21	23,22
117	30.29,99	52,81	22,82
118	31.10,29	33,11	22,82
119	31.22,99	45,81	22,82
120	31.34,89	58,11	23,22
121	31.49,19	72,21	23,02
122	32. 3,09	26,21	23,12

Fourth Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
123	15.32.46,59	69,71	23,12
124	32.59,29	82,51	23,22
125	33.12,29	35,21	22,92
126	33.24,79	47,81	23,02
127	33.36,99	60,61	23,62
128	33.51,79	74,31	22,52
129	34.34,99	58,21	23,22
130	34.48,99	72,21	23,22
131	35. 2,49	25,21	22,72
132	35.14,99	38,51	23,52
133	35.59,19	82,51	23,32
134	36.12,29	35,31	23,02
135	36.25,09	47,91	22,82
136	36.38,79	61,81	23,02
137	36.52,09	74,91	22,82
138	37. 4,99	27,81	22,82

Fourth Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
140	15.38. 2,79	25,61	22,82
141	38.15,99	38,71	22,72
142	38.30,19	53,51	23,32
143	38.42,79	65,51	22,72
144	38.55,09	78,21	23,12
145	39.36,89	59,51	22,62
146	39.47,99	71,21	23,22
147	40. 5,69	28,21	22,52
148	40.50,89	13,91	23,02
149	41. 5,59	28,41	22,82
150	41.16,99	39,91	22,92
151	41.29,99	52,81	22,82
152	41.44,99	68,11	23,12
153	41.57,09	20,31	23,22
154	42.10,09	33,31	23,22

Notes.

No. 5, T.'s observation has not been altered. No. 23, the Greenwich time has been increased by 1^{s}. No. 62, 63, and 65, the Greenwich times have each been increased by 1^{s}. (See the observations.) Nos. 111 and 113, the Cambridge time has been corrected by -1^{s} in the former, and by $+1^{s}$ in the other.

DETERMINATIONS OF THE LONGITUDE, BY THE SIGNALS OF MAY 18.

First Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
1	14.48,50,18	73,94	23,76
2	49. 4,68	27,84	23,16
3	49.18,48	41,94	23,46
4	49.32,48	55,84	23,36
5	49.46,38	69,44	23,06
6	50. 0,18	22,94	22,76
7	50.14,08	36,94	22,86
9	51.49,28	71,94	22,66
10	52. 0,18	22,94	22,76
11	52.10,08	33,04	22,96
12	52.18,38	41,14	22,76
13	53. 2,18	24,94	22,76
14	53.16,08	39,14	23,06

First Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
15	14.53.29,78	52,84	23,06
16	53.43,68	66,94	23,26
17	53.57,58	80,74	23,16
18	54.11,48	34,44	22,96
19	54.25,18	48,04	22,86
20	55.44,69	67,94	23,25
21	55.56,79	80,04	23,25
22	56. 6,59	29,94	23,35
23	56.19,89	43,04	23,15
24	57.57,39	80,24	22,85
25	58.12,19	35,04	22,85
26	58.27,39	50,34	22,95
27	58.38,39	61,14	22,75

First Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
28	14.58.52,49	75,34	22,85
29	59. 6,39	29,14	22,75
30	59.21,49	44,14	22,65
31	59.34,39	57,34	22,95
32	59.46,49	69,34	22,85

Second Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
33	15. 2. 1,59	24,94	23,35
34	2.12,79	35,94	23,15
35	2.26,49	49,84	23,35
36	2.38,99	61,94	22,95
37	2.53,19	76,04	22,85
38	3.56,19	78,94	22,75
39	4. 9,29	32,04	22,75
40	4.21,19	44,04	22,85
41	4.33,79	56,94	23,15
42	4.47,49	70,24	22,75
43	5. 0,19	23,24	23,05
44	5.13,89	36,94	23,05
45	6.19,69	42,14	22,45
46	6.31,19	54,04	22,85
47	6.43,09	66,04	22,95
48	6.56,09	79,14	23,05
49	8. 0,59	23,94	23,35
50	8.13,49	36,74	23,25
51	8.27,39	50,44	23,05
52	8.39,69	62,94	23,25
53	8.53,49	76,34	22,85
54	9.57,49	80,44	22,95
55	10.10,79	33,64	22,85
56	10.23,69	46,84	23,15
57	10.36,49	59,44	22,95
58	10.49,39	72,34	22,95
59	11. 3,99	27,04	23,05
60	12. 4,59	27,84	23,25
61	12.17,89	40,94	23,05
62	12.29,89	52,84	22,95
64	13.49,49	72,44	22,95
65	14. 1,69	24,84	23,15
66	14.14,89	37,74	22,85
67	14.28,99	51,94	22,95
68	14.41,89	64,94	23,05
69	14.54,29	77,04	22,75
70	15. 8,39	31,44	23,05

Third Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
72	15.17.34,70	57,93	23,23
73	17.46,40	69,33	22,93
74	17.58,20	81,13	22,93
75	18.10,10	32,93	22,83
76	18.23,10	46,03	22,93
77	18.35,10	57,93	22,83
78	20.33,30	56,23	22,93
79	20.46,80	70,13	23,33
80	21. 1,10	24,03	22,93
81	21.14,80	38,03	23,23
82	21.28,70	52,03	23,33
83	21.42,40	65,83	23,43
84	21.56,40	79,33	22,93
85	24.44,20	67,43	23,23
86	24.57,90	81,33	23,43
87	25.11,80	35,13	23,33
88	25.25,90	49,03	23,13
89	25.39,60	62,73	23,13
90	25.53,10	76,43	23,33
91	26. 7,00	30,43	23,43
92	28.21,70	45,03	23,33
93	28.35,60	58,53	22,93
94	28.49,20	72,43	23,23
95	29. 3,30	26,43	23,13
96	29.17,00	40,43	23,43
97	29.30,80	54,13	23,33
98	29,44,70	67,93	23,23

Fourth Set.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
99	15.31.46,70	69,93	23,23
100	32. 0,50	23,73	23,23
101	32.13,10	35,93	22,83
102	32.24,80	47,83	23,03
103	32.39,50	62,43	22,93
104	33.41,00	63,93	22,93
105	33.53,40	76,63	23,23
106	34. 6,90	29,83	22,93
107	34.20,90	43,93	23,03
108	34.33,30	56,13	22,83
109	34.46,50	69,43	22,93
110	35. 0,20	23,43	23,23
111	36.27,41	50,33	22,92
112	36.40,91	63,93	23,02

Fourth Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
113	15.36.53,91	76,93	23,02
114	37. 6,91	30,03	23,12
115	37.20.21	43,13	22,92
116	37.33,11	56,23	23,12
117	37.46,01	68,93	22,92
118	37.57,41	80,33	22,92
119	38. 9,61	33,93	23,32
120	39. 9,01	32,03	23,02
121	39.22,41	45,23	22,82
122	39.35,11	58,13	23,02
123	39.49,51	72,43	22,92
124	40. 3,01	26,23	23,22
125	41.10,41	33,43	23,02

Fourth Set—continued.

Reference Number.	Greenwich Sidereal Time.	Seconds of Cambridge Sidereal Time.	Excess of Cambridge Sidereal Time.
	h. m. s.	s.	s.
126	15.41.24,11	46,93	22,82
127	41.36,21	59,13	22,92
128	41.49,41	72,03	22,62
129	42. 3,21	26,43	23,22
130	42.16,21	39,23	23,02
132	44.49,31	72,33	23,02
133	45. 3,31	26,33	23,02
134	45.16,91	39,93	23,02
135	45.31,21	53,93	22,72
136	45.45,11	67,83	22,72
137	45.59,21	82,13	22,92
138	46.13,31	36,43	23,12
139	46.26,11	49,13	23,02

Notes.

Nos. 11 and 40, the Cambridge times have been increased 1ˢ. (See the observations). No. 48, the Greenwich time has been corrected by + 1ˢ. No. 72, the Cambridge time has been corrected by + 1ˢ. No. 97, the Cambridge time has been corrected by + 2ˢ.

The mean results given by the several sets and by the sets collectively are as follows :—

May 17.

	No. of signals.	Mean.
		s.
First set (I.)	26 ...	22,895
Third set (III.)......	34 ...	23,062
(I.) + (III.)	60 ...	22,990

	No. of signals.	Mean.
		s.
Second set (II.)......	41 ...	23,002
Fourth set (IV.) ...	45 ...	22,984
(II.) + (IV.)	86 ...	22,993

Mean from all the sets = 22ˢ,992.

May 18.

	No. of signals.	Mean.
		s.
First set (I.)	31 ...	23,004
Third set (III.)......	27 ...	23,161
(I.) + (III.)	58 ...	23,078

	No. of signals.	Mean.
		s.
Second set (II.)	37 ...	22,999
Fourth set (IV.) ...	40 ...	22,995
(II.) + (IV.)	77 ...	22,997

Mean from all the sets = 23ˢ,032.

May 17, Excess of (I.) + (III.) above (II.) + (IV.) = − 0ˢ,003
18, ... = + 0,081
Mean = + 0,039

Hence the interval apparently occupied by the transmission of the galvanic action is by this calculation $0^{s},0195$. I think the former result to be rather more trustworthy.

The results of the method (B) may be deduced from those of the method (A), by merely applying the differences of the errors of the transit-clocks at the same epochs as given by the two methods, since the rates may be assumed to be the same as determined by either method. Now on May 17, the Greenwich transit-clock was $0^{s},10$ slower, and the Cambridge transit-clock $0^{s},05$ slower, by method (B) than by method (A). Hence $-0^{s},05$ is to be applied to the longitude found by the latter method to obtain that given by the former. On May 18, the Greenwich transit-clock was $0^{s},06$ slower, and the Cambridge transit-clock $0^{s},16$ slower, by method (B) than by method (A), and the correction is consequently $+0^{s},10$. The final results are, therefore, as follows:—

East Longitude of the Cambridge Observatory.

Method (A).

	s.
May 17, by 146 signals......	22,922
18, by 135 signals......	23,038
Mean......	23,015

Method (B).

May 17, by 146 signals......	22,942
18, by 135 signals......	23,138
Mean......	23,040
Mean of the whole......	23,027

Shortly after the experiment was made I calculated the longitude provisionally, and communicated the results to the Astronomer Royal, stating at the same time that they would probably be modified by calculations I proposed to make respecting the effect of the forms of the pivots of the Cambridge Transit. The difference between these first results (which are published in the *Monthly Notices of the Royal Astronomical Society*, Vol. XIII. No. 8, p. 251) and those obtained above, is owing partly to slight mistakes discovered in the former calculation, and partly to the circumstance that it corrected in an imperfect manner for the forms of the pivots, whilst the present calculation supposes the transit-pivots to be cylindrical and equal. Having recently gone through a complete investigation of the corrections required to be applied for inequalities of the pivots, according to the method which I have descrided in the *Memoirs of the Royal Astronomical Society* (Vol. XIX. p. 103), I will now state the effect of these corrections on the determination of the longitude, and then add the evidence on which the necessity for such corrections rests.

The forms of the pivots being taken into account, the following are the errors of Hardy:—

Method (A).

	h. m		m. s.
May 17, at	9.49	sidereal time, Hardy slow	1.20,328
18, at	14. 9	....	1.22,243

Method (B).

		h. m.		m. s.
May 17,	at	11.49	sidereal time, Hardy slow	1.20,497
18,	at	14.17	....	1.22,380

Comparing these with the errors of Hardy already used, the corrections of the longitude are $-0^s,332$ for method (A) and $-0^s,343$ for method (B) on May 17, and $-0^s,307$ for method (A) and $-0^s,340$ for method (B) on May 18. The following are consequently the corrected determinations of the longitude:—

	s.
May 17 by method (A).........	22,660
18	22,731
Mean.........	22,696
	s.
May 17, by method (B).........	22,599
18,...........................	22,798
Mean.........	22,698
Concluded Mean	22,697

The accordance of the results by the two methods is sufficiently close on each day, and the mean results of the two days are extremely close. It will be seen that this agreement has been improved by the application of corrections for the forms of the pivots. I am at a loss to account for the mean excess $0^s,135$ of the determination of May 18 above that of May 17, as (at least under the same circumstances) there appeared to be no difference of personal equation between the observers.

It remains to give an account of the observations and calculations by which the corrections for the forms of the pivots were ascertained. Let ξ represent the angle which a line drawn from the middle wire of the transit-telescope, at the point where the transits are taken, and passing through the optical centre of the object-glass, makes with the plane of the meridian. This angle was ascertained for various zenith distances by the method of bisection of dots at the extremities of the pivots, as described in the communication to the Royal Astronomical Society already cited. The same angle was found on the supposition of equal and cylindrical pivots, the collimation and level errors being determined by the collimating eye-piece and by reversing the instrument. Let ξ_1 be its value thus determined. Then as exactly the same data (given by the collimating eye-piece and by transits of Polaris) were used in the two calculations, if the supposition of equal and cylindrical pivots had been correct, there should have been no constant difference between the values of ξ and ξ_1. The actual differences as determined by trials made in 1850 and 1854 are subjoined. A trial made in 1852 failed on account of accidental unsteadiness in the attachment of one of the dots to the pivot. The level error used in the observations of 1853, May 16—18, for the clock-errors on those days, was obtained in the ordinary manner by levelling with the spirit-level. By the mean of six comparisons of levellings obtained in this manner in 1852 and 1854, with an equal number obtained simultaneously by the collimating eye-piece and reversing the instrument, the constant excess of the former was found to be $-2'',65$, for the position of the instrument in which the lamp for illuminating the field of view is westward. In the subjoined table ξ_2 is the value of ξ_1 which would be given by levelling with the spirit-level, and $(\xi-\xi_1)+(\xi_1-\xi_2)$ or $\xi-\xi_2$ is the quantity required for calculating the effect of the forms of the pivots on the observations of transits.

Zenith Distance South.	Illumination East.		Illumination East.		Illumination West.		Illumin. W. $(\xi-\xi_2)$ 1854.	Correction to time of transit. 1854, Illum. W.
	$(\xi-\xi_1)$ June 19, 1850.	$(\xi-\xi_1)$ July 16, 1850.	$(\xi-\xi_1)$ April 4, 1854.	$(\xi-\xi_1)$ April 12, 1854.	$(\xi-\xi_1)$ April 6, 1854.	$(\xi-\xi_1)$ April 12, 1854.		
	″	″	″	″	″	″	″	s.
+ 90°	+ 3,9	+ 4,6	+ 5,6	+ 6,3	− 5,3	− 5,1	+ 5,2	+ 0,44
80	+ 3,8	+ 4,6	+ 5,7	+ 6,3	− 5,4	− 5,2	+ 5,4	+ 0,41
70	+ 3,7	+ 4,4	+ 5,4	+ 6,2	− 5,4	− 5,0	+ 5,7	+ 0,40
60	+ 3,9	+ 4,1	+ 6,1	+ 6,2	− 5,7	− 4,9	+ 5,4	+ 0,36
50	+ 4,8	+ 4,9	+ 6,8	+ 7,0	− 6,3	− 5,7	+ 4,3	+ 0,28
40	+ 4,1	+ 4,1	+ 6,4	+ 6,1	− 6,2	− 5,8	+ 3,7	+ 0,25
30	+ 2,9	+ 3,0	+ 4,6	+ 4,8	− 4,9	− 3,9	+ 4,4	+ 0,32
20	+ 1,9	+ 1,9	+ 3,6	+ 3,9	− 4,0	− 3,3	+ 4,1	+ 0,32
10	+ 1,1	+ 1,2	+ 2,6	+ 2,8	− 2,8	− 2,3	+ 4,1	+ 0,37
0	+ 0,8	+ 0,3	+ 1,6	+ 1,9	− 2,8	− 1,9	+ 3,0	+ 0,32
− 10	+ 0,2	− 0,5	+ 1,0	+ 1,1	− 1,3	− 1,3	+ 2,6	+ 0,38
20	− 0,5	− 0,7	+ 0,6	+ 0,8	− 0,6	− 0,3	+ 2,0	+ 0,44
30	− 0,7	− 0,7	+ 0,1	+ 0,1	− 0,1	+ 0,2	+ 1,1	+ 0,56
40	− 0,0	− 0,2	+ 0,0	+ 0,1	− 0,2	− 0,1	− 0,5	+ 0,79
50	− 1,1	− 1,2	− 1,0	− 1,2	+ 0,3	+ 0,5	− 1,2	+ 0,37
60	− 2,0	− 2,6	− 2,7	− 2,4	+ 1,7	+ 1,8	− 1,0	+ 0,17
70	− 2,4	− 3,1	− 3,8	− 4,1	+ 2,6	+ 2,3	− 1,3	+ 0,16
80	− 3,1	− 3,6	− 4,5	− 4,1	+ 2,9	+ 2,7	− 1,7	+ 0,17
− 90	− 2,9	− 3,6	− 4,6	− 4,4	+ 3,2	+ 2,7	− 2,1	+ 0,18

On April 4 and April 6 the measures for finding the value of ξ were taken for intervals of 5° of zenith distance. It is unnecessary to state here in more detail the calculations by which the corrections for the forms of the pivots were obtained, as a full account will appear in the published *Cambridge Observations*. The quantities actually employed for the correction of those transits that were taken for the determination of the longitude are contained in the following list:—

Zen. Dist. South.	Correction.	Zen. Dist. South.	Correction.	Zen. Dist. South.	Correction.
	s.		s.		s.
+ 90°	+ 0,44	+ 55	+ 0,34	+ 20	+ 0,32
85	+ 0,41	50	+ 0,28	− 36° . 19′ Polaris ...	+ 0,60
80	+ 0,41	45	+ 0,24		
75	+ 0,40	40	+ 0,25	− 39° . 15 Polaris SP.	+ 0,60
70	+ 0,40	35	+ 0,30		
65	+ 0,40	30°	+ 0,32		
60°	+ 0,36	25	+ 0,35		

It may be remarked that the effect of the forms of the pivots follows the same law in 1854 as in 1850, but is of larger amount. The difference may reasonably be ascribed to the wear of the pivots in the interval. Upon the whole I consider the final result obtained by taking the forms of the pivots into account, viz. $22^{s},70$ East Longitude, to be a trustworthy determination.

J. CHALLIS.

CAMBRIDGE OBSERVATORY,
May 15, 1854.

P. S. It may be remarked that the measures for the determination of the effects of the forms of the pivots were taken in April 1854, whilst the observations for the Longitude were made in May 1853. Some allowance is, therefore, required for the change of effect in the interval. By calculations recently gone through, of which an account will be given in the published *Cambridge Observations*, and by having regard to the time during which the Illumination was West in that interval, I find that the corrections for the forms of the pivots should be less by the mean quantity $0^{s},056$. The Longitude should therefore be increased by this quantity, and the corrected result is $22^{s},697 + 0^{s},056 = 22^{s},753$.

The value $22^{s},75$ East is now adopted for the Longitude of the Cambridge Observatory.

J. CHALLIS.

CAMBRIDGE OBSERVATORY,
Oct. 16, 1854.

XXII. *On some points in the Theory of Differential Equations. By* AUGUSTUS DE MORGAN, *of Trinity College, Secretary of the Royal Astronomical Society, and Professor of Mathematics in University College, London.*

[Read *March* 27, 1854.]

The miscellaneous character of this paper makes it desirable to prefix a brief table of contents.

1, Summary of peculiar notation and language employed. 2; Remark on *self-compensating* variables. 3, On some neglected theorems, and in particular, if *either* or both P and Q be infinite, $P:Q = P_x:Q_x$. 4, On the factor of a differential, its difficulties, and mode of avoiding them. 5, Reassertion of the entrance of arbitrary functions into the primitives intermediate between the final equation and the general primitive, and remark on the reason given by Lagrange for their rejection. 6, Connexion between the ordinary and singular (whether intraneous or extraneous) solution of a primordinal equation; results of my last paper obtained without geometry; new equation of connexion between the usual criteria. 7, Remarkable theorem of M. Cauchy for ascertaining whether a given solution be intraneous or extraneous. 8, Remark on the verification of the singular solution of an equation. 9, Connexion of the ordinary and singular solution of biordinal equations, including the extension of Cauchy's theorem. 10, Sketch of the theory of triordinal equations; entrance of what are called *determinants;* objection to this name. 11, System of *polar correlation.* 12, Connexion of the method of transformation for all differential equations given in a former paper, and here presented as the method of *polar transformation.* 13, Polar correlation of biordinal equations. 14, General account of the equation $\phi(x, y, z, y', z') = 0$. 15, Polar correlation of curves in space; limitation arising from the differential correlation being primordinal. 16, Remarks on Lagrange's primary solution of partial equations. 17, Primary solution of biordinal partial equations; failure of result in the attempt to make self-compensating variables of the constants. 18, Analogy of point and curve, chord through two points and developable surface through two curves; construction of a biordinal partial equation. 19, Remark on the general solution of a biordinal equation. 20, Polar correlation of surfaces; polar biordinal equation a case of $Q + Rr + Ss + Tt + U(s^2 - rt) = 0$; analogy of this equation with the most general biordinal equation of two variables, in its primordinals, and in its singular solution. 21, Method of polar transformation for partial differential equations. 22, Suggested notation for eliminants, and deduction of some primary properties. 23, Test of the equation $Q + Rr +$ &c. $= 0$ being polar, already arrived at, from another point of view, by Ampère; reduction of the integration to that of two primordinal equations, whose common solutions are those required. 24, Remarks on notation. *Appendix,* Convergence of Maclaurin's series, and Lagrange's theorem.

1. Further consideration has confirmed and extended some views on differential equations published in Vol. IX. Part II.: it has also indicated points of connexion between subjects there treated apart. Before entering on the subject, I dispose of some preliminary matters.

It is desirable to invent single words to stand for the phrases "of the first order," "of the second order," &c. I propose *primordinal*, *biordinal*, *triordinal*, &c. The word *differential* may be dispensed with, since these adjectives are understood to apply to differential equations only.

The symbol U_x, which I use for the partial differential coefficient of U with respect to x as explicitly contained in U, may be extended as follows. The complex coefficient, with reference to x directly, and to p, q, r, as functions of x contained in U, may be denoted by $U_{x|p,q,r}$, which thus stands for $U_x + U_p p_x + U_q q_x + U_r r_x$. The symbol $U_{x|x,p,q,r}$ might be more correct: but it very seldom happens that we want to exclude the explicit effect of a variable. Parentheses may, when required, be employed to separate all that relates to one differentiation: thus $U_{(x|p,q,r)(y|u,v,w)}$ denotes $d^2U : dxdy$, on the supposition that p, q, r, vary with x, and u, v, w, with y. Thus, if p, q, r, be not functions of y, nor u, v, w, of x, we have

$$U_{xy|p,q,r,u,v,w} = U_{(x|p,q,r)(y|u,v,w)}.$$

When the suffix notation is inconvenient, D may indicate differentiation, the specification being within brackets: as in

$$D^2[xy|p, q, r, u, v, w]U = D^2[(x|p, q, r)(y|u, v, w)]U.$$

When repeated specification is useless, we may preserve an indication of complexity by using $U_{x|}$, $U_{xy|}$, or $U_{(x|)(y|)}$, as required: or we may, with due warning, omit all symbols of complexity in any particular case.

Differentiations may be expressed as in

$$d_x U = U_x dx, \quad d_{x,y,z} U = U_x dx + U_y dy + U_z dz.$$

A relation, the character of which there is no occasion to distinguish, may be expressed as in $(x, y, z) = 0$, $(a, b, c) = 0$. A letter may be used for its own functional symbol: thus $u = u(x, y, z)$ may denote that u is a function of x, y, z. By the phrase "for u write $u(x, y, z)$" we may signify that we substitute for u its value in terms of x, y, z. Remembrance of distinctive functional letters, when the only idea needed is that of unspecified functional relation, is often a heavy tax on the reader's thought.

2. When a variable so enters that the partial differential coefficient vanishes, so that the same primordinal forms of relation exist which would have existed if that variable had been constant, it may be termed *self-compensating:* as in $\phi(x, y, a) = 0$, $\phi_a(x, y, a) = 0$. When two or more variables are so related to others, as in $\phi(x, y, a, b) = 0$, $\phi_a da + \phi_b db = 0$, they may be termed *compensating*, or *compensative:* all *primordinally*. When also

$$\phi_{a(x|y)} da + \phi_{b(x|y)} db = 0,$$

they may be called biordinally compensative: and so on. The whole system is always compensative: thus $\phi_x dx + \phi_y dy + \phi_a da = 0$ is merely the condition of permanence of the equation $\phi(x, y, a) = \text{const.}$, if the expressed letters be all which enter the form ϕ. When, having $\phi(x, y, a, b) = 0$, we deduce $\phi_x + \phi_y y' = 0$, which only *expresses* that x and y are compensative, we usually imply $da = 0$, $db = 0$. It must be remembered that the word compensation refers to *form*, not to *value*.

3. If y be a function of x and constants which becomes infinite for a finite value of x, then $y':y = \infty$, being $(\log y)'$; also $y'':y' = \infty$, being $(\log y')'$; and so on. These remain true, even if y', &c. become infinite by the reduction of $y = y(x, a, \&c.)$ to $x = m$, exclusive of y. That is, when a finite change in x makes an infinite change in y, it makes an infinite change in $y':y$, $y'':y'$, &c. This, as well as $y':y = \infty$ when $y = 0$, may be made visible from the subtangent of the curve represented, and demonstrably obvious both from the fluxional principle, and from the equation $\phi(x + h) = \phi x + \phi'(x + \theta h) . h$.

If, for a finite or infinite value of x, *either or both* P and Q become infinite, then $P:Q$ and $P_x:Q_x$ are *equal:* that is, both nothing, both *finite and equal*, or both infinite. (Would *equatable* be a good word to distinguish this set of alternatives from that in which the zeros or infinites *must* have the *ratio of equality?*) This is well known when *both* are infinite. Let P be finite or zero, Q being infinite. If P_x be also finite or zero, the theorem is obvious; if P_x be infinite, P being finite, it is clear that, x being finite, the *last increment* of Q must be described with an infinitely greater velocity than the last increment of P; so that $P_x:Q_x = 0$, as well as $P:Q$. This fluxional step may be easily reduced into the language of limits. If the critical value of x be infinite, then P cannot be finite or zero, unless $P_x = 0$; whence, Q_x being anything except 0, $P_x:Q_x = 0$, as well as $P:Q$ (this last because $Q_x (x = \infty)$ not being $= 0$, Q must be infinite). There only remains the case in which $Q_x = 0$ $(x = \infty)$, P being finite and therefore $P_x = 0$, and Q being infinite. Let x be taken very great, and let it then receive the increment Δx. The hypotheses show that $\Delta P:\Delta Q$ diminishes without limit, when Δx increases without limit: but this lies among the values of $P_x:Q_x$ as found in passing from x to $x + \Delta x$; if then $P_x:Q_x$ tend towards any final limit, that limit must be zero. In all that precedes, we suppose that some tendency is permanently established at last: in all other cases, the above theorem depends upon the interpretation of $\cos\infty$ and $\sin\infty$.

In my last paper I made use of the theorem that any relation between x and y under which u is finite and u_x infinite, gives either $u_y = \infty$ or $x = \text{const.}$ This is equally true when u is infinite; for the differential coefficient becomes infinite whenever the function becomes infinite on a finite hypothesis involving the variable of differentiation. And the theorem may be generalized thus: If v and w be any two of the subject-letters of a function u, then any supposition which makes u_v infinite either makes u_{vw} infinite, or is itself independent of w. Hence, if a relation containing x make $u_x = \infty$, we see that

$$u_x dx + u_y dy + u_z dz + \ldots = 0 \text{ and } u_{xx} dx + u_{xy} dy + u_{xz} dz + \ldots = 0$$

are the same equations: for they are

$$dx + \frac{u_y}{u_x} dy + \frac{u_z}{u_x} dz + \ldots = 0 \text{ and } dx + \frac{u_{xy}}{u_{xx}} dy + \frac{u_{xz}}{u_{xx}} dz + \ldots = 0.$$

A relation such as $u(x, y) = \infty$, gives the same form for y' as $u = \text{const.}$ generally; namely $y' = -u_x : u_y$. If any doubt should arise as to our right to make $c = \infty$ in $u = c$, or any question as to distinction between infinite *constants* and infinite *variables*, we may have recourse to $u^{-1} = 0$, which gives $y' = -u_x : u_y$.

4. If $M = 0$, or $M = \infty$, preserve P finite, then in $(MP)_x$ or $MP_x + PM_x$, MP_x is infinitely small compared with PM_x: the ratio is that of $(\log P)_x$ to $(\log M)_x$, $\log P$ being zero or finite, while $\log M$ is infinite; so that the theorem in § 3 applies, and proves the assertion. Generally, $M = 0$ and $M = \infty$ give to MP_x and PM_x the ratio of $\log P$ to $\log M$.

Let $Pdx + Qdy$ be made integrable by the factor M, so that $(MP)_y = (MQ)_x$. If $M = 0$ or ∞ make P and Q both finite, the last gives $PM_y : QM_x = 1$ or $P : Q :: M_x : M_y$. But $M = 0$ and $M = \infty$, being cases of $M = \text{const.}$, both give $y' = -M_x : M_y$; whence both are solutions of $P + Qy' = 0$. Hence the well known properties of the integrating factor; but the extreme cases offer some difficulty, which may be avoided as follows. Let U be a function of x and y, then $U = \text{const.}$ (∞ inclusive) gives $y' = -U_x : U_y$, and $U_x = \text{const.}$ and $U_y = \text{const.}$ (∞ included) give $y' = -U_{xx} : U_{xy}$ and $y' = -U_{xy} : U_{yy}$. Now if a relation between y and x make U_x and U_y *both vanish*, or make *either or both* infinite, the three forms of y' are either zeros, finite and equal, or infinites: so that the selected relation satisfies $U_x + U_y y' = 0$, the differential equation of $U = \text{const.}$

This reasoning might easily be extended to equations of n variables; and most easily by making $n - 2$ of them indeterminate functions of the $(n-1)^{\text{th}}$. The want of such considerations is rather a defect in the elementary treatment of differential equations. For example, let $U = U(x, y)$ give $dU = U_y(Pdx + dy)$. The differential equation of $U = \text{const.}$ is at once taken to be $Pdx + dy = 0$, be the constant what it may. But may there not be *singular exceptions*, in which $U =$ a specific const. produces $dU = 0$, not through $Pdx + dy = 0$, but through $U_y = 0$? The answer is, that $U_y = 0$, P not being infinite, is accompanied by $Pdx + dy = 0$, as shown.

5. In my last paper I pointed out that the differential equations intermediate between the original primitive and the one which has lost all the constants cannot be represented without the aid of arbitrary functions. Lagrange, I have since observed, has noticed this extension, and has rejected it on grounds which I shall presently discuss. The ordinary theory is complete as to the passage from the original primitive to the final equation, upon the supposition that all relation between the constants, *and between those constants and others*, is denied or forbidden. But it is neither true nor complete as to the passage from the final equation to the original primitive.

Let $xy = ax + by + c$. So long as a, b, c, are wholly unconnected, no primordinal equations can be deduced except

$$y + xy' = a + by', \quad y^2 = a(y - xy') - cy', \quad x^2y' = b(xy' - y) - c;$$

nor any biordinals except

$$yy'' - 2y'^2 = ay'', \quad 2y' + xy'' = by'', \quad 2xy'^2 - 2yy' - xyy'' = cy'',$$

nor any triordinal (in any case) except $2y'y''' - 3y''^2 = 0$. If we ascend from this last, then, the descending biordinals being $a = A$, $b = B$, $c = C$, the first primitive in ascent is $\phi(A, B, C) = 0$, where ϕ is arbitrary, and may contain any number of constants. It is obvious that $\phi = 0$ gives the triordinal common to $dA = 0$, $dB = 0$, $dC = 0$, and can give no other, even in the case of $\phi_A A_{y''} + \phi_B B_{y''} + \phi_C C_{y''} = \infty$, or $\phi_{y''|A,B,C} = \infty$. But the primitive of $\phi(A, B, C) = 0$, though it be $xy = ax + by + c$, is subject to $\phi(a, b, c) = 0$. This is not a restrictive relation between the *values* of a, b, c, since ϕ may contain arbitrary constants besides a, b, c. Thus, let the first primitive be $A = mB$ or $(y - mx)y'' - 2my' - 2y'^2 = 0$: the original is $xy = b(y + mx) + c$, or $xy = ax + by + c$, where $a = mb$. These two forms are coextensive: the only difference is, that according as a, b, c, or as m, b, c, are treated as independent constants, the curves contained under $2y'y''' - 3y''^2 = 0$ are grouped in one or another of two different ways. The primordinal is found by eliminating y'' between $\phi(A, B, C) = 0$, and $\chi(A, B, C) = 0$: and its original is $xy = ax + by + c$, subject to $\phi(a, b, c) = 0$, $\chi(a, b, c) = 0$ and a similar remark. The original primitive is found by eliminating y' and y'' between $\phi(A, B, C) = 0$, $\chi(A, B, C) = 0$, $\psi(A, B, C) = 0$, which amount to $A =$ const., $B =$ const., $C =$ const., and give $xy = ax + by + c$.

Lagrange, (*Leçons*, &c. p. 249 [1806]), after noting that, for instance, $\phi(y', xy' - y) = 0$ gives $y'' = 0$, does not admit it as a primitive, because it necessarily leads to $y' = a$, $xy' - y = b$. This would be a sufficient reason for rejecting one of the last pair, since either is the necessary consequence of the other. All the primordinals of any biordinal can be expressed by means of any two among them: in the case before us, simplicity would dictate $y' = a$, $xy' - y = b$, as fundamental forms. But more complicated biordinals have many primordinal forms of equal simplicity: and further, in ascent we are generally obliged to take what we can come by, without any means of telling whether or no we have got one of the forms we should have preferred, if we had known the original primitive. Having $2y'y''' = 3y''^2$, and finding, by some accident of the problem, that $(y' - 1):y''^2$ is an integrating factor, we are led to

$$(x + y)y'' + 2(y' - y'^2) = ay'',$$

which is not one of the forms selected by the common theory.

6. I now proceed to some consideration of primordinal equations. By a *singular* solution I mean any one, however obtained, which must, by the very method of solution, fail to contain the ordinal number of arbitrary constants: it may be called *intraneous* or *extraneous* according as it can or cannot be made a case of the ordinary primitive.

Let $y' = \chi(x, y)$ be the primordinal deduced from $y = \psi(x, a)$, or its supposed equivalent $a = A(x, y)$. Hence $dA = A_y(y' - \chi)dx$, and $\chi = -A_x:A_y$ are both identically true. All solutions of $y' = \chi$ not contained in $A =$ const. are in $A_y = \infty$; except only, it may be, those of the form $x =$ const., which, though they satisfy $y' = \chi$ by making both sides infinite, give $y' - \chi$ the form $\infty - \infty$. And, with the same exception, the converse is true, that $A_y = \infty$ satisfies $y' = \chi$: for it gives $y' = -A_{xy}:A_{yy}$ which (§ 3) agrees with $-A_x:A_y$, or χ, whenever

$A_y = \infty$ is produced by a *substitution** *for y;* that is, whenever it is produced by anything except $x = \text{const.}$ This last case, which satisfies $y' = \chi$ in the form $\infty = \infty$, requires special consideration; in what follows I suppose it excluded.

Since $y = \psi(x, A)$ is identically true, we have $1 = \psi_a A_y$, identically, ψ_a meaning $\psi_a(x, A)$. Hence $\psi_a = 0$ contains all singular solutions except those in $x = \text{const.}$ Again, $\chi(x, y)$ is identically $\psi_x(x, A)$: whence $\chi_y = \psi_{ax} A_y = \psi_{ax} : \psi_a = (\log \psi_a)_x$. Of this equation† I can find neither notice nor use: it may be directly deduced thus;—

$$\frac{\psi_{ax}}{\psi_a} = \left(\frac{1}{A_y}\right)_x . A_y = -\frac{A_{xy}}{A_y} - \frac{A_{yy}}{A_y} y' = -\frac{A_{xy}}{A_y} + \frac{A_x A_{yy}}{A_y^2} = -\left(\frac{A_x}{A_y}\right)_y = \chi_y.$$

It follows that when ψ_a vanishes under a substitution for x, $(\log \psi_a)_x$ becomes infinite with $\log \psi_a$; and therefore also χ_y. But when ψ_a vanishes under a constant value of a, $\psi_a = \infty$ does not necessarily give $(\psi_a)_x = \infty$. Consequently, $\chi_y = \infty$ contains all the *extraneous* solutions, and may, and frequently will, contain intraneous singular solutions: in particular, it must contain all those intraneous solutions which, besides being deducible from a specific constant value of a, are also deducible from $y = \psi(x, a)$ by substituting for a a function of x. So that $\chi_y = \infty$ must give all the solutions, whether intraneous or extraneous, which have contacts with ordinary solutions.

The exception may be thus illustrated. Since $\psi_a = 0$ gives $\psi_{ax} = -\psi_{aa} . a_x$ we have in that case $\chi_y = -(\log \psi_a)_a . a_x$. Now $\psi_a = 0$, under any substitution for a, gives $(\log \psi_a)_a = \infty$: but if $a = \text{const.}$, or $a_x = 0$, χ_y takes the form $\infty \times 0$, and is not necessarily infinite.

Next, χ_y or $(\log \psi_a)_x$ may be infinite without $\psi_a = 0$: as when $\psi_a = \infty$, or when $\psi_{ax} = \infty$ without $\psi_a = \infty$. But whenever the result of $\chi_y = \infty$ is not a solution of $y' = \chi$, the relation between a and x which turns $y = \psi(x, a)$ into that result gives $\psi_{xx} = \infty$: that is, an ordinary solution can only meet $\chi_y = \infty$, when itself not a solution, in points of infinite curvature. This was proved in my last paper, but the following is shorter. First, let $\psi_a = \infty$, ψ being finite: then, except when $a = \text{const.}$ (a case here against hypothesis) we have $\psi_x = \infty$, and therefore $\psi_{xx} = \infty$. Next, let $\psi_{ax} = \infty$, and not ψ_a: then, unless $x = \text{const.}$ (a case throughout excluded), ψ is not infinite; so that ψ_{ax} being infinite, ψ_{xx} is infinite, except in the extra-hypothetical case of $a = \text{const.}$ Now ψ_{xx}, or y'', is $\chi_x + \chi_y \cdot \chi$, so that all the relations resulting from $\chi_y = \infty$ which make $\chi_x + \chi_y \cdot \chi$ finite or zero, are necessarily solutions: as otherwise proved in my last paper.

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* It must be remembered that the agreement of $A : B$ and $A_x : B_x$ when A and B both vanish, or when either is infinite, has only been proved when the singular state is produced, as it were, *by action upon x*, that is, by substituting for x a function of other letters. It is easy, when the state is produced by a relation between x and other variables, to forget that the particular case in which x disappears from the relation may be, and probably is, an exception.

A caution of the same kind is necessary in another case. The function and differential coefficient are so connected that the second becomes infinite with the first, for a finite value of the variable: but not when the function becomes infinite without action on the variable.

† The more complicated relations between χ_y and the original primitive, mostly derived by Lagrange, have the disadvantage of connecting ψ_a and χ_y with other functions. A great number may be found in Lagrange's paper *Sur les Intégrales particulières* (Berlin Memoirs, 1774, pp. 197-275) and in the *Leçons sur le Calcul des Fonctions.* If, which is more than I know, the equation in the text has ever been given, it is curious indeed that it has not become common as the mode of connecting the two best known and most widely used tests of singular solution.

Whenever two solutions have a common point, χ_y is infinite at that point, if y' be finite. Let $y = P$, $y = P + Q$ be two solutions, and let P' be finite when $Q = 0$. The equations

$$P' = \chi(x, P), \quad P' + Q' = \chi(x, P + Q)$$

are identically true; whence

$$Q' = \chi(x, P + Q) - \chi(x, P) = \chi_y(x, P + \theta Q) \, . \, Q, \qquad (\theta < 1),$$

if $\chi_y(x, P)$ be finite. Hence $(\log Q)'$ is finite when $Q = 0$, which, Q being really a function of x, cannot be: so that $\chi_y(x, P)$ is infinite.

It has been commonly asserted that when $\chi_y = \infty$ gives a solution, that solution is extraneous. The well known proof (Lacroix, Vol. II. pp. 383—385) depends on a use of expansions in the failing case of Taylor's Theorem. But the assertion itself is not merely incorrect, but falsified by many ordinary examples. Frequent instances occur in which a case of $y = \psi(x, \text{const.})$ has at each point contact with another case: and further, we have seen that $\chi_y = \infty$ may include ordinary solutions which are not even deducible from $\psi_a = 0$, that is, solutions which are not singular even in the sense in which I have used the word.

The following remark may suggest the means of inquiring further into this case. It appears that when ψ_a and $(\log \psi_a)_x$ are both made infinite by the same constant value of a, there is a solution of $y' = \chi$ which is wholly ordinary, contained in $\chi_y = \infty$, the relation which has been asserted to give *only extraneous* solutions, and not even the rest of the cases which come under $\psi_a = 0$. If this be the case, as from $\chi_y = (\log \psi_a)_x$ it certainly is, how comes it that instances have not appeared in the multitude of questions which have been exhibited in elementary works? The answer is, I think, of the same kind as must be made to the question why the aptitude of Maclaurin's development to exhibit certain *finite* funtions in the form $0 + 0 \, . \, x + 0 \, . \, x^2 + \ldots$ has never been noticed until our own time.

A function of a and x may become 0 or ∞ for a value of a independent of x in two distinct ways. First, *factorially*, when the function $f(x, a)$ can be exhibited as

$$Fa \times \{f(x, a) : Fa\},$$

in which Fa becomes 0 or ∞ and $f(x, a) : Fa$ remains finite: as in $\epsilon^{ax} - 1$ which vanishes with a, and is $a \times \{(\epsilon^{ax} - 1) : a\}$, or $0 \times x$. Secondly, *non-factorially*, where this transformation cannot be made, as in $\epsilon^{-a^{-2}x}$, which vanishes with a, but cannot be reduced to a factorial form in which the evanescence arises *solely* from that of a function of a alone. When ψ_a becomes 0 or ∞ factorially, the function of a disappears in $\psi_{ax} : \psi_a$, and χ_y may be finite: when non-factorially, no such disappearance occurs, and $\psi_{ax} : \psi_a$, which then becomes $0 : 0$ or $\infty : \infty$, may be finite or infinite, as it shall happen.

7. M. Cauchy (Moigno, Vol. II. pp. 445—454) has given a criterion for distinguishing between extraneous and intraneous singular solutions. This criterion I believe to be universally valid, and fully demonstrated: but as it seems to have attracted little or no attention in this country, and as the demonstration may be much shortened without material alteration, I subjoin my own account of it.

The question whether $y = P$, a solution of $y' = \chi(x, y)$, P' and $\chi(x, P)$ being identical, is intraneous or extraneous to $y = \psi(x, a)$, the general solution, is clearly the same as the question whether $y = 0$, a solution of $y' = \chi(x, P + y) - \chi(x, P)$, is intraneous or extraneous to $y + P = \psi(x, a)$, the general solution of the last. We need therefore only consider the case in which $y = 0$ solves $y' = \chi(x, y)$, or in which $\chi(x, 0)$ vanishes independently of x.

It is first to be shown that when $y = 0$ *is* extraneous, then $\int_0^\beta \{\chi(x, y)\}^{-1} dy$ *begins finite*, that is, will be finite for some finite extent of value of β. Since $\chi = -A_x : A_y$, we have

$$\int_0^\beta A_y dy \text{ or } A(x, \beta) - A(x, 0) = -\int_0^\beta \frac{A_x(x, y)}{\chi(x, y)} dy, \quad (x \text{ constant}).$$

The general solution of $y' = \chi$ is $a = A(x, y)$, and $y = 0$ is extraneous, so that $A(x, 0)$ is neither constant nor infinite for all values of x, but is a function varying with x. Hence, for some extent of value of β, $\int_0^\beta A_x \chi^{-1} dy$ is finite. If then A_x begin infinite at $y = 0$, it is clear that $\int_0 A_x \chi^{-1} dy$ (which begins finite) begins with elements infinitely great compared with those of $\int_0 \chi^{-1} dy$: so that in this case $\int_0 \chi^{-1} dy$ begins finite. But if A_x begin finite, then, so long as A_x and χ retain each one sign,

$$\int_0^\beta \frac{A_x}{\chi} dy = A_x(x, \theta\beta) \int_0^\beta \frac{dy}{\chi}, \quad (\theta < 1), \quad \text{or} \int_0^\beta \frac{dy}{\chi} = -\frac{A(x, \beta) - A(x, 0)}{A_x(x, \theta\beta)},$$

when $A(x, 0)$ is not $= 0$, this vanishes with β, and begins finite: and $A(x, 0) = 0$, independently of x, cannot be admitted, since it brings $y = 0$ into the general solution. Hence $\int_0 \chi^{-1} dy$ begins finite whenever $y = 0$ is an extraneous solution.

The converse must now be shown: namely, that whenever $\int_0 \chi^{-1} dy$ begins finite, $\chi(x, 0)$ vanishing independently of x, it follows that $y = 0$ is extraneous. Let $\int_0 \chi^{-1} dy = f(x, y)$, or $d_y f(x, y) = dy : \chi(x, y)$, independently of x: so that, m being any quantity,

$$d_y f(m, y) = dy : \chi(m, y).$$

Since $dy = \chi(x, y) \,.\, dx$ is made identical by $y = \psi(x, a)$, and since $df\{m, \psi(x, a)\}$ is identical with $d\psi(x, a) : \chi\{m, \psi(x, a)\}$, we have

$$df\{m, \psi(x, a)\} = \frac{\chi\{x, \psi(x, a)\}}{\chi\{m, \psi(x, a)\}} dx, \text{ identically.}$$

Now it is known that $\phi(x + h) - \phi x = \phi'(x + \theta h) \,.\, h$, $(\theta < 1)$, provided that *between* $x + h$ and x there be no value which makes $\phi' x$ infinite: and this though $\phi' x$ should be infinite, ϕx being finite, at either extremity.

Hence,

$$f\{m, \psi(x + h, a)\} - f\{m, \psi(x, a)\} = \frac{\chi\{x + \theta h, \psi(x + \theta h, a)\}}{\chi\{m, \psi(x + \theta h, a)\}} h, \quad (\theta < 1),$$

for all values of x, h, and m, with such casual exceptions as may arise if the second side become infinite between x and $x + h$, though these may be avoided by taking h sufficiently small. Since $f(m, 0) = 0$ for all values of m, we see that if it were possible to assign a value to a

which should make $\psi(x, a) = 0$ independently of x (that is, if $y = 0$ could be an intraneous solution of $y' = \chi$) the first side would vanish identically, the second becoming

$$\chi(x + \theta h, 0) \, . \, h : \chi(m, 0).$$

That is, $\chi(x + \theta h, y) : \chi(m, y)$ vanishes with y, whatever $x + \theta h$ and m may be, and therefore even though their values should change places, which is absurd, and, with it, the supposition that $y = 0$ is an intraneous solution.

Returning to the general case, it appears that, P' and $\chi(x, P)$ being identical, $y = P$ is an extraneous or intraneous solution of $y' = \chi(x, y)$ according as

$$\int_0^{\beta} \frac{dz}{\chi(x, P + z) - \chi(x, P)}, \quad \text{or} \quad \int_P^{P+\beta} \frac{dy}{\chi(x, y) - \chi(x, P)}$$

begins finite or infinite. It is easily shown that this integral cannot begin finite unless $\chi_y(x, P)$ be infinite; so that $\chi_y = \infty$ includes all extraneous solutions. But the converse is not true. To take M. Cauchy's example, let $y' = y \log y$. Here $y = 0$ makes $\chi_y = \infty$, but it also makes $\int_0 \chi^{-1} dy = \infty$, whence $y = 0$ is intraneous: the primitive is $y = \epsilon^{a\epsilon^x}$, and $y = 0$ belongs to $a = -\infty$.

This theorem of M. Cauchy is a beautiful instance of the *discriminating power* of the process of integration: and I cannot help looking upon it as one of the most remarkable accessions of this century to the theory of differential equations.

8. It is not generally noticed that the legitimacy of the extraneous solution often depends on the convention* by which $A = B$ is considered as satisfied in the form $0 = 0$, or $\infty = \infty$, even though $A{:}B = 1$ be not satisfied. Thus, let $y' = \phi' x + 2\sqrt{(y - \phi x)}$, whose ordinary primitive is $y = \phi x + (x + a)^2$, the singular (here extraneous) solution being $y = \phi x$. Now we have

$$y'' = \phi'' x + \frac{y' - \phi' x}{\sqrt{(y - \phi x)}} = \phi'' x + 2 \text{ or } \phi'' x + \frac{0}{0}.$$

The first value belongs to the ordinary primitive. If the differential equation *always* mean $(y' - \phi' x) : \sqrt{(y - \phi x)} = 2$, then $y'' = \phi'' x + 2$ is permanent, and $y = \phi x$ cannot be admitted. The extraneous solution requires that in differentiating $\sqrt{(y - \phi x)}$, we should attend to the effect of the precedent condition $y = \phi x$ upon the process of differentiation. We are to find the limit of $\sqrt{(y - \phi x + \Delta y - \Delta\phi x)} : \Delta x$, that is, of $\sqrt{(\Delta y - \Delta\phi x)} : \Delta x$, which is infinite except when $y' - \phi' x = 0$, and then depends upon the mode in which $\Delta y : \Delta x$ and $\Delta\phi x : \Delta x$ approach equality. If, as by hypothesis, $\Delta y = \Delta\phi x$ be maintained during the diminution of Δy and Δx, then $\sqrt{(\Delta y - \Delta\phi x)} : \Delta x$, being always $= 0$, is 0 at the limit. This modification of differentiation is never noticed in the case of one variable, because its effect upon value is always the production of 0 or ∞, and value is generally enough for the purpose, without consideration of form.

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* I consider this point in another communication, an abstract of which has already appeared. I will here remark that the difficulty usually depends on the form of the equation only. Thus $x^2 = x + 2$ is unambiguously satisfied by $x = 2$: but not the consequence $x^2 - 4 = x - 2$; this last does not give $A{:}B = 1$, when made $0 = 0$.

It is very important, in this subject, to distinguish between the means of detecting singular solutions, and that of establishing their general properties. For instance, the primary being $\phi(x, y, a) = 0$, we cannot be certain of all the singular solutions until (besides the case of $x = \text{const.}$) we have completely examined $\phi_a : \phi_y = 0$, both in $\phi_a = 0$ and $\phi_y = \infty$. Nevertheless, we may be sure of the general properties of those solutions by reasoning either on $\phi_a = 0$ or on $\phi_y = \infty$: for $\phi(x, y, a) = 0$ can be examined with perfect generality either in the form $y - \psi(x, a) = 0$, or $A(x, y) - a = 0$. Consequently, $\phi_a : \phi_y = 0$ may be distinguished, under the name of the criterion of detection, from $\phi_a = 0$ or $\phi_y = \infty$, the general property.

9. I shall now treat biordinal equations fully, and triordinals to such an extent as will show how the general theory commences. An n-ordinal equation has 2^n distinct kinds of primitives possible *à priori*, there being the choice of ordinary or extraneous singular at each step of the ascent. Generally, the singular solution of a singular solution is not a solution of the final equation.

The common theory contains a defect which ought to have been striking. If $\phi(x, y, a, b) = 0$ give $y'' = \chi(x, y, y')$, it is clear that if a and b be made compensatory to the second order, that is, if $d_{a,b}\phi = 0$, $d_{a,b}\phi_{x|y} = 0$, then $y'' = \chi$ will remain true: and also that a singular primordinal is produced, satisfying $y'' = \chi$, and having no arbitrary constant. It is assumed that *all* the singular primordinals are thus obtained. This is true, but to the inference we have no more right than to the deduction of $A = 0$, $B = 0$, from $A + B = 0$.

Let $y = \psi(x, a, b)$ be the primitive, which, combined with $y' = \psi_x$, gives $a = A(x, y, y')$, $b = B(x, y, y')$. The primordinal of $y'' = \chi(x, y, y')$ is $f(A, B) = 0$, f being arbitrary. Any given curve $y = \varpi x$ may be made to solve some one form of the primordinal, namely, that obtained by eliminating x between $A = A(x, \varpi x, \varpi' x)$ and $B = B(x, \varpi x, \varpi' x)$. But as $y = \varpi x$, taken at hazard, will not satisfy $y'' = \chi$, we may presume that it is a singular solution of the primordinal. And this may be verified as follows. Let $f(A, B) = 0$ be the result of the elimination; then $dB : dA = -f_A : f_B$. But, as we shall see, $dB : dA = B_{y'} : A_{y'}$; whence $f_A A_{y'} + f_B B_{y'} = 0$. If $f(A, B) = 0$ give $y' = \kappa(x, y)$, we have $\kappa_y = -(f_A A_y + f_B B_y) : (f_A A_{y'} + f_B B_{y'})$; whence not only $y = \varpi x$, but any solution of $f(A, B) = 0$, seems to give $\kappa_y = \infty$. The truth is that *all* solutions which make variables of A and B, *are singular*: for, $f(A, B) = 0$ being $B = \mu A$, the ordinary primitive is $y = \psi(x, a, \mu a)$, which gives $A = a$, $B = \mu a$.

We must not suppose that we are here dealing with a peculiar or restricted class of equations. Every primordinal is in an infinite number of ways reducible to $f(A, B) = 0$, where A and B are made constants by the ordinary primitive. Thus $y'^2 = 1$ is coextensive with $y^2 + x^2 - 2xyy' = (yy' - x)^2$, the ordinary primitive of which makes both sides constant. It is obtained by throwing $\pm y = x + a$ into the form $y^2 = x^2 + 2ax + b$, where $b = a^2$.

Given $y = \varpi x$, required a key to the primordinal equations which have $y = \varpi x$ for a singular solution. Take *any* equation $y = \psi(x, a, b)$, determine $a = A(x, y, y')$, $b = B(x, y, y')$, eliminate x between $a = A(x, \varpi x, \varpi' x)$ and $b = B(x, \varpi x, \varpi' x)$, and write $A(x, y, y')$ and $B(x, y, y')$ for a and b in the result.

We know that $a = A$ and $b = B$ give but one biordinal equation $y'' = \chi(x, y, y')$, where $\chi = -(A_x + A_y y') : A_{y'} = -(B_x + B_y y') : B_{y'}$. We need not appeal to the original primitive to prove that $a = A$ satisfies $y'' = \chi$, so that we have nothing to do with the connexion between y and y', except to note that $a = A(x, y, z)$ produces $dz - \chi(x, y, z) \, . \, dx = 0$ if we substitute zdx for dy. I noted this in my last paper, as the means of showing that the primordinal equation really involves an arbitrary function.

I need not fully repeat the proof that $dA = A_{y'}(y'' - \chi)\,dx$ gives $A_{y'} = 0$ as containing ordinary primordinal solutions of $y'' = \chi$, and $A_{y'} = \infty$ as containing all the singular solutions. If we take $A_{y'} = \infty$ we have y'' as obtained from $A_{xy'} + A_{yy'}y' + A_{y'y'}y'' = 0$. Now if $A_{y'} = \infty$ be a relation which really involves y', and is not merely $(x, y) = 0$, this last equation gives the same ratios of partial coefficients, and value of y'', which occurs in $A_x + A_y y' + A_{y'}y'' = 0$: that is, $A_{y'} = \infty$ gives $y'' = \chi$. The case of $(x, y) = 0$ must be excluded, and considered apart. We are not to infer that $A_{y'} = \infty$ must, *per se*, reduce $y'' - \chi$ to 0, or $dA : A_{y'}$ to an infinitely small quantity of a higher order than dx. It is allowable to confound $1:\infty$ with 0 in comparing infinitely small quantities with finite ones: but $1 : 0$ is really an infinite of an unattainable, or infinite order, just as ∞ itself is a magnitude of unattainable value.

The biordinal $y'' = \chi$ is satisfied by the ordinary solution of $A_{y'} = \infty$, but (as will presently be shown) not necessarily by the singular solution: for the ordinary and singular solutions of a primordinal usually give different forms and values to y''. Since $y = \psi(x, A, B)$, $y' = \psi_x(x, A, B)$ are identical, we have

$$\begin{aligned} \psi_a A_{y'} + \psi_b B_{y'} &= 0, \\ \psi_{ax} A_{y'} + \psi_{bx} B_{y'} &= 1, \end{aligned} \qquad A_{y'} = \frac{-\psi_b}{\psi_a\psi_{bx} - \psi_b\psi_{ax}}, \qquad B_{y'} = \frac{\psi_a}{\psi_a\psi_{bx} - \psi_b\psi_{ax}}.$$

Consequently, the singular primordinals are all such as satisfy either (and therefore both) of the equations

$$\frac{\psi_a}{\psi_b}\psi_{bx} - \psi_{ax} = 0, \qquad \frac{\psi_b}{\psi_a}\psi_{ax} - \psi_{bx} = 0.$$

Hence $\psi_a\psi_{bx} - \psi_b\psi_{ax} = 0$ is the criterion of detection, subject to further examination when it is satisfied by $\psi_a = 0$, $\psi_b = 0$, which, combined with $y = \psi$, give a relation between x and y independent of y'. In every other case, the combination of $y = \psi$, $y' = \psi_x$, and the criterion, gives the singular primordinal by elimination of a and b. If it should happen that x does not appear in the criterion, which then takes the form $(a, b) = 0$, the singular solution is $(A, B) = 0$, and is intraneous.

Again, since $y = \psi$, $y' = \psi_x$, $\chi = \psi_{xx}$, are all made identical by substitution of A and B for a and b, we have

$$\begin{aligned} &\psi_a \; A_{y'} + \psi_b \; B_{y'} = 0, && (\psi_a\psi_{bx} - \psi_b\psi_{ax})\,\chi_{y'} + (\psi_{axx}\psi_b - \psi_{bxx}\psi_a) = 0, \\ &\psi_{ax} A_{y'} + \psi_{bx} B_{y'} = 1, && \chi_{y'} = \{\log(\psi_a\psi_{bx} - \psi_b\psi_{ax})\}_x. \\ &\psi_{axx} A_{y'} + \psi_{bxx} B_{y'} = \chi_{y'}, && \end{aligned}$$

Hence $\chi_{y'} = \infty$ includes all the singular primordinals in which the criterion contains x, but not necessarily those in which the intraneous form, $(A, B) = 0$, is brought out by absence of x. But $\chi_{y'} = \infty$ includes cases in which $\psi_a\psi_{bx} - \psi_b\psi_{ax}$ is infinite; and in all such cases, those in $(A, B) = 0$ excepted, we have $\psi_{xxx} = \infty$, or $\chi_x + \chi_y y' + \chi_{y'}\chi = \infty$. If $\psi = \infty$, or

$\psi_x = \infty$, or $\psi_{xx} = \infty$, this is obvious. If ψ and ψ_x be finite, and ψ_a or ψ_b infinite, the producing relation can only be $(a, b) = 0$. If ψ_x be finite and ψ_{ax} infinite, either $\psi_{xx} = \infty$, or $(a, b) = 0$. If ψ_{axx} be infinite and not ψ_{xx}, either $\psi_{xxx} = \infty$, or $(a, b) = 0$. So that $\chi_{y'} = \infty$ may either not show certain intraneous singular solutions, or may show intraneous solutions which are not singular: but it must contain all extraneous solutions; and every case of it which makes $\chi_x + \chi_y y' + \chi_{y'} \chi$ finite or zero must be singular.

M. Cauchy's theorem may be extended to biordinal equations, so far as regards a case of the primitive, by a proof which may be copied from that of the case already given, and need not be inserted here. If $y = P$ be a solution of $y'' = \chi(x, y, y')$, P being a function of x, the solution is extraneous or intraneous according as $\int_0 dx : \}\chi(x, P, P_x + x) - \chi(x, P, P_x)\}$ begins finite or infinite, x being the only variable.

Returning to the criterion, let elimination of a and b from $y = \psi$, $y' = \psi_x$, and $\psi_a \psi_{bx} - \psi_b \psi_{ax} = 0$ give $y' = \varpi(x, y)$, the singular primordinal: we also obtain a and b each in terms of x and y. These values of a, b, y', substituted in $y = \psi$, $y' = \psi_x$, make them identical; whence

$$\psi_a \frac{da}{dy} + \psi_b \frac{db}{dy} = 1, \qquad \psi_a \varpi_y - \psi_{ax} = (\psi_a \psi_{bx} - \psi_b \psi_{ax}) \frac{db}{dy},$$

$$\psi_{ax} \frac{da}{dy} + \psi_{bx} \frac{db}{dy} = \varpi_y, \qquad \varpi_y = (\log \psi_a)_x + \frac{1}{B_y} \frac{db}{dy}.$$

If either or both $B_y = \infty$, $b =$ constant, give $\psi_a = 0$, we have, if ψ_a in this case undergo substitution for x, $\varpi_y = \infty$: that is, attempts at solution made through $\psi_a = 0$, give either singular primitives of the singular primordinal, or the failures, already noticed, in which y'' is infinite. And $\psi_a = 0$, $\psi_b = 0$, combined with $y = \psi$, give either the singular primitive of the singular primordinal, or the failure adverted to.

The following question may be asked, and at first presents some difficulty:—Since $dA : A_y = (y'' - \chi)\, dx$ is an absolute identity, how is it that some solutions of $A_y = \infty$, the ordinary ones, give $y'' - \chi = 0$, and others, the singular ones, do not? What can it matter to the deduction of $y'' = \chi$ from $dA : \infty = (y'' - \chi)\, dx$, whether the symbol ∞ arose in one way, or another? The answer is, that independently of our not being sure that $A_{y'} = \infty$ gives $y'' = \chi$, it may happen that a differential relation the ordinary solutions of which make $A_{y'}$ infinite, may have singular solutions which *do not* make $A_{y'}$ infinite. Let $A_{y'} = \infty$ be produced by $v(x, y, y') = 0$, and by help of $v = v(x, y, y')$ throw A into the form $V(x, y, v)$. We have then $A_{y'} = V_v v_{y'}$. Now, $v = 0$ giving $V_v = \infty$, gives $A_{y'} = \infty$, except only when accompanied by $v_{y'} = 0$. And $v_{y'} = 0$ is precisely the condition under which the singular solutions of $v = 0$ must be looked for.

Again, if $v = 0$ satisfy $y'' = \chi$, it means that $v = 0$ makes $v_x + v_y y' + v_{y'} y'' = 0$ identical with $y'' = \chi$. In the case of $v_{y'} = 0$, this identity can no longer be affirmed; here $v = 0$ makes $v_x + v_y y'$ vanish, and y'' can only be determined by another differentiation.

We may now prove the assumption on which the ordinary view of the theory is founded. The coexistence of $y = \psi$, $y' = \psi_x$ demands that a and b shall either be constants, or compensatory; as in $\psi_a da + \psi_b db = 0$. Now $\psi_a \psi_{bx} - \psi_b \psi_{ax} = 0$, except when satisfied by $\psi_a = 0$,

$\psi_b = 0$, is the condition under which $\psi_{ax}da + \psi_{bx}db = 0$ is satisfied whenever $\psi_a da + \psi_b db = 0$ is satisfied. Consequently, the criterion of detection, when satisfied, completes all that is wanted to make a and b compensatory to the second order.

10. I defer for a moment some details peculiar to the biordinal equation, and I give a sketch of the method of treating triordinals. Let $y = \psi(x, a, b, c)$, combined with $y' = \psi_x$, $y'' = \psi_{xx}$, give $a = A(x, y, y', y'')$, $b = B(x, \&c.)$, $c = C(x, \&c.)$; and let $y''' = \chi(x, y, y', y'')$ be the deduced triordinal. As before, the identities $dA : A_{y''} = (y''' - \chi)\,dx$, &c., show that $A_{y''} = \infty$, &c. severally give singular solutions, except when satisfied by a relation void of y'': that is, they give all the singular biordinals. Substitute A for a, &c., in $y = \psi$, $y' = \psi_x$, $y'' = \psi_{xx}$, which makes them identities, and gives $A_{y''}$ from the first three of the four equations following:

$$\begin{aligned}
\psi_a\, A_{y''} + \psi_b\, B_{y''} + \psi_c\, C_{y''} &= 0,\\
\psi_{ax}\, A_{y''} + \psi_{bx}\, B_{y''} + \psi_{cx}\, C_{y''} &= 0,\\
\psi_{axx}\, A_{y''} + \psi_{bxx}\, B_{y''} + \psi_{cxx}\, C_{y''} &= 1,\\
\psi_{axxx} A_{y''} + \psi_{bxxx} B_{y''} + \psi_{cxxx} C_{y''} &= \chi_{y''},
\end{aligned}$$

$$\begin{aligned}
A_{y''} &= \frac{\psi_b\psi_{cx} - \psi_c\psi_{bx}}{T_{\prime\prime\prime}},\\
T_{\prime\prime\prime} &= \psi_a\psi_{bx}\psi_{cxx} + \psi_{ax}\psi_{bxx}\psi_c + \psi_{axx}\psi_b\psi_{cx}\\
&\quad - \psi_c\psi_{bx}\psi_{axx} - \psi_{cx}\psi_{bxx}\psi_a - \psi_{cxx}\psi_b\psi_{ax}.
\end{aligned}$$

The general property of the singular biordinals is $T_{\prime\prime\prime} = 0$, and cases involving $\psi_b\psi_{cx} - \psi_c\psi_{bx} = 0$, &c. must be considered apart, and are related to singular primordinals of the singular biordinals. Make the substitutions in $y''' = \psi_{xxx}$, which brings ψ_{xxx} to identity with χ, and we have the fourth of the equations. In $T_{\prime\prime\prime}$, write one more x in each suffix, advance each other letter one place in the cycle *abca*, and change the sign: thus $\psi_a\psi_{bx}\psi_{cxx}$ becomes $-\psi_{bx}\psi_{cxx}\psi_{axxx}$, &c.: call the result T. Repeat this process twice, striking off xs when they reach four in number in any suffix: call the results $T_\prime$ and $T_{\prime\prime}$. Multiply the four equations by T, $T_\prime$, $T_{\prime\prime}$, $T_{\prime\prime\prime}$, and add, which gives $T_{\prime\prime} + T_{\prime\prime\prime}\chi_{y''} = 0$. Now it will be found that $T_{\prime\prime} = -(T_{\prime\prime\prime})_x$; whence $\chi_{y''} = (\log T_{\prime\prime\prime})_x$. Hence may be established, relatively to $\chi_{y''}$, conclusions similar to those of the preceding cases: and M. Cauchy's theorem may be extended.

It thus appears that the generalizations of form which a complete theory would require may be easily obtained by those who have studied the new and as yet isolated method of ***determinants***[*]; that is, by all who have taken full advantage of existing facilities.

The implicit forms $\phi(x, y, a) = 0$, $\phi(x, y, a, b) = 0$, &c., may easily be substituted for $y = \psi(x, a)$, &c. We find

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[*] I should rarely venture to resist the term introduced by investigators into their own results, except by passive refusal to follow them: but the present case will allow even a remonstrance. Very good reason ought to be shown before the participle *determinant* is separated from the verb *determine*: and till such reason is manifest, I, for one, feel unwilling to abandon the right to say that co-ordinates are determinants of a point, and coefficients of an equation determinants of the roots. That some instances of such separation are too well established to be shaken, is unfortunately true: the evils which they create, in thinking and in teaching, are warning enough against the introduction of more. The rule of invention, hitherto, has been to let the name express the most striking *distinctive* character of the thing signified. *Determination* is no distinctive property of the so-called determinants: all functions *determine*. But eliminating power *is* their distinctive property, and elimination is the process in which they were *first* thought worthy of a name. Until those who have a right to speak with authority propose a *distinctive* appellation, I intend to call them *eliminants*. But, though objecting to the word *determinant*, I feel the highest admiration of the system; it is difficult to say how much will hereafter be held due to those who have laid its foundations.

$$\psi_x = -\frac{\phi_x}{\phi_y}\quad \psi_a = -\frac{\phi_a}{\phi_y}, \text{ \&c.}: \quad \psi_{ax} = -\left(\frac{\phi_x}{\phi_y}\right)_a - \left(\frac{\phi_x}{\phi_y}\right)_y \cdot \psi_a,$$

$$\psi_a\psi_{bx} - \psi_b\psi_{ax} = \frac{\phi_a}{\phi_y}\left(\frac{\phi_x}{\phi_y}\right)_b - \frac{\phi_b}{\phi_y}\left(\frac{\phi_x}{\phi_y}\right)_a = \frac{\phi_a\phi_{bx|y} - \phi_b\phi_{ax|y}}{\phi_y^{\ 2}},$$

$$\frac{1}{\phi_y^{\ 2}}\left\{\phi_x\left(\frac{\phi_a}{\phi_b}\right)_y - \phi_y\left(\frac{\phi_a}{\phi_b}\right)_x\right\} = \frac{1}{\phi_b^{\ 2}}\left\{\phi_a\left(\frac{\phi_x}{\phi_y}\right)_b - \phi_b\left(\frac{\phi_x}{\phi_y}\right)_a\right\}.$$

In biordinals, $A_{y'} = \dfrac{\phi_b\phi_y}{\phi_a\phi_{bx|y} - \phi_b\phi_{ax|y}}, \quad B_{y'} = \dfrac{-\phi_a\phi_y}{\phi_a\phi_{bx|y} - \phi_b\phi_{ax|y}}.$

But it must be remembered that we have no right to substitute these results, containing y and y', in such formulæ as $\chi_{y'} = \{\log(\psi_a\psi_{bx} - \psi_b\psi_{ax})\}_x$. If we apply to $\phi(x, y, a, b) = 0$ the process we have applied to the more convenient form $y = \psi(x, a, b)$, we shall find

$$\chi_{y'} = \left\{\log\left(\frac{\phi_a\phi_{bx|y} - \phi_b\phi_{ax|y}}{\phi_y^{\ 2}}\right)\right\}_{x|y} + \frac{\phi_a\phi_{by} - \phi_b\phi_{ay}}{\phi_a\phi_{bx|y} - \phi_b\phi_{ax|y}}\cdot\chi$$

where $U_{x|y}$, even when U is a function of y', means only $U_x + U_y \cdot y'$. Thus $(xy' - y)_{x|y}$ is to mean $y' + (-1) . y'$, or 0.

11. In the biordinal equation, the number of constants is equal to that of variables: or rather, the term constant being only relative, the numbers of variables in the two compensating systems are equal. Hence arise facilities for the conception* and use of a system which may be called that of *polar correlation*, of which the well known theory of poles and polar curves is a particular case.

If there be equations involving $m + n$ variables, of which m compensate each other, and also the remaining n, and if we fix each of the m variables, and then satisfy the equations in every possible way by the remaining n variables; and if, merely to give a name, we say that the fixture determines a point, and the variation a curve; we may say that the (m)-point is the *pole* of the (n)-curve. By the converse process, we may make an (n)-point the pole of an (m)-curve. And the fundamental reciprocal property is obvious, namely, that all the points of any curve have polar curves which contain the pole of that curve.

Let (x, y) and (a, b) be two points: it is not necessary that the co-ordinates a, b, should be on the same system as x, y. Let the fixed point (a, b) be the pole of the (xy)-curve $\phi(x, y, a, b) = 0$; and let the fixed point (x, y) be the pole of the (ab)-curve $\phi(x, y, a, b) = 0$. Every point of an (xy)-curve is the pole of an (ab)-curve which passes through the pole of that (xy)-curve: and *vice versâ*. As a simple instance, suppose $x^2 + y^2 = ax + by$. Let both systems be rectangular, let O be the origin, and P any point. If P be (x, y), its polar (ab)-curve is a straight line through P perpendicular to OP: if P be (a, b), its polar

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* When I first wrote this paper, in July, 1853, I did not know that Professor Boole had distinctly conceived and used this system: see the *Cambridge and Dublin Journal*, vol. vii., p. 156. Our lines of application, however, differ widely. Mr Boole, as he states, was equally unaware that the subject of polarity, in as general a point of view, had been presented by Mr Druckenmüller, in Crelle's journal.

(xy)-curve is a circle on the diameter OP. Any point Q on this circle has its (ab)-curve passing through P: that is, OQP is a right angle. Any point Q on the straight line has a polar circle passing through P. That is, the direct and converse properties of the angle in a semicircle are only expressions of the reciprocal polar property of $x^2 + y^2 = ax + by$. Again, if the point (a, b) describe a directing curve, then, since $a, b, db{:}da$, determine $x, y, dy{:}dx$, the (xy)-curves produced all touch, with a certain order of contact, a certain connecting curve: if this connecting curve be made a directing curve for (x, y), that which was the directing curve of (a, b) becomes, with the same order of contact, the connecting curve of the (ab)-curves produced. If $\phi(x, y, a, b)$ and $\phi(a, b, x, y)$ be identical, then for a given pole the two polar curves are identical: and if the pole be taken on the curve $\phi(x, y, x, y) = 0$, the polar curve touches this last curve in the pole.

12. The points (a, b) and (x, y) are not determinants of each other: but a, b, b' (or $db{:}da$) determine x, y, y' (or $dy{:}dx$). That is, any *element* of any curve determines an *element* of another curve: and these curves are the reciprocal curves of direction and connexion just now described, which may be called *reciprocal polar curves*[*], the *modular equation* being $\phi(x, y, a, b) = 0$. Let this give $a = A(x, y, y')$, $b = B$: then $db{:}da$ or $b' = B_{y'}{:}A_{y'}$, $y'' - \chi$ disappearing. Accordingly, a, b, b', depend upon x, y, y'. Hence b'', or $db'{:}da$, depends upon x, y, y', y''; and so on. The method in my last paper, by which every instance of $\phi(x, y, a, b) = 0$ is made a means of transforming any equation of two variables into another of the same order, amounts to a reference of the curves defined by the given equation to their polar reciprocals, upon a modular equation taken at pleasure. Let $\phi(x, y, a, b) = 0$ give $x = X(a, b, b')$, $y = Y(a, b, b')$, $y' = Y_{b'}{:}X_{b'}$, &c.: substitute these values in any given equation $f(x, y, y', y'', \text{\&c.}) = 0$. If the result can be integrated into $F(a, b) = 0$, then $x = X$ and $y = Y$ may be made to express x and y in terms of a; whence the primitive of $f = 0$ can be found. This may be called the method of *polar transformation*[†]: under it may be brought nearly all the isolated methods which end in algebraic elimination.

The modular equation being $\phi(x, y, a, b) = 0$, and a curve $y = fx$ being given, the co-ordinates a, b of its polar reciprocal are seen in $a = A(x, fx, f'x)$ and $b = B(x, fx, f'x)$. Two curves, $y = fx$, $n = Fm$, being given, any one modular equation, $\phi(x, y, a, b) = 0$, may be made the means of making these curves polar reciprocals, with such an amount of conditions to spare as may suggest the means of solving many problems. Assume $m = \mu(a, b)$, $n = \nu(a, b)$: and thence $a = \alpha(m, n)$, $b = \beta(m, n)$. We now see that $m = \mu(A, B)$, $n = \nu(A, B)$ are primordinals of $\phi(x, y, \alpha, \beta) = 0$. And since $y = fx$ makes A, B, determinate functions of x, we see that the problem is solved for every way in which $\nu(A, B) = F\mu(A, B)$ is capable of being identically satisfied. Assume the form μ, then $F\mu(A, B)$ becomes a determinate function of x, say κx; and $\nu(A, B) = \kappa x$ can then be satisfied in an infinite number of ways. Choosing one of these, we have $m = \mu(a, b)$, $n = \nu(a, b)$ from which to determine α and β, and $\phi(x, y, \alpha, \beta) = 0$ is a modular equation to which $y = fx$, $n = Fm$ are polar reciprocals. Accordingly, we can in every case dictate one of the two species of reciprocal curves, subject to the usual difficulties of extreme cases.

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[*] Thus the equation $x^2 + y^2 = ax + by$ gives the following theorem;—If the curve A be touched by the perpendiculars to the radii of B, the curve B is touched by the circles whose diameters are the radii of A. This very obvious theorem is not common: had it been so, surely the following would have been a common exercise for beginners;—The circle described on any focal radius of a conic section touches the circle described on the major axis.

[†] The most remarkable case, deduced from $y = ax - b$, may be called *linear*. When I gave (in 1848) the polar *linear* transformation for *partial* differential equations, which I found, before the paper was printed, to have been fully given by M. Chasles, I noted that Legendre had made an application to a certain partial biordinal, and that Lacroix had made an imperfect (or all but perfect) transformation of a partial primordinal. I may now add that Ampère, in a paper presently cited, has given more than one instance of linear transformation for *partial* equations, but only with reference to one class of equations, and without any hint of general application. I have as yet found no trace whatever of any thing beyond the linear method even for partial equations, and none of any case whatever for equations of two variables.

13. The primitive $\phi(x, y, a, b) = 0$ gives two correlative biordinal equations, say $y'' = \chi(x, y, y')$ and $b'' = \alpha(a, b, b')$. And we have

$$\frac{1}{A_{y'}} = \frac{1}{\phi_b}\left\{\phi_a\left(\frac{\phi_x}{\phi_y}\right)_b - \phi_b\left(\frac{\phi_x}{\phi_y}\right)_a\right\}, \quad \frac{1}{X_{b'}} = \frac{1}{\phi_y}\left\{\phi_x\left(\frac{\phi_a}{\phi_b}\right)_y - \phi_y\left(\frac{\phi_a}{\phi_b}\right)_x\right\},$$

whence (§ 10) $A_{y'}\phi_b = X_{b'}\phi_y$, and the condition of primordinal singular solution is the same both for $y'' = \chi$ and $b'' = \alpha$. Let this condition be $\sigma(x, y, a, b) = 0$.

$$\begin{tikzcd}[column sep=small]
 & & & \phi(x, y, a, b) = 0 \arrow[dl, dash] \arrow[dr, dash] \arrow[ddlll, dash] \arrow[ddrrr, dash] & & & \\
\sigma(x, y, a, b) = 0 \arrow[d, dash] & & \phi_x + \phi_y y' = 0 \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] & & \phi_a + \phi_b b' = 0 \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \sigma(x, y, a, b) = 0 \arrow[d, dash] \\
y' = \varpi(x, y) \arrow[d, dash] & a = A(x, y, y') \arrow[dr, dash] & b = B(x, y, y') \arrow[d, dash] & & x = X(a, b, b') \arrow[d, dash] & y = Y(a, b, b') \arrow[dl, dash] & b' = \lambda(a, b) \arrow[d, dash] \\
y = \Pi(x, C) & & y'' = \chi(x, y, y') & & b'' = \alpha(a, b, b') & & b = \Lambda(a, Z)
\end{tikzcd}$$

If we eliminate a and b between $\phi = 0$, $\sigma = 0$, $\phi_{x|y} = 0$, we deduce $y' = \varpi$, $y = \Pi$, the singular primordinal and singular primitive of $y'' = \chi$. If we eliminate x and y between $\phi = 0$, $\sigma = 0$, $\phi_{a|b} = 0$, we have $b' = \lambda$, $b = \Lambda$, the same of $b'' = \alpha$. Each singular primitive of $y'' = \chi$, for each value of C, is the (xy)-curve which touches all in $\phi\{x, y, a, \Lambda(a, Z)\} = 0$ for one value of Z and every value of a: and each singular primitive of $b'' = \alpha$, for each value of Z, is the (ab)-curve which touches all contained in $\phi\{x, \Pi(x, C), a, b\}$ for one value of C and every value of x. The contacts are of the second order, and the *ordinary* primitives of $y' = \varpi$ and $b' = \lambda$ are polar correlatives, each to each. This is not necessarily the case with the *singular* primitives, which are determined from $\phi_a = 0$, &c. independently of y' and b'. But if it should happen that the singular primitives *are* polar correlatives, it follows that they solve the biordinals, and have contacts of the third order with original primitives. The proof of this is in my last paper, at the end of the first section: but the result is not correctly stated. The contact there mentioned is of the third order, if as high as the second: the closest primitives touch the singular primitive of the singular primordinal with a contact of the first or third order, not of the second at most.

The above statements contain the assertion that $\phi = 0$, $\sigma = 0$, $y = \Pi$, $b = \Lambda$, containing x, y, a, b, C, Z, though only four equations, admit of the elimination of the four quantities x, y, a, b, and lead to a relation between C and Z only. To show this, throw $y = \Pi$, $b = \Lambda$, into the forms $C = C(x, y)$, $Z = Z(a, b)$: then if for a and b we substitute in terms of

x, y, y', and write $\varpi(x, y)$ for y', $Z(a, b)$ will become a function of $C(x, y)$. This, we know, will happen or not, according as $Z_{x|a,b}C_y - Z_{y|a,b}C_x$ does or does not vanish. Now since ϖ and $-C_x:C_y$ are identical, this vanishes when $Z_{x|a,b} + Z_{y|a,b}\,.\,\varpi$ vanishes, and this is what

$$Z_a\left\{A_x + A_y\varpi + A_{y'}(\varpi_x + \varpi_y\,.\,\varpi)\right\} + Z_b\left\{B_x + B_y\varpi + B_{y'}(\varpi_x + \varpi_y\,.\,\varpi)\right\}$$

becomes, after substitution of ϖ for y' in A_x, &c. Now $y'' = \chi$ is the same as

$$A_x + A_y y' + A_{y'} y'' = 0, \text{ and as } B_x + B_y y' + B_{y'} y'' = 0:$$

so that, as $y' = \varpi$ satisfies $y'' = \chi$, the coefficients of Z_a and Z_b vanish; while, the primitive of $b' = \lambda$ employed not being singular, neither Z_a nor Z_b is infinite. It might be proved by direct means that $y = \Pi$ is a consequence of $\phi = 0$, $\sigma = 0$, $b = \Lambda$, from which therefore it might be obtained by elimination.

I have hardly a right, according to my own definitions, to make $Z_{x|a,b}$ mean all I have made it mean in the preceding process. Strictly, I ought to have made the suffix, not $x\,|\,a, b$, but $x\,|\,(a\,|\,x, y' = \varpi)\,(b\,|\,x, y' = \varpi)$, or the like. When it is desirable to make part only of the implications prominent in the symbol, some general warning may be given, such as in $Z_{x|a,b\ldots}$

I now apply the preceding to Lagrange's example, which has been repeated by all writers.

$$\phi(x, y, a, b) = -y + \tfrac{1}{2}ax^2 + bx + a^2 + b^2.$$

$$\text{Let}\quad P = 16(1 + x^2)y + x^4 - (8x^3 + 16x)y' - 16y'^2,$$

$$Q = b'^2 - 4bb' - 4a,$$

$$\phi_x + \phi_y y' = -y' + ax + b, \qquad \phi_a + \phi_b b' = (\tfrac{1}{2}x^2 + 2a) + (x + 2b)b',$$

$$\sigma(x, y, a, b) = x^2 + 4bx - 4a.$$

$$A = \frac{4xy' + x^3 + \sqrt{P}}{4(1 + x^2)}, \quad A_{y'} = \frac{x}{1 + x^2} - \frac{x^3 + 2x + 4y'}{(1 + x^2)\sqrt{P}} = \frac{2(x + 2b)}{x^2 + 4bx - 4a},$$

$$B = \frac{4y' - x^3 - x\sqrt{P}}{4(1 + x^2)}, \quad B_{y'} = \frac{1}{1 + x^2} + \frac{x^4 + 2x^2 + 4xy'}{(1 + x^2)\sqrt{P}} = -\frac{(x^2 + 4a)}{x^2 + 4bx - 4a},$$

$$X = -b' + \sqrt{Q}, \qquad X_{b'} = -\frac{2(x + 2b)^2}{x^2 + 4bx - 4a},$$

$$Y = ab'^2 - (2ab + b)b' + b^2 - a^2 + (b - ab')\sqrt{Q}, \qquad Y_{b'} = -\frac{2(x + 2b)^2(ax + b)}{x^2 + 4bx - 4a}.$$

$$y'' - \chi = y'' - \frac{4xy' + x^2 + \sqrt{P}}{4(1 + x^2)}; \text{ singular solutions, } P = 0,\ 1 + x^2 = 0,$$

$$b'' - a = b'' - \frac{2(1 + b'^2)}{Q - (2b - b')^2}(2b - b' + \sqrt{Q}); \text{ singular solution, } Q = 0.$$

For the singular solutions we have

$$a = \frac{4xy' + x^2}{4(1+x^2)} = -\frac{x^2}{4} + \frac{x}{4}\sqrt{\frac{16y + 4x^2 + x^4}{1 + x^2}},$$

$$b = \frac{4y' - x^3}{4(1+x^2)} = -\frac{x}{2} + \frac{1}{4}\sqrt{\frac{16y + 4x^2 + x^4}{1 + x^2}},$$

$$x = -b' = 2\{b + \surd(b^2 + a)\},$$

$$y = ab'^2 - (2ab + b)\,b' + b^2 - a^2 = 12ab^2 + 3(a^2 + b^2) + (12ab + 2b)\surd(b^2 + a^2),$$

the primitive of $Q = 0$ (obtained by linear polar transformation), is

$$2\left(b + \sqrt{b^2 + a}\right).\surd\left(1 + 4a + 8b^2 - 8b\sqrt{b^2 + a}\right)$$

$$+ \log\left\{2\left(b - \sqrt{b^2 + a}\right) + \surd\left(1 + 4a + 8b^2 - 8b\sqrt{b^2 + a}\right)\right\} = Z.$$

Substitute in this the values of a and b, and we have for the primitive of $P = 0$,

$$2\left(-\frac{x}{2} + \frac{1}{2}\sqrt{\frac{16y + 4x^2 + x^4}{1 + x^2}}\right).\surd(1 + x^2) + \log\{\surd(1 + x^2) - x\} = C,$$

as otherwise found by Lagrange.

The singular primitive of $P = 0$ is $16y + 4x^2 + x^4 = 0$, which may be obtained from $\phi = 0$, $\phi_a = 0$, $\phi_b = 0$: it does not solve $y'' = \chi$. There is no finite singular solution of $Q = 0$. The relation $1 + x^2 = 0$ happens to be a singular solution of $y'' = \chi$.

By choosing instances in which the multipliers of constants are functions of only one of the variables, we are obliged to adopt cases of considerable complication, if we would exhibit all the phenomena. If we take as our example

$$a^2y - axy + mx = b, \text{ with } a^2y + mx - 2ma = 0$$

as the condition of singular solution, we find for the biordinals

$$(m + xy)\,y''^2 + 2y'(y - xy')\,y'' - 4y'^3 = 0, \qquad a^2b''^2 - 2(2ma + ab' - b)\,b'' + 4mb = 0,$$

for the singular primordinals and singular primitives

$$(xy' + y)^2 - 4my' = 0, \qquad (2ma + ab' - b)^2 - 4ma^2b' = 0,$$

$$C = \log\left(\frac{\surd m + \surd(m - xy)}{x}\right) - \sqrt{\frac{m - xy}{m}}, \qquad b = ma\{1 + (Z + \log a)^2\},$$

in which $C = Z + \log\dfrac{\surd m}{2}$. The singular solutions of the singular primitives are

$$y = 0, \quad x = 0, \quad a = 0, \quad b = ma.$$

14. The substitution of compensating variables in place of constants is the key to many of the general properties of differential equations. What effect this substitution shall have, as to alteration in extent of solution, depends upon the number of constants in relation to the number of equations of compensation: and this effect may consist in diminution, permanence,

or augmentation, of the generality of the solution. The arbitrary function certainly appears in every case in which the number of equations of compensation falls short of the number of compensating variables: and the converse seems to be true. The difference between the two numbers determines the number of variables in the arbitrary function; that is, the number of definite forms which may be arbitrarily combined: thus $\phi(xyz, x + y + z)$ is a function of two variables, though each is a function of three.

A system of polar curves in space will be a good illustration of the great differences which exist between the laws of compensated systems. When Monge (*Mem. Acad. Sci.* 1784, p. 502) undertook to give meaning to the equation $f(x, y, z, y', z') = 0$, y and z being functions of x, he found, as he states, that such an equation as $dz^2 = dx^2 + dy^2$ was considered as absolutely absurd, and also $Pdx + Qdy + Rdz = 0$, except when integrable. This opinion was founded, probably, upon the supposition that one equation must result from one primitive. But it is easily seen, *à priori*, that a point moving so as to satisfy $f(x, y, z, y', z') = 0$, cannot move at pleasure upon any possible surface, but may describe a curve upon ***any surface whatsoever.*** This point, when at (l, m, n) must begin to move along a generating line of the cone $f\{l, m, n, (y - m):(x - l), (z - n):(x - l)\} = 0$: consequently, to suppose unrestricted motion upon some surface, is to suppose a surface every point of which has full contact of the first order with some cone ***at the vertex.*** This can only be when the cone is a plane, or when f has the form $P + Qy' + Rz'$: and even then, as we otherwise know, only on further conditions. Nevertheless, an infinity of curves may be described, the only condition being that the curve shall pass through each of its points in the direction of some generating line of the cone which has its vertex at that point. And such a curve may be drawn through any point upon any surface, provided the tangent plane cut the cone in more than the vertex: if not, there is still an imaginary curve, which satisfies the condition algebraically. This is easily seen, in the usual way, from the equation itself.

We want a term to express a surface composed, as it were, of curves, in the same manner as we imagine a line composed of points, but coupled with the condition that we may not (or at least do not) pass from one curve to another, though we may choose any one curve. The word *ruled* is appropriated; but *shaded* is not.

If $\chi(x, y, z) = 0$ be obtained by eliminating a from $\phi(x, y, z, a) = 0$, $\psi(x, y, z, a) = 0$, we may say that the surface $\chi = 0$ is obtained by the mode of *shading* which is marked out upon it by either $\phi = 0$, or $\psi = 0$. And thus we may say that the equation

$$f(x, y, z, y', z') = 0$$

belongs to any given surface, provided one assignable mode of shading be adopted: with this addition, that when the form is $P + Qy' + Rz' = 0$ accompanied by the usual conditions of integrability, there is *one* family of surfaces on which *any* mode of shading may be employed. And conversely, any mode of shading may be chosen, provided one of the right family of surfaces be taken.

15. Choose two equations between x, y, z, a, b, c, which reduce to $y = \Phi(x, a, b, c)$, $z = \Psi(x, a, b, c)$. Between these and $y' = \Phi_x$, $z' = \Psi_x$, eliminate a, b, c, producing

$$f(x, y, z, y', z') = 0.$$

This last is the only primordinal which coexists with $y = \Phi$, $z = \Psi$: the joint primitives of $f = 0$ and any other can have no more than *two* arbitrary constants. Any other primitive of $f = 0$ must be deducible from $y = \Phi$, $z = \Psi$, by making a, b, c, compensatory, which gives, b and c being functions of a, $\Phi_a + \Phi_b b' + \Phi_c c' = 0$, $\Psi_a +$ &c. $= 0$. Take any surface

$$\omega (x, y, z) = 0,$$

and eliminate x between $\Phi_{a|b,c} = 0$, $\Psi_{a|b,c} = 0$ and $\omega (x, \Phi, \Psi) = 0$. We have then two results of the form $(a, b, c, b', c') = 0$, from which we get $b = b(a, G, H)$, $c = c(a, G, H)$, G and H being arbitrary constants.

Substitute in $y = \Phi$, $z = \Psi$ and we have a new form of the primitive, involving a, G, H: and thus every form of ω gives its own variation of $y = \Phi$, $z = \Psi$. But we are not to infer that the curves thus obtained form a shading of the surface $\omega = 0$ from which they are obtained. This will happen only under two relations between a, G, H.

If we simply eliminate x from $\Phi_{a|b,c} = 0$, $\Psi_{a|b,c} = 0$, we have $(a, b, c, b', c') = 0$, of which we might assume a primitive as general as that of $f = 0$, say $b = B(a, L, M, N)$, $c = C(a, L, M, N)$. From these, by help of $\Phi_{a|b,c} = 0$, we get $x = (a, L, M, N)$, whence $a = (x, L, M, N)$, and similarly for b and c. Hence $y = \Phi$, $z = \Psi$, give forms of the same character as we began with. Any one of these new primitives, selected by assigning L, M, N, connects, by contact of the first order, all the original primitives which are selected by $b = B$, $c = C$: and all ordinary primitives are found, in an infinite number of ways, among the connecting curves of others. There is a singular solution, a curve of contact to *all* primitives, when it happens that the six equations $\Phi_a = 0$, $\Phi_b = 0$, $\Phi_c = 0$, $\Psi_a = 0$, $\Psi_b = 0$, $\Psi_c = 0$, can be satisfied by one assignment of a, b, and c, each in terms of x: the result is obtained by substitution in $y = \Phi$, $z = \Psi$.

Let the pole (a, b, c) be restricted to the surface $c = c(a, b)$. If we take $\Phi_{a|c} da + \Phi_{b|c} db = 0$, and $\Psi_{a|c} da + \Psi_{b|c} db = 0$, which give $T = 0$, T being $\Phi_{a|} \Psi_{b|} - \Phi_{b|} \Psi_{a|}$, we may eliminate a and b between $y = \Phi$, $z = \Psi$, $T = 0$, and obtain $z = z(x, y)$. When the pole, then, is on $c = c(a, b)$, the polar curve touches the surface $z = z(x, y)$ in a point determined by obtaining x, y, z, in terms of a and b from $y = \Phi$, $z = \Psi$, $T = 0$. Substitute these values in either of the compensating equations, and integrate, giving $b = b(a, \text{const.})$. This is a mode of shading the surface $c = c(a, b)$, so that if the pole move along one of the lines of shading, the polar curve touches a corresponding line of shading on $z = z(x, y)$. And all these properties are reciprocal. The original equations being $\Phi (x, y, z, a, b, c) = 0$, $\Psi (x,$ &c.$) = 0$, if we make (x, y, z) the pole, and restrict it to the surface $z = z(x, y)$ which resulted above, and then eliminate x and y between $\Phi = 0$, $\Psi = 0$, $\Phi_{x|z} \Psi_{y|z} - \Phi_{y|z} \Psi_{x|z} = 0$ we shall find $c = c(a, b)$ as at first assumed: and so on. The main point of the proof is as follows. If in $\Phi = 0$, $\Psi = 0$, we suppose $c = c(a, b)$, $z = z(x, y)$, and a, b, therefore, each an implied function of x and y, we have $\Phi_{x|} + \Phi_{a|} a_x + \Phi_{b|} b_x = 0$, and $\Psi_{x|} + \Psi_{a|} a_x + \Psi_{b|} b_x = 0$, to determine a_x and b_x. But if $\Phi_{a|} \Psi_{b|} - \Phi_{b|} \Psi_{a|} = 0$ and a_x and b_x be not infinite, we must have $\Phi_{a|} \Psi_{x|} - \Phi_{x|} \Psi_{a|} = 0$. Similar reasoning on y gives $\Phi_{a|} \Psi_{y|} - \Phi_{y|} \Psi_{a|} = 0$, whence $\Phi_{x|} \Psi_{y|} - \Phi_{y|} \Psi_{x|} = 0$. It is not in our power to take any surface we please, and find another with the properties of a polar reciprocal: the reason being, that there is a *primordinal* equation, $f(x, y, z, y', z') = 0$, necessarily consequent upon $\Phi = 0$, $\Psi = 0$; while, as to $\phi(x, y, a, b) = 0$, taken alone, the first equation which is clear of constants is biordinal. That every surface dictates another surface, and a mode of shading both, which makes each line of shading on one surface a polar reciprocal of some line of shading on the other, is the nearest resemblance of properties which we can make.

16. The simple theory of Lagrange for partial primordinal equations, is a case in which the transition from constants to compensating variables is a considerable generalization. If $f(x, y, z, p, q) = 0$ arise from elimination of a and b between $\phi(x, y, z, a, b) = 0$, $\phi_x + \phi_z p = 0$, $\phi_y + \phi_z q = 0$, a and b being constants, it equally arises when a and b are compensatory of each other, as in $\phi_a da + \phi_b db = 0$. If $b = Fa$, and a be eliminated between $\phi = 0$, $\phi_a + \phi_b F'a = 0$, we have the general primitive, involving one arbitrary function. Constructive processes also show that we cannot have more: the surface $f = 0$ is determined so soon as one curve in it is determined; just as the curve $y' = \chi(x, y)$ is determined so soon as one point is determined. This tempting analogy between the arbitrary function and the arbitrary constant has caused much confusion, though it soon became evident that it breaks down as to biordinal equations. Arbitrary constants, considered as compensating variables, are in all cases sufficiently general. Hence Lagrange may be justified in calling $\phi(x, y, z, a, b) = 0$ the *complete* solution of $f(x, y, z, p, q) = 0$: though it may seem strange that $z = a(x + y) + b$ should be so called with reference to $p = q$, when $z = F(x + y)$ does equally well for any form of F. But the two forms are co-extensive when a and b are made compensating variables. Nevertheless, I prefer to call $\phi(x, y, z, a, b) = 0$, a *primary* solution.

The singular solution of $\phi(x, y, z, a, b) = 0$, or $z = \psi(x, y, a, b)$, is obtained by eliminating a and b from $\psi = 0$, $\psi_a = 0$, $\psi_b = 0$, or $\phi = 0$, $-\phi_a{:}\phi_z = 0$, $-\phi_b{:}\phi_z = 0$. That is, every singular solution except those in which z is absent, as in $(x, y) = 0$. If $\phi_{x|z} = 0$, $\phi_{y|z} = 0$, give $a = A(x, y, z, p, q)$, $b = B(\,)$, so that $z = \psi(x, y, z, A, B)$ is identical with $f(x, y, z, p, q) = 0$, we see that the expressions written in the following lines are proportionals,

$$\frac{\psi_x + \psi_a A_x + \psi_b B_x}{f_x},\quad \frac{\psi_y + \psi_a A \;+ \psi_b B_y}{f_y},\quad \frac{-1 + \psi_a A_z + \psi_b B_z}{f_z},\quad \frac{\psi_a A_p + \psi_b B_p}{f_p},\quad \frac{\psi_a A_q + \psi_b B_q}{f_q}.$$

Usually, then, $f_p = 0$ and $f_q = 0$, or $f_z = \infty$, determine the singular solution. If $M(x, y, z, p, q) = 0$ be the supposition on trial, we differentiate $f = 0$ and $M = 0$, each with respect to x and to y separately, and from the six equations we eliminate p, q, r, s, t; the relation $(x, y, z) = 0$ produced, if a solution at all, is singular.

The structure of methods of compensation (speaking of the first order only) is threefold. First, when each variable is made self-compensating in every equation. When there is one equation only, this method introduces an equation for every new variable, and gives singular solutions. Secondly, when the new variables are made mutually compensative, and the number of equations and variables is so related that elimination produces equations between the new variables only, which can be integrated and new constants introduced. In this case the forms obtained are of the same character as those which produced them. Thirdly, when elimination does not produce equations between the new variables only, so that the only method of satisfying the equations of compensation is by so many arbitrary assumptions of relation among the new variables as will eliminate the differentials, and leave only algebraic relations between the new and old variables, to be combined with the assumed relations. In this case arbitrary functions enter the final result.

17. The primary solution of the biordinal partial equation contains five arbitrary constants. Having given $U(x, y, z, a, b, c, e, h) = 0$, we usually obtain a single biordinal equation by eliminating $a, b, \ldots$ between the six equations $U = 0$, $U_{x|z} = 0$, $U_{y|z} = 0$, $U_{xx|z} = 0$, $U_{xy|z} = 0$, $U_{yy|z} = 0$. But it is not true that every form of U gives one biordinal and one only: there must be a certain *irreducibility* in the mode of entrance of the constants which does not come under any obvious criterion. A case of exception may be easily given. Every primordinal satisfies an infinite number of biordinals: and $U(x, y, z, a, b, c, e, h) = 0$, whenever it is reducible to $(V, W, a, b, c, e, h) = 0$, V and W being functions of x, y, z, and not of a, b, &c., leads to a primordinal which is also independent of a, b, &c.

Let us suppose, however, that $U = 0$ leads to one biordinal equation. To obtain the same equation by making a, b, &c. compensative, we must use $d_{a,b,c,e,h}U = 0$, $d_{a,b,c,e,h}U_{x|z} = 0$, $d_{a,b,c,e,h}U_{y|z} = 0$, which are reducible to $dc = Cda + C_{,}db$, $de = Eda + E_{,}db$, $dh = Hda + H_{,}db$. First, suppose these equations integrable. Eliminate x, y, z, h, between the five equations

$$U = 0,\quad \frac{dc}{da} = C,\quad \frac{dc}{db} = C_{,},\quad \frac{de}{da} = E,\quad \frac{de}{db} = E_{,},$$

by which we obtain $\left(a, b, c, e, \dfrac{dc}{da}, \dfrac{dc}{db}, \dfrac{de}{da}, \dfrac{de}{db}\right) = 0$.

Again, from any four of the five equations determine x, y, z, h, in terms of the rest, substitute in $dh = Hda + H_{,}db$, and then form the criterion $(H)_{b|c,e} = (H_{,})_{a|c,e}$, which is biordinal with respect to c and e. If we can integrate jointly this last equation and $(a, b, \&c.) = 0$, then, in the integral of $dh = Hda + H_{,}db$ and the five equations, we have the means of eliminating a, b, c, e, h, and producing the general integral of the biordinal derived from $U = 0$. In order, then, completely to characterize the general integral of a biordinal of three variables, direct method requires that we should ascertain the integrals of two equations, one primordinal and one biordinal, between four variables.

Secondly, we attempt the question without requiring the equations of compensation to present integrable forms. The primitives must be five equations between the eight variables x, y, z, a, b, c, e, h, and five new constants. These and their differentiations give ten equations, which elimination of constants reduces to five. By two of these and $d_{x,y,z}U = 0$, we determine dx, dy, dz, in terms of da, db, &c., and substitute in the rest, which gives three equations between da, db, &c.; and these must be jointly identical with the three equations of compensation. Between the five primitives and $U = 0$, we eliminate a, b, c, e, h, and so get another primary solution, namely, an equation between x, y, z, and the five new constants.

18. The usual geometrical analogies give no assistance in determining the general form of the primitive of a biordinal partial equation; those of a less usual character, though of no greater success, will at least point out the conditions under which the solution is reduced to

one definite surface. A curve upon a surface corresponds to a point upon a curve: by giving one point, we select one curve from the infinite number contained in $y' = \chi(x, y)$; by giving one curve, we select one surface from the infinite number contained in $f(x, y, z, p, q) = 0$. Through any two curves can be drawn one or a definite number of developable surfaces: the developable drawn through two curves of a surface answers to the chord drawn through two points of a curve. Two points infinitely near determine a tangent: two curves infinitely near determine a tangent developable, which (and not the tangent plane) is the correlative to the tangent of a curve. When in two dimensions we have an ascending succession of notions A, B, C, and in three dimensions a corresponding succession P, Q, R, S, the new notion which change of dimension introduces is not S, but P: and A, B, C, answer to Q, R, S. Thus a polygon presents angular points, sides, and area; a polyhedron presents corners, edges, faces, and volume: of which angular points answer to edges, sides to faces, area to volume; the corner being a new introduction for which space of two dimensions has no analogue. The angular point of a plane angle has its correlative, not in the angular point of a solid angle, but in the containing straight lines: and generally, of correlatives in two and three dimensions, the second is a dimension higher than the first.

To a curve described by points answers what I have called a *shaded* surface, a surface covered by curves according to a law. The curve which has the tangent $y = ax + fa$ is determined by making a self-compensating. If we shade the surface $z = \phi(x, y)$ in the manner dictated by $y = \psi(x, a)$, we obtain the developable which touches the surface in the line of shading belonging to any value of a, by eliminating x, after substitution for y and z, between $\zeta - z = p(\xi - x) + q(\eta - y)$ and $(r + s\psi_x)(\xi - x) + (s + t\psi_y)(\eta - y) = 0$. If in the first of these we make a a self-compensating variable, assuming x a function of a derived from the second, we have $p_a(\xi - x) + q_a(\eta - y) = 0$. If $z = \phi$ be itself not a developable surface, this third equation cannot consist with the second, except upon the condition $\xi = x$, $\eta = y$: for any other supposition gives $p_{x|y}q_a - q_{x|y}p_a = 0$, or $q = fp$, making $y = \phi$ developable. Hence $\zeta = \phi(\xi, \eta)$, the original surface, is the connecting surface of all the developables which touch it in curves.

Constructive reasoning shows that by assigning two points we assign one, or a definite number, of the curves in $y'' = \chi(x, y, y')$: when the two points are infinitely near they determine y', and $y'' = \chi$ determines the curvature. Let us now suppose we know, as to a surface, one curve, the tangent developable through that curve, and the equation

$$f(x, y, z, p, q, r, s, t) = 0.$$

Let the curve be one line of shading, and taking its projection on xy, draw successive projections for other lines of shading, each infinitely near the preceding. On the given curve, the tangent planes are known from the developable, whence p and q are known: the curvature of the curve gives one relation between r, s, t, throughout the curve; the equation $q = Fp$, or $dq = F'p \,.\, dp$ of the developable gives another; and $f = 0$ is a third; whence r, s, t, are known on the curve. [Or thus;—if the curve give $y = \alpha x$, $z = \beta x$, then $\beta' x = p + q\alpha' x$, which, combined with $q = Fp$ gives $p = \mu x$, $q = \nu x$. Hence $\mu' x = r + s\alpha' x$, $\nu' x = s + t\alpha' x$, which, with $f = 0$, determine r, s, t, in terms of x]. Hence we can establish, with infinitely small errors of the second order, the relations between p and q which exist on the next line of shading, by which, with like errors, we determine the tangent developable of that second line; on which we may repeat the process. It is thus shown that a surface is determined when we know one curve, the tangent developable belonging to that curve, and a biordinal partial equation which belongs to the whole surface. This is an *à priori* proof that two, and no more, relations of form are requisite for the determination of each surface in $f = 0$.

Let $\phi(x, y, z) = 0$ be the complete primitive, and $y = \alpha x$ an assigned projection on xy: let $z = \beta x$ then determine the curve, and $q = Fp$ the developable. Then $\phi(x, \alpha x, \beta x) = 0$, $-\phi_y : \phi_z = F(-\phi_x : \phi_z)$, independently of x; from which the forms of β and F are to be determined. The conditions of the question thus make it apparent that algebraical expression of a geometrical construction which certainly determines one surface, gives conditions enough to determine two unknown forms of function: but this does not show that any given mode of entrance of arbitrary functions is consistent with an equation between x, y, z, p, q, r, s, and t, independently of the forms of those functions.

19. The restrictions under which two arbitrary functions must enter, so as to necessitate a biordinal equation which is free of those functions, are wholly unknown. The simple relation $z = x\phi y + \psi(x + y)$ does not succeed: but[*] if

$$3z = v^4 x - 2v^2 y + 4\int v^3 \phi' v\, dv + 3\psi(vx + v^{-1} y + \phi v), \quad y = v^2(x + \phi' v),$$

we may deduce $r - pt = q\sqrt{rt}$. The mere appearance of an arbitrary form entering the subject of another, and the possibility of a continuance of this mode of entrance, as in

$$\psi[A + \phi\{B + \psi(C + \phi(D + \&c.))\}],$$

joined to the failure of every attempt to obtain the general form, may lead us to suspect that the general integral and the biordinal equation have what might be called a *functio-functional* relation, or a functional relation *of the second order*. I mean such a relation as $\int_0^x \phi x\, dx$ bears to ϕx: since the first depends wholly on the second, we may write

$$\int_0^x \phi x\, dx = (F . \phi)\, x,$$

but not $F . (\phi x)$; F brings out a form depending on the form of ϕ, just as ϕ in ϕx brings out a value depending on the value of x.

20. It can be shown that the case which is fully analogous to the most general biordinal equation of two variables is not the *general* biordinal equation of three variables, but a very limited instance of it. This result belongs to Ampère: but the polar relation led me to it

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[*] This instance is taken from a paper by Ampère on partial differential equations, in the *Journal de l'Ecole Polytechnique*, vol. x. pp. 549-611, and vol. xii. pp. 1-188. Few have had the courage to read this voluminous paper, the notation of which appears, though it really is not, unusually intricate. It would be valuable to any investigator who is concerned with a case in which one variable is in two ways a function of two others, one of which is in both, as in $z = \phi(x, y) = \psi(x, t)$. Ampère denoted the two pairs of differential coefficients thence resulting by

$$\frac{dz}{dx(y} \quad \frac{dz}{dy(x} \quad \text{and} \quad \frac{dz}{dx(t} \quad \text{and} \quad \frac{dz}{dt(x}$$

of which it is more easy to see the cumbrous effect than the means of avoiding it which would secure general assent.

before I had examined his paper. Let the primitive equation be $\phi(x, y, z, a, b, c) = 0$. It is not necessary formally to enunciate the definition of (a, b, c) as the pole of the xyz-surface $\phi = 0$, &c., nor to state the theorems with which the geometrical reader is familiar in the usual case. By $\phi = 0$, $\phi_{x|z} = 0$, $\phi_{y|z} = 0$, we produce $a = A(x, y, z, p, q)$, $b = B(\;)$, $c = C(\;)$. These equations must identically satisfy $\phi_a dA + \phi_b dB + \phi_c dC = 0$: for $\phi = 0$ gives $d_{x,y,z}\phi + d_{a,b,c}\phi = 0$, and $a = A$, &c. satisfy $\phi = 0$, $d_{x,y,z}\phi = 0$, identically. If then one of the three, say A, be made constant, we have $dB{:}dC = -\phi_c{:}\phi_b$ and is primordinal. But $dA = 0$ or $A_{x|z,p,q}dx + A_{y|z,p,q}dy = 0$, substituted in $dB{:}dC$, gives

$$\frac{dB}{dC} = -\frac{A_{x|}B_{y|} - A_{y|}B_{x|}}{C_{x|}A_{y|} - C_{y|}A_{x|}} = -\frac{\phi_c}{\phi_b},$$

the two last functions becoming identical when A, B, C, are written for a, b, c. It follows that $A_{x|}B_{y|} - A_{y|}B_{x|}$, $B_x C_{y|} - B_{y|}C_x$, $C_{x|}A_{y|} - C_{y|}A_{x|}$, have a common biordinal factor, all factors not common being primordinal. Hence $\psi(A, B, C) = 0$ is a primordinal equation which has but one biordinal consequence independent of ψ. From it we deduce

$$\begin{aligned}\psi_A A_{x|} + \psi_B B_{x|} + \psi_C C_{x|} = 0,\\ \psi_A A_{y|} + \psi_B B_{y|} + \psi_C C_{y|} = 0,\end{aligned} \qquad \psi_A(C_{x|}A_{y|} - C_{y|}A_{x|}) - \psi_B(B_{x|}C_{y|} - B_{y|}C_{x|}) = 0,$$

which is satisfied, independently of ψ, by the biordinal factor just deduced. Now $\psi = 0$ has for a primary solution $\phi(x, y, z, a, b, c) = 0$, subject to $\psi(a, b, c) = 0$: but we are not to conclude that the primitive of the biordinal contains an arbitrary function of two variables; for $C_x A_{y|} - C_{y|}A_{x|}$, $A_{x|}B_{y|} - A_{y|}B_{x|}$, $B_{x|}C_{y|} - B_{y|}C_{x|}$ severally vanish under $C = \beta A$, $A = \gamma B$, $B = \alpha C$, β, γ, α, being any forms whatever. There are then two primordinal equations of the form $\psi(A, B, C) = 0$, reducible to $A = \gamma B$, $B = \alpha C$. And $\phi(x, y, z, a, \lambda a, \mu a) = 0$ is a solution of $B = \lambda A$ and of $C = \mu A$, as well as of their common biordinal. If a be made self-compensating, by derivation from $\phi_a + \phi_b\lambda' a + \phi_c\mu' a = 0$, the primordinals remain true, the biordinal follows as before, and substitution of the value of a in $\phi = 0$ gives the complete primitive of the biordinal.

This biordinal itself is of a very limited form. From $B = \lambda A$ we obtain

$$A_{x|}B_{y|} - A_{y|}B_x = 0, \text{ or}$$

$$(A_x + A_z p + A_p r + A_q s)(B_y + B_z q + B_p s + B_q t) - (A_y + A_z q + A_p s + A_q t)(B_x + B_z p + B_p r + B_q s) = 0,$$

$$\text{or} \quad Q + Rr + Ss + Tt + U(s^2 - rt) = 0,$$

where Q, R, &c. are functions of x, y, z, p, q. But it is only a certain species of this general form which is fully analogous to the biordinal of two variables: and this species may be called *polar*.

If the pole (a, b, c) be placed upon the curve $b = \lambda a$, $c = \mu a$, the polar xyz-surface is made subject to $B = \lambda A$, $C = \mu A$. If the pole move along that curve, the polar surface always touches the surface obtained by eliminating a from $\phi = 0$, $\phi_a + \phi_b\lambda' a + \phi_c\mu' a = 0$, and touches in the whole curve which has these two equations. That is, every abc-curve has a *shaded surface* which is its polar reciprocal; and if the pole (x, y, z) be made to move along any line of shading of this surface, the reciprocal *abc*-surface has the original curve for one of its lines of shading.

Next, let the pole be placed on the surface $c = \kappa(a, b)$. The polar *xyz*-surface is then subject to the primordinal equation $C = \kappa(A, B)$. If a and b be made compensating, by help of $\phi_a + \phi_c\kappa_a = 0$, $\phi_b + \phi_c\kappa_b = 0$, substitution in $\phi(x, y, z, a, b, \kappa) = 0$, gives an *xyz*-surface reciprocal to $c = \kappa(a, b)$, and subject to $C = \kappa(A, B)$. Let this surface be $z = \omega(x, y)$: then any point on $c = \kappa$ has a polar surface which touches $z = \omega$ in a point; and if this last point be made a pole, its polar *abc*-surface touches $c = \kappa$ in the original point.

In the following sketch of the determination of the singular solution extreme cases, such as arise from the absence of a variable, &c. are purposely omitted. A relation which makes any differential coefficient of A, say A_p, infinite, gives

$$A_x + A_z p + A_p r + A_q s \text{ and } A_y + A_z q + A_p s + A_q t$$

proportional, independently of r, s, t, to

$$A_{px} + A_{pz}p + A_{pp}r + A_{pq}s \text{ and } A_{py} + A_{pz}q + A_{pp}s + A_{pq}t.$$

That is, $A_{x|z,p,q}$ and $A_{y|z,p,q}$ are proportional to $A_{p(x|z,p,q)}$ and $A_{p(y|z,p,q)}$. If B_p be also infinite, then A_p and B_p are related as if they were constant together.

We have then

$$\frac{B_{p(x|z,p,q)}}{A_{p(x|z,p,q)}} - \frac{B_{p(y|z,p,q)}}{A_{p(y|z,p,q)}} = 0, \quad \text{whence } \frac{B_{x|z,p,q}}{A_{x|z,p,q}} - \frac{B_{y|z,p,q}}{A_{y|z,p,q}} = 0.$$

Hence the biordinal is satisfied by a singular solution obtained by making differential coefficients of A and B simultaneously infinite. As before, one general formula will suffice for investigation of properties. Since $\phi = 0$, $\phi_{x|z} = 0$, $\phi_{y|z} = 0$ are made identical by $a = A$, $b = B$, $c = C$, the equations on the left determine A_p, B_p, C_p. (Remember that $\phi_p = 0$, $\phi_{p(x|z)} = \phi_z$, $\phi_{p(y|z)} = 0$.)

$$\begin{array}{ll}
0 + \phi_a \quad A_p + \phi_b \quad B_p + \phi_c \quad C_p = & \phi_a(\phi_{bx|}\phi_{cy|} - \phi_{cx|}\phi_{by|}) \\
\phi_z + \phi_{a(x|z)}A_p + \phi_{b(x|z)}B_p + \phi_{c(x|z)}C_p = 0 & + \phi_b(\phi_{cx|}\phi_{ay|} - \phi_{ax|}\phi_{cy|}) = 0. \\
0 + \phi_{a(y|z)}A_p + \phi_{b(y|z)}B_p + \phi_{c(y|z)}C_p = 0 & + \phi_c(\phi_{ax|}\phi_{by|} - \phi_{bx|}\phi_{ay|})
\end{array}$$

On the right is the ordinary condition under which A_p, B_p, C_p, become infinite: this is the condition of singular solution answering to $\phi_a\phi_{bx|} - \phi_b\phi_{ax|} = 0$ in equations of two variables. By eliminating a, b, c, between this condition and $\phi = 0$, $\phi_{x|z} = 0$, $\phi_{y|z} = 0$, we obtain a primordinal singular solution. The ordinary primitives of this satisfy the biordinal: not so the singular primitive, obtained from $\phi = 0$, $\phi_a = 0$, $\phi_b = 0$, $\phi_c = 0$.

If a, b, c, be now treated as variable, and x, y, z, as constants, correlative primordinal and biordinal equations may be obtained. The criterion of singular solution is the same for both equations. In the criterion obtained above, write the expansion of each complex term, as $\phi_{bx} + \phi_{bz}p$ for $\phi_{bx|}$: substitute $-\phi_x{:}\phi_z$ and $-\phi_y{:}\phi_z$ for p and q, and we shall obtain $T = 0$, T consisting of nine terms, of which $\phi_a\phi_x(\phi_{by}\phi_{cz} - \phi_{bz}\phi_{cy})$ and $\phi_a\phi_y(\phi_{bz}\phi_{cx} - \phi_{bx}\phi_{cz})$

are two, and the rest formed on the same type. This expression remains unaltered when a and x, b and y, c and z, are simultaneously interchanged: which proves the property asserted. The same kind of demonstration might be applied in the case of two variables.

21. The general method of transforming partial equations, given in the Appendix to my last paper, and which I here call that of *polar transformation*, amounts to the investigation of surfaces which possess a certain differential property by investigating their reciprocal polar surfaces from the correlative polar property. We take any *modular* equation we please, $\phi(x, y, z, \xi, \eta, \zeta) = 0$, and make each set of co-ordinates compensatory; so that $d_{x,y,z}\phi = 0$, $d_{\xi,\eta,\zeta}\phi = 0$. Of this it has been shown to be a consequence that $x = x(\xi, \eta, \zeta, \varpi, \kappa)$ where $\varpi = \zeta_\xi$, $\kappa = \zeta_\eta$: and the same of y, z, p, q: and *vice versâ*. We determine ξ, η, ζ, in terms of x, &c. from $\phi = 0$, $\phi_{x|z} = 0$, $\phi_{y|z} = 0$: and in all results the letters of the two systems are interchangeable. We determine p and q (when $-\phi_x : \phi_z$ and $-\phi_y : \phi_z$ followed by substitution are not more convenient) from

$$p = -\frac{y_{\xi|}z_{\eta|} - y_{\eta|}z_{\xi|}}{x_{\xi|}y_{\eta|} - x_{\eta|}y_{\xi|}}, \qquad q = -\frac{z_{\xi|}x_{\eta|} - z_{\eta|}x_{\xi|}}{x_{\xi|}y_{\eta|} - x_{\eta|}y_{\xi|}},$$

from which, as proved, ρ, σ, τ, (answering to r, s, t), disappear. We find r by complete differentiation of p, on the supposition that $y = \text{const.}$, or $y_{\xi|}d\xi + y_{\eta|}d\eta = 0$, which gives

$$r = \frac{d \, . \, p}{d \, . \, x} = \frac{p_{\xi|}d\xi + p_{\eta|}d\eta}{x_{\xi|}d\xi + x_{\eta|}d\eta} = -\frac{y_{\xi|}p_{\eta|} - y_{\eta|}p_{\xi|}}{x_{\xi|}y_{\eta|} - x_{\eta|}y_{\xi|}},$$

and so on. In $f(x, y, z, p, q, r, s, t, \ldots) = 0$ we substitute for x, y, z, &c., and thus produce $F(\xi, \eta, \zeta, \varpi, \kappa, \rho, \sigma, \tau, \ldots)$ of the same order as $f = 0$. If $F = 0$ can be integrated, we can find ζ, ϖ, κ, &c. in terms of ξ and η: and we then arrive at the primitive of $f = 0$ by eliminating ξ and η between $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, $z = z(\xi, \eta)$ or else by eliminating ξ, η, ζ, p, q between the integral found and the expressions for $\xi, \eta, \zeta, \varpi, \kappa$ in terms of x, y, z, p, q.

22. The notation by which *eliminants* (commonly called *determinants*) are expressed in writings which treat on their properties, is judiciously made to be as full as possible: but it is inconvenient for such applications as usually occur in connection with other symbols. I find it convenient to denote an eliminant, say of the fourth order, by $(ABCD_{pqrs)}$ or $(ABCD^{0\,\prime\,\prime\prime\,\prime\prime\prime)}$ according to the notation used for the component symbols.

Thus,

$$(A_{p)} \qquad = A_p,$$

$$(AB_{pq)} \qquad = A_p(B_{q)} - B_p(A_{q)} = A_pB_q - B_pA_q,$$

$$(ABC_{pqr)} \qquad = A_p(BC_{qr)} + B_p(CA_{qr)} + C_p(AB_{qr)},$$

$$(ABCD_{pqrs)} \quad = A_p(BCD_{qrs)} - B_p(CDA_{qrs)} + C_p(DAB_{qrs)} - D_p(ABC_{qrs)},$$

$$(ABCDE_{pqrst)} = A_p(BCDE_{qrst)} + B_p(CDEA_{qrst)} + C_p(DEAB_{qrst)} + D_p(EABC_{qrst)} + E_p(ABCD_{qrst)}.$$

It matters nothing which set of symbols is made progressive in the formation: thus

$$(ABC_{pqr)} \text{ is } A_p(BC_{qr)} + A_q(BC_{rp)} + A_r(BC_{pq)}.$$

If necessary, the two modes of progression may be distinguished as in $((ABC_{pqr)}$ and $(ABC_{pqr))}$, or as in $(ABC_{pqr)}$ and ${}_{(pqr}ABC)$.

In $(ABCD_{pqrs)}$ we have constructed the 24 ways of distributing p, q, r, s, among A, B, C, D, in such manner that $A_pB_qC_rD_s$, suggested by the order in the symbol, shall be positive, and that any two terms which are formed from each other by *one* interchange of contiguous symbols shall have different signs. Thus, $B_pC_qA_rD_s$ and $B_pA_qC_rD_s$ have different signs. When this is always true, it is clear that one contiguous interchange, such as is seen in passing from $(ABCD_{pqrs)}$ to $(ACBD_{pqrs)}$ changes the sign of the whole: whence* any odd number changes the sign, while any even number restores it. Hence it is easily proved that if the property be true for the n^{th} eliminant, it is true for the $(n+1)^{\text{th}}$. Suppose, for instance, that it is found true for every fifth eliminant. Then, since $(ABCDEF_{pqrstu)} = A_p(BCDEF_{qrstu)} - \ldots$ it is clear that the theorem is true for all interchanges which take place in fifth eliminants without affecting the multipliers. When the interchange affects a multiplier, take

$$(ABCDEF_{pqrstu)} = A_p(BCDEF_{qrstu)} - B_p(CDEFA_{qrstu)} + \ldots$$

For $(CDEFA_{qrstu)}$ write its equal $(ACDEF_{qrstu)}$ formed by four contiguous interchanges, and then resolve the fifth eliminants one step, which gives

$$A_pB_q(CDEF_{rstu)} + \ldots - B_pA_q(CDEF_{rstu)} - \ldots$$

whence the theorem is obvious as to terms beginning with A_pB_q and B_pA_q.

In passing from an even eliminant to the following odd one, the change of sign arises from the process which brings A from the end to the beginning, requiring an odd number of interchanges. This incomplete outline of a proof is merely given to exhibit the facility with which the notation applies.

It immediately follows that when two symbols of either set are the same, the eliminant vanishes. Let there be n homogeneous linear equations, $A_p\alpha + B_p\beta + \ldots = 0$, $A_q\alpha + B_q\beta + \ldots$, between $n+1$ quantities $\alpha, \beta, \ldots$: for instance, five equations between six quantities. Multiply the equations by $(CDEF_{qrst)}$, $(CDEF_{rstp)}$, &c. and add, as the terms of fifth eliminants are all additive.

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* In order to ascertain, by a mechanical process, and in any case however complicated, whether the number of contiguous interchanges which convert one arrangement into another be even or odd, proceed as follows. Write down one arrangement under the other, and beginning at one letter in one of the lines, mark the companion letter in the other line, pass on to that companion letter in the first line, mark its companion, and so on, until we arrive at a letter already marked. Call this sequence a *chain*, each mark being one *link*. Having formed one chain, begin at a letter not yet used and form another chain; and so on, until every letter has been used. Then, according as the number of chains with *even* numbers of links is odd or even, the number of displacements required is odd or even. The demonstration is very easy. As an example take

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
H	M	O	G	Q	B	K	L	J	P	F	C	I	N	A	D	E
1	2	1	2	3	2	2	1	2	2	2	1	2	4	1	2	3

Under A is H, under H is L, under L is C, under C is O, under O is A, already taken: the first chain has five links. The second is found to have nine, the third two, the fourth one. The number of chains having even links is *one*, an odd number; hence the number of contiguous interchanges by which either arrangement is converted into the other, is odd.

Then, since $(CCDEF_{pqrst)} = 0$, &c., we have

$$(ACDEF_{pqrst)} \cdot \alpha + (BCDEF_{pqrst)} \cdot \beta = 0, \text{ or } (CDEFA_{pqrst)} \cdot \alpha + (BCDEF_{pqrst)} \beta = 0.$$

And thus it will be found that the six quantities α, β, γ, &c. are in the proportion of the sixth eliminants of the fifth order $(BCDEF_{pqrst)}$, $(CDEFA_{pqrst)}$, $(DEFAB_{pqrst)}$, &c., with the alternate signs which they take in the formation of one of the sixth order.

In writing, and even in printing, the parentheses of the symbol $(ABC_{pqr)}$ or $(ABC^{\circ\,\prime\,\prime\prime)}$ may be omitted. The two sets of symbols having the same number in each is a guide sufficient to prevent mistake.

23. I now return to $Q + Rr + Ss + Tt + U(s^2 - rt) = 0$, which includes the polar form of the biordinal equation, and which must follow, independently of ϖ, from

$$B(x, y, z, p, q) = \varpi\{A(x, y, z, p, q)\}.$$

Some cases have no such primordinal solution, some have one only, and some have two: these last are *polar*, that is, have a solution of the form $\phi(x, y, z, a, b, c) = 0$, which may be called the *polar primitive*. The biordinal itself, derived from $B = \varpi A$, is

$$A_{x|z,p,q} B_{y|z,p,q} - A_{y|z,p,q} B_{x|z,p,q} = 0,$$

$$\text{or} \quad (AB_{xy)} + (AB_{py)} r + \{(AB_{qy)} + (AB_{xp)}\} s + (AB_{xq)} t + (AB_{qp)} (s^2 - rt) = 0,$$

where $(AB_{xy)}$ means $A_x B_y - A_y B_x$, &c. But A_x and A_y signify $A_{x|z}$ and $A_{y|z}$ or $A_x + A_z p$ and $A_y + A_z q$.

Let $B = \varpi A$ and $D = \kappa C$ be two primordinal solutions of one and the same biordinal. We have then

$$AB_{xy} = M \cdot CD_{xy}, \quad AB_{py} = M \cdot CD_{py}, \quad AB_{xq} = M \cdot CD_{xq}, \quad AB_{qp} = M \cdot CD_{qp},$$
$$AB_{qy} + AB_{xp} = M(CD_{qy} + CD_{xp}),$$

M being $M(x, y, z, p, q)$. Consequently,

$$AB_{xq} AB_{py} + AB_{xy} AB_{qp} = M^2 (CD_{xq} CD_{py} + CD_{xy} CD_{qp})$$
$$\text{or* } \quad AB_{xp} AB_{qy} = M^2 CD_{xp} CD_{qy}.$$

From this, and the fifth equation preceding, we see that either $AB_{xp} = M \cdot CD_{xp}$ and $AB_{qy} = M \cdot CD_{qy}$ or else $AB_{xp} = M \cdot CD_{qy}$ and $AB_{qy} = M \cdot CD_{xp}$. In the first case, all the six AB eliminants are proportional to the six CD eliminants; whence $(ABCD_{xypq)}$ becomes $2M$ multiplied by $AB_{xy} AB_{pq} - AB_{xp} AB_{yq} + AB_{xq} AB_{yp}$ and vanishes. Now $(ABCD_{xypq)} = 0$ is the criterion which shows that $f(A, B, C, D)$ can be identically satisfied by some form of f, z entering only as a function of x and y. But we shall presently discuss this result in a more satisfactory manner.

If we now look only at $B = \varpi A$, as a solution of $Q + Rr + \text{\&c.} = 0$, we have $AB_{xy} = MQ$, $AB_{py} = MR$, &c., whence we get, as before, $AB_{xp} + AB_{qy} = MS$, $AB_{xp} AB_{qy} = M^2 (RT + QU)$.

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[*] The two following equations are most essential in what follows:

$$ABCD_{abcd} = (AB_{ab} CD_{cd} + AB_{cd} CD_{ab}) - (AB_{ac} CD_{bd} + AB_{bd} CD_{ac}) + (AB_{ad} CD_{bc} + AB_{bc} CD_{ad})$$
$$AB_{ab} AB_{cd} = AB_{ac} AB_{bd} + AB_{ad} AB_{cb}.$$

Let $AB_{xp} = k . AB_{qy}$; then $kS^2 = (1 + k)^2 (RT + QU)$. The five equations of connexion may now be replaced by

$$AB_{py} = MR,\ AB_{xq} = MT,\ AB_{qp} = MU,\ AB_{xp} = \frac{k}{1+k} . MS,\ AB_{qy} = \frac{1}{1+k} . MS,$$

from which $AAB_{xpq} = 0$, $AAB_{ypq} = 0$, give

$$UA_x + TA_p - \frac{k}{1+k} SA_q = 0, \quad UA_y + RA_q - \frac{1}{1+k} SA_p = 0,$$

in which B may be written for A, and k (which must be the same in both equations) is either solution of $kS^2 = (1 + k)^2 (RT + QU)$. Hence the following theorem;—To determine the primordinals, if any, of $Q + Rr + Ss + Tt + U (s^2 - rt) = 0$, in which an arbitrary function explicitly occurs, as in $B (x, y, z, p, q) = \varpi A (x, y, z, p, q)$, find separately, by common methods, the complete solutions of

$$\left.\begin{aligned} U\left(\frac{dv}{dx} + p\frac{dv}{dz}\right) + T\frac{dv}{dp} - \frac{k}{1+k} S \frac{dv}{dq} = 0, \\ U\left(\frac{dv}{dy} + q\frac{dv}{dz}\right) + R\frac{dv}{dq} - \frac{1}{1+k} S \frac{dv}{dp} = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (v)$$

x, y, z, p, q being five independent variables, and k being one of the roots of

$$kS^2 = (1 + k)^2 (RT + QU).$$

If A and B be any functions of x, y, z, p, q, *each* of which satisfies *both* equations, then $B = \varpi A$ is one primordinal solution required. If no such common solution, or one only, should exist, there can be no primordinal of the form required.

For example, let $pq + (px + qy) s + xy (s^2 - rt) = 0$. Here k is either $px : qy$, or $qy : px$. We have, for the first,

$$y\frac{dv}{dx} + yp\frac{dv}{dz} - p\frac{dv}{dq} = 0, \qquad v = \phi (y, p, z - px, px + qy),$$

$$x\frac{dv}{dy} + xq\frac{dv}{dz} - q\frac{dv}{dp} = 0, \qquad v = \psi (x, q, z - qy, px + qy),$$

of which $v = px + qy$ is the only fundamental common solution. The second value of k gives

$$x\frac{dv}{dx} + xp\frac{dv}{dz} - q\frac{dv}{dq} = 0, \qquad v = \phi (y, p, qx, z - px),$$

$$y\frac{dv}{dy} + qy\frac{dv}{dz} - p\frac{dv}{dp} = 0, \qquad v = \psi (x, q, py, z - qy),$$

of which $v = qx$, $v = py$ are fundamental common solutions, and the only ones. Hence the given equation has a primordinal solution of the required form, $qx = \varpi (py)$, and no other.

The particular cases $A = \text{const.}$, $B = \text{const.}$, are solutions. If in $B = \varpi A$, we assume ϖA to be a function which approximates towards a constant, then $A_{x|}B_{y|} - A_{y|}B_{x|}$ will have a factor independent of r, s, t, which diminishes without limit. But if a division by this factor be made, the quotient, $Q + Rr +$ &c. is always $= 0$, absolutely, when $B = \varpi A$. This then is true at the limit, or when $B = \text{const.}$ But, though $A = \text{const.}$ and $B = \text{const.}$, are solutions, it does not follow that they can *coexist* as solutions. Thus in the preceding example, $py = a$, $qx = b$, are solutions, but cannot coexist; for $ay^{-1}dx + bx^{-1}dy = dz$ is not integrable. Coexistence requires that s, as obtained from p in $A = a$, $B = b$, should agree with s as obtained from q. Now, A_x still meaning $A_x + pA_z$, &c., we find from

$$A_x + A_pr + A_qs = 0 \text{ and } B_x + B_pr + B_qs = 0, \text{ that } AB_{xp} + AB_{qp}.s = 0;$$

we also find $AB_{yq} + AB_{pq}.s = 0$. Hence $AB_{xp} = AB_{qy}$ is the condition necessary to the coexistence of $A = a$ and $B = b$. Hence $k = 1$, the two values of k are equal, and $S^2 = 4(RT + QU)$ is a necessary equation of condition between the coefficients of the biordinal.

We cannot have two *coexisting* primordinals $P(A, B) = 0$, $Q(C, D) = 0$, unless $A = \text{const.}$, $B = \text{const.}$, $C = \text{const.}$, $D = \text{const.}$ can coexist. That the first and second pairs may coexist is obvious from the equations $P = 0$, $Q = 0$. Again, the coexistence of these last equations gives

$$PQ_{xp} = PQ_{qy} \text{ or } P_xQ_p - P_pQ_x = P_qQ_y - P_yQ_q.$$

Writing for P_x its value $P_AA_x + P_BB_x$, &c. we find

$$P_AQ_C(AC_{xp} - AC_{qy}) + P_BQ_C(BC_{xp} - BC_{qy}) + P_AQ_D(AD_{xp} - AD_{qy}) + P_BQ_D(BD_{xp} - BD_{qy}) = 0.$$

This cannot be always true, independently of the forms of P and Q, unless $AC_{xp} = AC_{qy}$, &c.

It is also clear that four distinct solutions $A = \text{const.}$ &c. cannot coexist, unless either of the four, A, B, C, D be identically a function of the other three. For, otherwise, elimination of p, q, z, would give a relation between the independent variables x and y.

When $Q + Rr +$ &c. is polar, that is, derivable from $\phi(x, y, z, a, b, c) = 0$, there are three coexistent solutions $A = a$, $B = b$, $C = c$. Consequently $S^2 = 4(RT + QU)$ is a necessary condition. And if three such solutions can be found, from the equations marked (v), it follows that $B = \varpi A$, $C = \kappa B$ satisfy $Q + Rr +$ &c. $= 0$. Substitution of A and B in (v) will give four equations, from which common elimination will restore the *four* relations seen in the *five* equations $AB_{yy} = MR$, &c. It may be easily verified that $A = a$, $B = b$, $C = c$, when satisfying (v), give a polar primitive by elimination of p and q. The relation between x, y, z, a, b, c thence obtained gives $A_x + A_zz_x + A_pp_x + A_qq_x = 0$, and similarly for B and C: in order then that $z_x = p$, we must have $A_x + A_zp + A_pp_x + A_qq_x = 0$, or, making the abbreviation hitherto used, $A_x + A_pp_x + A_qq_x = 0$. Eliminating p_x and q_x from the three equations thus obtained, we have $ABC_{xpq} = 0$: and similarly we obtain $ABC_{ypq} = 0$. These equations, resolved into $A_x.BC_{pq} +$ &c. $= 0$, $A_y.BC_{pq} +$ &c. $= 0$, give AB_{xy}, BC_{xy}, CA_{xy}, proportional to AB_{pq}, BC_{pq}, CA_{pq}, which is known to be true, the proportion being that of Q to $-U$.

Ampère has noted that $S^2 = 4(RT + QU)$ is a condition necessary in order that

$$Q + Rr + \&c. = 0$$

may have a primitive containing two arbitrary functions of one variable, such as we obtain from the polar form $\phi(x, y, z, a, b, c) = 0$, by assuming $b = \lambda a$, $c = \mu a$, and obtaining a from $\phi_a + \phi_b \lambda' + \phi_c \mu' = 0$.

The preceding investigation will sufficiently illustrate the restrictions under which equations of three variables are analogous to those of two. In the latter case, there is always *coexistence* of intermediate primitives, but not always so in the former: and the former cannot be treated co-ordinately with the latter except in the cases in which intermediate primitives coexist.

24. The notation for differential coefficients which I have now used for more than twelve years, with the extensions here proposed, will bring this paper into half the space which it would have filled, if all the formulæ had been written in the usual way. But this is not the only advantage. The equation which I have written as

$$(AB_{xy)} + (AB_{py)}\,r + \left\{(AB_{qy)} + (AB_{xp)}\right\} s + (AB_{xq)}\,t + (AB_{qp)}\,(s^2 - rt) = 0,$$

would, written in the ordinary way, have been

$$\begin{aligned}
&\left(\frac{dA}{dx} + \frac{dA}{dz}p\right)\left(\frac{dB}{dy} + \frac{dB}{dz}q\right) - \left(\frac{dA}{dy} + \frac{dA}{dz}q\right)\left(\frac{dB}{dx} + \frac{dB}{dz}p\right) \\
&\qquad + \left\{\frac{dA}{dp}\left(\frac{dB}{dy} + \frac{dB}{dz}q\right) - \frac{dB}{dp}\left(\frac{dA}{dy} + \frac{dA}{dz}q\right)\right\} r \\
&+ \left\{\frac{dA}{dq}\left(\frac{dB}{dy} + \frac{dB}{dz}q\right) - \frac{dB}{dq}\left(\frac{dA}{dy} + \frac{dA}{dz}q\right) + \frac{dB}{dp}\left(\frac{dA}{dx} + \frac{dA}{dz}p\right) - \frac{dA}{dp}\left(\frac{dB}{dx} + \frac{dB}{dz}p\right)\right\} s \\
&+ \left\{\frac{dB}{dq}\left(\frac{dA}{dx} + \frac{dA}{dz}p\right) - \frac{dA}{dq}\left(\frac{dB}{dx} + \frac{dB}{dz}p\right)\right\} t + \left(\frac{dA}{dq}\frac{dB}{dp} - \frac{dA}{dp}\frac{dB}{dq}\right)(s^2 - rt) = 0,
\end{aligned}$$

the first containing 55 types, without the parentheses 43, the second 258. And it would have been exceedingly difficult to have made the common form of the coefficients an object of thought, if the coefficients had been either detailed or represented by single letters.

The adoption of *occasional* symbols in abbreviation of formulæ, though often necessary, naturally belongs to cases in which the forms are not of frequent occurrence. In some few cases, as in the well known use of p, q, r, s, t, an occasional notation has become permanent: but this cannot often happen. The want of some system of abbreviation drives many writers to a frequent adoption of occasional symbols. For example, the following passage occurs in the proof which M. Cauchy (*Moigno*, Vol. II. p. 447) gives of his remarkable theorem already mentioned. "Représentons toujours par $F(x, y) = C$ l'intégrale générale de l'équation

$$dy = f(x, y)\,dx,$$

et soient d'ailleurs $\Phi(x, y)$, $X(x, y)$ les deux dérivées partielles de la fonction $F(x, y)$ par rapport aux deux variables x et y. De l'équation connue $\dfrac{\frac{du}{dx}}{M} = \dfrac{\frac{du}{dy}}{N}$, en y remplaçant u par $F(x, y)$, N par l'unité, et M par $-f(x, y)$, on tirera

$$X(x, y) = -\frac{\Phi(x, y)}{f(x, y)}."$$

This means that $F(x, y) = C$ gives $F_x + F_y . y' = 0$, so that $y' = f(x, y)$ gives

$$F_y = -F_x : f.$$

But the constant substitution, throughout the work, of distinct letters for differential coefficients, has established $Mdx + Ndy = 0$ as the representative of the differentiation of

$$F(x, y) = C.$$

If it be thought necessary to mark differentiation by a symbol, an accent may be added, as in y'_x used for $dy : dx$. This accent may be omitted in working or in continuous writing, and may be supplied in revising the manuscript.

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March 25, 1854.

APPENDIX.

ON THE CONVERGENCY OF MACLAURIN'S SERIES.

A REMARKABLE theorem was given by M. Cauchy in 1831 or 1832, and repeated in 1840, (*Exercices d'Analyse...* Paris, 1840, 4to. pp. 269–279; Moigno, *Leçons...* Paris, 1840, Vol. I. pp. 150–161), which appears to have attracted* considerable attention, and to have met with some objection. It is as follows:—The development of ϕx in powers of x continues convergent from $x = 0$ until x becomes equal to the least of all the moduli in the roots of the equations $\phi x\ \infty$, $\phi' x = \infty$: the modulus of a quantity, real or imaginary, being a positive quantity r, when the quantity is exhibited as $r(\cos\theta + \sin\theta . \sqrt{-1})$. I think M. Cauchy means to imply that divergence *must* begin when this limit is past: but I cannot find any express statement to this effect.

That this theorem is not generally true, may be at once established by the simple instance $\phi x = (1 + x)^{\frac{1}{2}}$, which satisfies the definition of continuity cited in the preceding note, and gives no finite real or imaginary root to either $\phi x = \infty$ or $\phi' x = \infty$. It ought then to have a convergent development for all values of x; but it is very well known to have a divergent development for all values of x greater than unity.

I shall show that this theorem cannot be true unless $\phi^{(n)} x = \infty$ be introduced into the conditions, for all values of n: and that both the direct and converse theorem may then be established. I think it probable that the assent which M. Cauchy's enunciation has received, arose from the impression that by a *continuous function*, between given limits, must have been intended a function which remains finite and determinate between those limits, and *all* its differential coefficients also. This seems to me to be the best definition of the term; and if my supposition as to the mode in which the imperfect theorem gained assent be correct, there is good reason to think that, by express statement, this definition† might soon be established. In this case it might be desirable to say that there is continuity of the n^{th} order, when between

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[*] In the *Exercices* M. Cauchy says "Parmi les théorèmes nouveaux, que contiennent mes mémoires de 1831 et 1832, sur la mécanique céleste, l'un des plus singuliers, et en même temps l'un de ceux auxquels les géométres paraissent attacher le plus de prix, est celui qui donne immédiatement les règles de la convergence des séries fournies par le développement des fonctions explicites, et réduit simplement la loi de convergence à la loi de continuité, la définition des fonctions continues n'étant pas celle qui a été longtems admise par les auteurs des traités d'algèbre, mais bien celle que j'ai adoptée dans mon *Analyse Algébrique*, et suivant laquelle une fonction est continue entre des limites données de la variable, lorsque entre ces limites elle conserve constamment une valeur finie et déterminée, et qu'à un accroissement infiniment petit de la variable correspond un accroissement infiniment petit de la fonction elle-même." I may also add that M. Cournot, in his excellent work on the infinitesimal calculus (Paris, 1841) states the theorem, in the form in which M. Cauchy gave it, and without any doubt of its perfect generality. The definition of continuity here given supposes all values, real and imaginary, of the variable.

[†] Any definition would be better than none at all. At present it is (to me at least) a most difficult task, in reading a work, first, to find out the author's definition of continuity, next, to find out whether he sticks to it. M. Moigno, in his enumeration of cases under M. Cauchy's theorem, sets down $(1+x)^{m \div n}$, generally, as discontinuous when $x = 1$: here is an implied definition of continuity differing from that on which the theorem is constructed.

the given limits neither the function itself, nor any one of the first n differential coefficients becomes infinite or indeterminate.

There has been some controversy on M. Cauchy's demonstration in Liouville's journal. With this I have nothing do, as my own demonstration has no affinity whatever with that in dispute. That M. Cauchy has not fully stated the conditions, is evident from instances alone. But further, it is plain from the last paragraph in this page, that if $\phi' x = \infty$ enters the conditions, $\phi'' x = \infty$ must do the same: and so on.

Let $x = r(\cos\theta + \sin\theta . \sqrt{-1})$, and let r be called the *modulus* of x, and let θ be called the *directing angle*, or simply the *angle* of x. Let the modulus[*] be always taken as positive; the simple negative quantity requiring the angle π, or one of equivalent direction. It would be little more than legitimate extension if we were to call r the *value* of x, and $\cos\theta + \sin\theta . \sqrt{-1}$ its *sign*: but in that little would lie the seed of much confusion.

Let any value of x, real or imaginary, which makes ϕx, or *any one of its differential coefficients*, infinite, be called a *critical value* of ϕx itself. Thus $x = \pm\sqrt{-1}$ are the only critical values of $(1 + x^2)^{\frac{1}{2}}$. The theorem which I now proceed to prove is as follows:—The development of ϕx, in ascending powers of x, remains convergent from $x = 0$ up to $x =$ the least of the critical moduli; after which it is always divergent.

I avoid all discussion of the different kinds of convergency, and of the effect produced upon it by variations of sign in the terms, by the following explanation. The given series being $a_0 \pm a_1 x \pm a_2 x^2 \pm \ldots\ldots$, the series $a_0 + a_1 x + a_2 x^2 + \ldots\ldots$ may be always convergent when $x < h$, and divergent when $x > h$. The value of h is the object of the inquiry, without any consideration of whether there be convergency or divergency when $x = h$. When $a_{n+1} : a_n$ finally obtains a permanent approach to a fixed limit, the limit is h^{-1}, and after $x = h$ the terms finally increase without limit. But when the value of $a_{n+1} : a_n$ never ceases to *undulate*, it may be possible that from $x = h$ up to some other definite value, the undulations may prevent $a_n x^n$ from finally increasing without limit, and may perhaps even allow it to diminish without limit, the series being still divergent. I omit all such inquiry, because it is not essential here; merely remarking that undulating convergency and divergency, in series with all their terms positive, have received so little consideration, that it would not be safe to hazard any assertion upon how they might or might not take place in a series of the form $a_0 + a_1 x + \ldots\ldots$, in passing from one value of x to another.

In the series $a_0 + a_1 x + a_2 x^2 + \ldots\ldots$ in which all the letters are positive, the limit of convergency is the same for the series and all its differential coefficients. First, a divergent series gives a divergent differential coefficient, for $a_1 + 2a_2 x + \ldots\ldots$ is of the same character as $a_1 x + 2a_1 x^2 + \ldots\ldots$ which certainly diverges with $a_0 + a_1 x + \ldots\ldots$ Next, if $a_0 + a_1 x + \ldots\ldots$ be convergent, that is, if $x < h$, it is convergent when we write $x + m$ for x, $x + m$ being still $< h$: hence $a_0 + a_1 x + \ldots\ldots + (a_1 + 2a_2 x + \ldots\ldots) m + \ldots\ldots$ can have no divergent coefficient, and all the differential coefficients of $a_0 + a_1 x + \ldots\ldots$ are convergent.

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[*] We often want to signify the value of a real quantity, independently of its sign: that is, to speak of the quantity, if it be positive, and of the quantity with its sign changed, if it be negative. What we ought to mention, then, is the *modulus* of the quantity.

Let $a_0 \pm a_1x \pm \ldots\ldots$ be one of the differential coefficients of ϕx, and let

$$r(\cos\theta + \sin\theta \,.\, \sqrt{-1})$$

be a value which makes it infinite. Consequently, one of the two series

$$a_0 \pm a_1\cos\theta.r \pm a_2\cos 2\theta.r^2 \pm \ldots\ldots \text{ and } \pm a_1\sin\theta.r \pm a_2\sin 2\theta.r^2 \pm \ldots\ldots$$

must be infinite. Hence $a_0 + a_1r + a_2r^2 + \ldots\ldots$ must be divergent: whence the original series for ϕx must be divergent when x is greater than r. We cannot undertake to say this when $x = r$: *at* the point of transition, convergency or divergency must be discovered by the closer rules.

So much as this is then proved, namely, that ϕx cannot remain convergent after x has passed any one critical modulus whatsoever. It remains to shew the converse, that ϕx cannot become divergent until it *has* passed a critical modulus: that is, ϕx remains convergent until x reaches the least of the critical moduli.

First, let all the coefficients be positive, or let $\phi x = a_0 + a_1x + a_2x^2 + \ldots\ldots$ Let $P_{n+1}x^{n+1} = a_{n+1}x^{n+1} + \ldots\ldots$ and let the point of transition be at $x = h$. Let ξ be a quantity which may be as small as we please, and take n so great that $a_0 + a_1x + \ldots\ldots + a_nx^n$, at $x = h + \xi$, is greater than ϕx, if ϕx be real, or than the real part of ϕx, if ϕx become imaginary after $x = h$. Let ϕh be finite, for if ϕh be infinite, the theorem is obvious. Consequently, at $x = h - \xi$, $P_{n+1}x^{n+1}$ is positive, being the sum of a converging series of positive terms: but at $x = h + \xi$, $P_{n+1}x^{n+1}$ is either negative, or imaginary with a negative real part. This change, which begins at $x = h$, since we may suppose ξ of any finite degree of smallness, is not made by passage of P_{n+1} through zero. For if it were so, we should have $\phi h = a_0 + \ldots\ldots + a_nh^n$, which is absurd if the series $a_0 + a_1h + \ldots\ldots$ be convergent; and equally so if it be divergent, for we may begin by supposing that n is so great as to make $a_0 + \ldots\ldots + a_nh^n$ greater than ϕh. The change then must be made, either through infinity, or by an absolute discontinuity. In either case $\phi^{(n+1)}h$ is infinite, for if it were finite (since we know that $\phi^{(n+1)}x$ never becomes infinite *before* $x = h$ by the previous part of the theorem) we should have $P_{n+1} = \phi^{(n+1)}(\lambda h) : 2.3\ldots(n+1)$, where $\lambda < 1$, that is, P_{n+1} would be finite and continuous at $x = h$, and for some subsequent interval.

Consequently, when all the coefficients of development are positive, it follows that *before* divergency begins, some differential coefficient of ϕx becomes infinite, and for a real and positive value of x. This may mean *when* divergency begins: that is, the series may become divergent *at* the transition value of x. And it is evident that the coefficients need not be positive from the outset: it is enough that they are never otherwise after a finite number of terms. And hence ultimate permanence of positive sign in the differential coefficients of ϕx, when $x = 0$, indicates one of two things; either the series is never divergent, or the root which gives the least critical value is real and positive.

Next let us suppose, in $a_0 \pm a_1x \pm a_2x^2 \pm \ldots$, that instead of having all the signs positive, the signs are varied, but in such a manner that there is ultimately a recurring cycle of signs, l in number; as for instance, $l = 6$, the cycle being $(+\ +\ -\ +\ -\ +)\,(+\ +\ -\ +\ -\ +)$ &c. If this cycle be permanently established after a finite number of terms, the alterations necessary to establish a cycle commencing with a_0 will not effect the critical values of ϕx, since they can but amount to the addition or subtraction of certain terms of the form $2a_kx^k$. Let us suppose it, then, established from the beginning: that is, whatever sign the term in x^k may have, $(k < l)$, the terms in x^{k+l}, x^{k+2l}, &c. have the same. If these terms be negative, in order to make them all positive, we must add $2(a_kx^k + a_{k+l}x^{k+l} + \ldots)$ to ϕx. This, $1, \lambda, \lambda^2, \ldots \lambda^{l-1}$ being the lth roots* of unity, is

$$\frac{2}{l}\{\phi x + \lambda^{l-k}\phi(\lambda x) + (\lambda^2)^{l-k}\phi(\lambda^2 x) + \ldots\ldots\},$$

and for every negative term which occurs in the first l terms, a similar addition must be made. Hence, if ϕx be of the form $a_0 \pm a_1x \pm a_2x^2 \pm \ldots\ldots$, with a cycle of l in the signs, the series $a_0 + a_1x + a_2x^2 + \ldots\ldots$ in which all the terms are positive, is of the form

$$A_0\phi x + A_1\phi(\lambda x) + A_2\phi(\lambda^2 x) + \ldots\ldots + A_{l-1}\phi(\lambda^{l-1}x).$$

From this it follows, by the last case of the theorem, that divergence, if it ever begin, begins immediately at or after a real and positive value of x which makes some differential coefficient of the above infinite.

One or more cases, then, of $\phi^{(n)}(\lambda^c x)$ must become infinite for a real and positive value of x, and the theorem is proved for every case in which a cycle of signs is ultimately established, with the addition that if the cycle of signs go through an extent of l terms,

$$\cos\theta + \sin\theta \,.\, \sqrt{-1}$$

must be an lth root of unity in the critical value which gives the least modulus.

Lastly, suppose that no cycle of signs is ever established. Let fx be the series of positive terms $a_0 + a_1x + \ldots\ldots$, and let Fx be the series which has cycles of signs, with l in each cycle, and each cycle agreeing with the first l terms of ϕx. A process similar to the one already used will give

$$Fx = A_0fx + A_1f(\lambda x) + A_2f(\lambda^2 x) + \ldots\ldots + A_{l-1}f(\lambda^{l-1}x),$$

and the greater l is taken, the more nearly will Fx represent ϕx through the whole period of convergency, and, consequently, for all values of x, unless we suppose ϕx discontinuous. Many terms may disappear by the vanishing of their coefficients; and, in particular, the cases in which $A_0 = 0$ will be very frequent. If $f^{(m)}h = \infty$, we shall always have $f^{(m)}(\lambda^c x) = \infty$ when $x = h\lambda^{-c}$, and thence $F^m(h\lambda^{-c}) = \infty$. And if $f^{(m)}x = \infty$ cannot be true for any modulus of x less than h, then neither can $F^{(m)}x = \infty$ for any modulus less than h. Up to the limit then, and at the limit, when l is infinite, and $Fx = \phi x$, the theorem is true, the directing angle being possibly incommensurable with 2π when the case gives an infinite value of c.

It is thus proved that divergence cannot begin before the lowest critical modulus of ϕx; it has been proved that convergence cannot last beyond that value: and these two assertions contain the whole theorem.

When the series is always divergent, we might at first suppose it indicated that ϕx or some differential coefficient becomes infinite when $x = 0$: that is, we might suppose it proved that Maclaurin's theorem can never yield an always divergent development with finite coefficients. And, so far as functions of finite form are concerned, this is true.

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* This ought to be called *Simpson's Theorem*. It was given by Thomas Simpson in the Philosophical Transactions, as read November 16, 1758.

So far as is known, those functions of x which yield developments of permanent divergency can be nothing but divergents of another form, for instance, definite integrals the subjects of which become infinite between the limits. As an example, let

$$\phi x = \int_0^\infty \frac{\epsilon^{-t} dt}{1 - xt} = 1 + x + 2x^2 + 2 \,.\, 3 \,.\, x^3 + 2 \,.\, 3 \,.\, 4x^4 + \ldots$$

Here, x having any positive value, however small, ϕx is an integral the subject of which becomes infinite when $t = x^{-1}$. Those who would contend that this definite integral always represents *infinity* when x has a positive value will see a confirmation of their theorem: those who suspend their opinion on this point must wait for other representation of ϕx before they see either confirmation or refutation.

Should the preceding demonstration be held sound, it is clear that it may be, with advantage, introduced into elementary instruction. As an instance of its effective use, take $\phi x = x(\epsilon^x - 1)^{-1}$. We see beforehand that the development will be convergent up to $x = 2\pi$, as is otherwise known. If my memory be correct, I have seen it stated that this development is always divergent.

It is equally obvious that $(1 + \epsilon^x)^{-1}$ is convergent only up to $x = \pi$.

Among remarkable consequences we find that the product of a number of series is convergent only so long as they are all convergent. But if a division by a series, ϕx, be introduced, then $\phi x = 0$, $\phi' x = \infty$, $\phi'' x = \infty$, &c. are introduced among the *competing* equations.

When ϕx is a rational fraction, a verification may be procured with ease, if the least modulus belong to a real root. Let $\phi x = (1 - \alpha x)^{-1}(1 - \beta x)^{-1}$, whence we find

$$a_n = \alpha^n + \alpha^{n-1}\beta + \ldots + \beta^n.$$

By a well known theorem, the series is convergent up to $x =$ the limit of $a_{n-1} : a_n$, if this fraction finally tend to a limit. Now when α and β are real, $a_{n-1} : a_n$ does tend to a finite limit, but which cannot be determinately expressed; it is α^{-1} or β^{-1}, whichever is numerically the less of the two, or has the lesser modulus. This agrees with the theorem. But when α^{-1} and β^{-1} are of the form

$$r(\cos\theta \pm \sin\theta \,.\, \sqrt{-1}), \text{ then } a_{n-1} : a_n \text{ or } (\alpha^n - \beta^n) : (\alpha^{n+1} - \beta^{n+1}) \text{ becomes}$$

$$\sin n\theta : r \sin(n+1)\theta, \text{ or } r^{-1}(\cos\theta + \tan n\theta \,.\, \sin\theta)^{-1}.$$

The ordinary theorem here fails, because there is no definite limit; the theorem in this paper settles the point. If, as I believe, $\tan \infty = \pm\sqrt{-1}$ be a true result, the limit of $a_{n-1} : a_n$ is the same in form when α and β are imaginary as when they are real, namely, α^{-1} or β^{-1}, indeterminately.

M. Cauchy's form of the theorem, though imperfect in the statement of the conditions, does not affect the validity of his application to Lagrange's theorem. If $y = z + x\phi y$, we have

$$(\psi y)_x = \frac{\psi' y \phi y}{1 - x\phi' y}.$$

If ϕy and ψy and their derived functions continue finite, the development of ψy continues convergent so long as $1 - x\phi' y$ does not vanish, but becomes divergent for all values of x greater than the first which gives $1 - x\phi' y = 0$. Now the development sets out with $y = z$ when $x = 0$, and consequently represents the value of ψy calculated by means of that root of $y = z + x\phi y$ which becomes z when $x = 0$. When $1 - x\phi' y = 0$, two roots become equal, one of them being the value of y hitherto used. Hence the convergency of the series does not cease until another root, as it were, meets that particular root which is z when $x = 0$. When ϕy is a function which can only become infinite when y is infinite, the root selected by the series, while convergent, is the least of all the real roots of its kind, positive or negative. For when x is very small, the only way in which $y = z + x\phi y$ can give a value of y sensibly different from z is by ϕy being very great, that is, by y being very great. Here, then, the root selected is the least of all the roots, and since convergency cannot cease until another root meets it, and since also the other roots begin with infinite values, it follows that it continues to be the least root during the whole convergency. [*April* 29, 1854].

The following is a method of establishing and extending Lagrange's theorem. When I say there is an extension, I mean in the sense in which the development of $\phi(x, y, z)$ in powers of x, y, z, is an extension of Maclaurin's theorem. Undoubtedly $\phi(xv, yv, zv)$ developed in powers of v, making $v = 1$ after the process, is a case of one variable only: and in like manner any result of the following method may be seen from the usual form of Laplace's extension (if in this paragraph I may call it an extension) of Lagrange's theorem.

Let all the capital letters used be functions of u: and let u be such a function of w, x, y, z,... as will be constant only under a linear relation between these variables. That is, let

$$wW + xX + yY + zZ + \dots\dots = 0.$$

A term dependent only on u is merely a term with an unused variable which at last becomes unity.

From the above we easily obtain

$$\frac{1}{W}\frac{du}{dw} = \frac{1}{X}\frac{du}{dx} = \frac{1}{Y}\frac{du}{dy}, \dots\dots; \qquad \frac{d}{dw}\left(V\frac{du}{dx}\right) = \frac{d}{dx}\left(V\frac{du}{dw}\right) \dots\dots\dots\dots(1)$$

in which for u may be written any function of u. Hence, in any differential coefficient of $V(dU:dx)$ the independent variable of one operation may be changed into any other independent variable, provided that V be multiplied by the coefficient (in $Ww + \dots = 0$) of the extruded variable, and divided by the coefficient of the intruded variable. For

$$\frac{d}{dw}\left(V\frac{dU}{dx}\right) = \frac{d}{dw}\left(\frac{VX}{Z}\frac{dU}{dz}\right) = \frac{d}{dx}\left(V\frac{dU}{dw}\right) = \frac{d}{dx}\left(\frac{VW}{Z}\frac{dU}{dz}\right) = \frac{d}{dz}\left(\frac{VW}{Z}\frac{dU}{dx}\right), \text{ \&c.}$$

Hence, $$\frac{d^{m+n+p}}{dw^m dx^n dy^p}\left(V\frac{dU}{dw}\right) = \frac{d^{m+n+p}}{dz^{m+n+p}}\left(\frac{VW^{m+1}X^n Y^p}{Z^{m+n+p+1}}\frac{dU}{dz}\right).$$

If, after differentiation, $w = 0$, $x = 0$, $y = 0$, these suppositions may be made on the second side before differentiation; that is, u may be obtained from $zZ + \dots = 0$, for substitution in U, V, W, &c.

If we develope U in powers of w, x, y, we have for the coefficient of $w^m x^n y^p : (1 . 2 \ldots m)(1 . 2 \ldots n)(1 . 2 \ldots p)$ the value ($w = 0$, $x = 0$, $y = 0$) of

$$\frac{d^{m+n+p}U}{dw^m dx^n dy^p} \text{ or } \frac{d^{m+n+p-1}}{dw^m dx^n dy^{p-1}}\left(\frac{dU}{dy}\right), \text{ or } \frac{d^{m+n+p-1}}{dz^{m+n+p-1}}\left(\frac{W^m X^n Y^p}{Z^{m+n+p}}\frac{dU}{dz}\right);$$

u, in the last, being derived from $zZ + \ldots = 0$. This residual equation ought to be of finite solution, though cases might possibly arise in which it would be desirable to apply the whole theorem to the residual equation itself, and to obtain a sufficient number of cases of $W^m \ldots\ldots (dU : dz)$ expanded in powers of z.

The following is a real extension. Let u be defined by the equation

$$F(u,\ wW + xX + \ldots,\ w_1W_1 + x_1X_1 + \ldots,\ w_2W_2 + x_2X_2 + \ldots,\ \ldots\ldots) = 0.$$

The first theorems in (1), with the rule of intrusion and extrusion, remain true for all variables which are in the same set: the second theorems for all variables whatsoever. Hence U may be expanded, from a residual equation, in powers of any of the variables. For example, suppose

$$u = F(x + \xi\phi u,\ y + \eta\chi u,\ z + \zeta\psi u).$$

The general term of the development of U has for the coefficient of $\xi^l \eta^m \zeta^n \div (2 . 3 \ldots l)$ &c. either

$$\frac{d^{l+m+n-1}}{dx^l dy^m dz^{n-1}}\left(\overline{\phi u}^{\,l}\,\overline{\chi u}^{\,m}\,\overline{\psi u}^{\,n}\frac{dU}{dz}\right) \text{ or } \frac{d^{l+m+n-1}}{dx^l dy^{m-1} dz^n}\left(\overline{\phi u}^{\,l}\,\overline{\chi u}^{\,m}\,\overline{\psi u}^{\,n}\frac{dU}{dy}\right), \text{ &c.:}$$

in which $u = F(x, y, z)$. The extension given by Laplace (Berlin Memoirs, 1777, p. 116; Lacroix, Vol. I. p. 282), which exhibits the case of expansion of a function of u, v, &c., where $u = F(x + X\xi)$, $v = G(y + Y\eta)$, &c., in which X, Y, &c. are functions of u, v, &c., may be both simplified and extended by the above mode of treatment.

A. DE MORGAN.

December 20, 1854.

XXIII. *On the Purbeck Strata of Dorsetshire. By the* Rev. OSMOND FISHER, F.C.P.S. F.G.S. *Fellow and Tutor of Jesus College.*

[Read *Nov.* 13, 1854.]

I HAVE been induced to offer the present paper to the Society, because it describes the occurrence of the interesting fossil remains of insects, which it was my good fortune to disinter, and which now form a portion of our noble Woodwardian Collection.

The occurrence of the remains of insects in the Dorsetshire Purbecks was first observed in 1851, by my friend the Rev. J. H. Austen, of Jesus College, who was then residing at Langton, near Swanage. He was led to search for them in Durlstone bay in consequence of having read Mr Brodie's work on fossil insects. Specimens were subsequently obtained by the members of the Geological Survey, who were engaged at that time upon one of the Dorsetshire sheets of their maps. In the spring of 1852, while on a visit to Mr Austen, I collected my first set of specimens, and I afterwards obtained fossil insects from other localities where the Purbecks appear in the neighbourhood of Weymouth, distant about twenty miles from Durlstone bay*.

I believe it to be now the general opinion that the Purbeck beds are an estuary deposit belonging to the Oolitic system: and this opinion appears to me to be borne out by their always, as far as I know, accompanying the Portland beds; for wherever the latter occur they are found capped with the freshwater and estuary limestones and clays of the Purbeck series. This is the case in Wiltshire and Buckinghamshire as well as in Dorsetshire. There is also a resemblance in lithological composition between the Portland beds and the lower portion of the Purbecks, and both have a peculiar bituminous smell when rubbed or struck. Their junction also in Dorsetshire is often so intimate that the same block of stone, as it comes out of the quarry, contains the marine fossils of the Portland series in its mass, and about four inches of freshwater limestone on its upper face full of Cyprides and Cyclas.

The immediate superposition of the sands and clays of the Hastings series on the Purbeck is a very remarkable fact, shewing that the same area which formed the embouchure of the Purbeck river performed still the same office for the Hastings river. But the entirely different character of the deposit shews at least that the soil of the country drained by the latter was different from that which supplied the former; and it is evident also that in the district under consideration the motion of the water of the Hastings river was much more rapid, from the abundance of sand, coarse quartz, and gravel, with some pebbles as large as a pigeon's egg in the heavier drifts.

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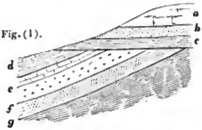
* The specimens were exhibited.

I am not however inclined to consider the Purbecks as a portion of the Wealden series, though in immediate sequence; and I am strengthened in this view by the fact of never having seen any of the bones of the gigantic Saurians of the Weald from any part of the Purbeck beds*.

It is evident that in the Dorsetshire district both the Purbecks and the Wealden beds are quite unconformable to the Cretaceous system, which not only overlaps them gradually, as it does all the Oolites in its extension westward, but also covers them very suddenly at some spots. At Osmington Mills, near Weymouth, we have within less than a square mile the upper Green sand in contact, 1st, with the Hastings sand, 2dly, with the Purbecks, 3dly, with the Kimmeridge clay. And at White Nose Cliff this very instructive section is exposed, and with a telescope may be observed from Weymouth.

(*a*) Chalk.
(*b*) Green sand.
(*c*) Galt.
(*d*) Purbeck.
(*e*) Portland stone.
(*f*) Portland sand.
(*g*) Kimmeridge clay.

Fig. (1).



The late Dr Mantell states in his Wonders of Geology, that he has never been able to obtain pebbles of any but the smallest size from the Wealden strata of Sussex. I can affirm the same with respect to the Isle of Wight, and in Swanage bay there is very little that can be called coarse drift, but more than in the Island. When, however, we reach Worbarrow bay, a few miles westward, there is much coarse drift, sometimes assuming even the character of a fine conglomerate, and at Lulworth I have found pebbles as large as a pigeon's egg, with coarse grains of mica. The same coarse character of the detritus may be remarked at Ridgway-hill; and it is curious to observe how suddenly the coarse drift is frequently superimposed upon a fine clay or sand, and then gradually diminishes in coarseness until it passes again into a fine sand; thus relating as plainly as if it were written the sudden swelling of the ancient torrent, and its then gradually subsiding into its normal state. The larger grains consist of quartz, and the pebbles when not of quartz seem always referable to crystalline rocks. There is also much mica†. Yet the abundance of iron has sometimes led me to speculate upon the new red sandstone of Devonshire as the most suitable deposit to have furnished the requisite materials; for that deposit abounds in layers of waterworn pebbles from the older rocks.

The difference thus observable between the state of the detritus in the Hastings sands of Sussex, Hants, and Dorset, seems to lead to the conclusion, that the stream flowed from the west towards the east. This supposition is also favoured by the rapid increase of thickness in the deposit as we proceed eastward. In the Isle of Wight, the lowest beds which appear at Brixton bay are some beds of white sandstone, the apparent equivalents of which are not half way down the Swanage section, while the impression conveyed to the mind is, that the beds exposed in the former locality are nearly as thick as the whole Swanage series. At Worbarrow bay the thickness is much less than it is at Swanage; and again on the west side of Worbarrow less than on the east. The diminution is very striking as we pass on to Lulworth Cove. At Ridgway-hill the fault cuts off the upper portion, but the part that is left (about 350 feet) does not militate against the truth of this assertion[*].

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* Perhaps, however, the quieter waters could not bring them down.

† The firmness of sea-sand seems due to mica. Contrast the sands at Bournmouth derived from the lower Bagshots with those at Brixton or Swanage from the Hastings series.

It must be owned, however, that the unconformable superposition of the upper green sand and galt, which form the superior limit of the Hastings sand, somewhat lessens the strength of the argument drawn from the change in thickness. The disappearance of the marine bands in the upper part of the series which occur in the Island, and at Punfield in Swanage bay, is in accordance with either supposition, that we are receding from the ancient sea, or that the upper beds have been cut off as we proceed westward[†].

I am not able to give any information respecting the part of the series in which the Iguanodon's bones have been found at any of the localities I have mentioned except Ridgway. There they occurred rather abundantly in a stiff grey clay near the base of the series, just above the passage beds which connect these sands and clays with the upper Purbecks.

With these few remarks upon the Hastings sands as seen in Dorsetshire, I will now proceed to describe the Purbeck beds.

The localities to which I shall refer are Durlstone bay, close to the town of Swanage; Worbarrow forming the Eastern headland of the bay of that name, ten miles westward; Mupes bay and Bacon hole, at the west of the same bay, a mile and a half across; Lulworth Cove, a mile further, with Ridgway and Upwey ten miles still further to the west. Hence in the whole district we include a range of about three and twenty miles. As might be expected, the lithological character of the deposit varies considerably within these limits. The grandest development is undoubtedly that of Durlstone bay. At Worbarrow the development still resembles that at Durlstone bay, but in crossing to Bacon hole the deterioration of the series is very perceptible, and the development on the whole is not greater than that at Ridgway—the other extremity of our district.

Professor E. Forbes has divided the Purbeck series into upper, middle, and lower, and has found that these several divisions are "each marked by a peculiar assemblage of organic remains[‡]." Following his division, we shall find that the upper and middle divisions are most largely developed at Durlstone bay, while the lower (at least towards the bottom of it) is most developed at Ridgway, and to a casual observer it is striking that all the stone-quarries at Swanage are in the middle Purbecks (except the marble in the upper), while at Ridgway all the quarries are in the lower. The great mass of the stone-beds is due to animal secretion, so that, somewhat paradoxically, we should have the greater deposit where the water was clearer, and vice versa; and this accounts for the upper portion of the lower Purbecks, contrary to the general condition of that part of the series, being more developed at Durlstone, because at both Durlstone and Ridgway these beds (which are in fact the insect beds) are a sedimentary deposit from water turbid with calcareous mud. In other words, where the shelly and cypris beds are thinner there the sedimentary will be thicker.

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[*] Unios occur both at Swanage and Ridgway in some hard, conglomerate-looking bands.

[†] At Punfield there is undoubtedly a fault separating the galt from the Hastings sand. I saw it distinctly in a small sand-pit on the 30th of June, 1853. The galt following the chalk and green sand is vertical, while the Hastings sands dip at a high angle towards the north.

[‡] British Association Reports, 1850, p. 79.

It is difficult to obtain a clear notion of the thickness of the upper division at Swavage, on account of the very disturbed position of that portion of the series, and the want of a deep section; but the general impression conveyed to the mind is that of a series of bands of clay and stone, about 30 feet thick, with very little sand or drift, the clays being charged with cypris shales, and the stone bands with Unios and Paludinas. Mr Austen has given a detached section of them at page 9 of his tract*, including about eight feet, but not joining on to any general section. This however is evident—that they are far less sandy, and contain far less drift, than the same beds at Ridgway. *Here* too there is a thick band of Paludina marble. At Ridgway it is very inadequately rather hinted at than represented, while the shells are individually very small, the sandy bottom not having suited them so well. At Mupes bay this part of the series bears a close resemblance to its development at Durlstone, but is not so thick. The marble beds are also thinner. There is, however, a greater variety of species of Cypris here than at Durlstone bay.

At Ridgway these upper beds are on the whole much coarser and more sandy in their texture, and contain a larger proportion of the spoils of the land, in the shape of lignite and plants, &c. There are, however, a few beds of fine-grained marls, and some clays with Cyprides, but on the whole there is evidence of a greater transporting power in the water than in the more eastern localities of our district; and also (what is generally observable at Ridgway) of a more variable and frequently changing condition of it. The best specimens of Unios have been obtained from this locality. I have not been able to obtain any insect remains from the Upper Purbecks.

The middle division of the Purbecks is very different in its lithological character from that which we have just considered. The ordinary condition of its limestone beds is that of a congeries of bivalve shells (the paving of Neville's Court at Trinity College, and of the Hall of Pembroke College is from these beds); and they shew several alternations of fresh, brackish, and salt-water types. Comparing these beds at the same three localities at which we have before compared the upper division—at Durlstone the middle beds attain the greatest thickness, and these form the most important division of the whole Purbeck series. In them are situated the extensive quarries for which the so-called Island has long been famous; and though the chief supply of paving for the London market has now fallen into the hands of Yorkshire stone-merchants, large quantities of this stone are still raised for the supply of Portsmouth and other places in the south. The first bed of stone is that to which Professor Forbes has given the name of the "broken shell limestone." It is composed of shells of a species of Cyrena, and attains a thickness in Durlstone bay of no less than nine feet. Next succeeds a series of about twenty-five feet of shales and clays, with layers of crushed bivalve shells and cyprides. Among these beds occur bands of fibrous carbonate of lime, called locally "beef" and "horseflesh." The crystals of this mineral are usually found shooting upwards from a band of perished bivalves, and appear due to a change in the condition of the shells on which they rest. The connexion of these bands of "beef" with deposits of bivalve shells is most strikingly seen in the neighbourhood of Osmington.

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* A Guide to the Geology of the Isle of Purbeck. W. Shipp, Blandford, 1852.

Next we come upon a mass of strata containing bands of apparently marine origin, interstratified amidst others still denoting the prevalence of a brackish condition of the estuary waters. These marine bands abound in a species of Corbula, with at least three species of Melania (or perhaps rather Ceritheum), a Protocardium, &c. These "Corbula beds" attain the thickness of about eight feet.

We now descend upon a portion of the deposit decidedly marine, as the change in the very appearance of the rocks at once leads the observer to suspect. The stone becomes earthy, coarser, and more rubbly in its texture, and the shells are Pectens, Oysters, Cardium, Perna, Modiola[*], &c.; and it is amongst these that we find a marly limestone, sometimes blue and sometimes olive brown, the first band containing *insect* remains,—a remarkable fact, as, *à priori*, I do not think we should have sought for these fragile and terrestrial forms, where both marine and fluviatile beds exist, by preference amongst the former. Yet it is singular that hitherto all the chief insect limestones of the Purbecks have been found in the marine parts of the series.

The rocks of this part of the series are many of them of a dark greenish hue; others brown; and towards the bottom of them Corbula bands again occur, as we approach the freshwater beds beneath. One of these often exhibits slabs, covered with a fine Paludina, so striking in appearance as to arrest the attention of the quarrymen. The total thickness of these "Pecten beds" is about forty feet.

A very important division of the middle Purbecks is the next in descending order. It is that from which all the best stone is quarried, and in which the abundant and interesting fish and turtle remains are chiefly obtained. The Macrorynchus also occurs in this horizon[†]. The limestones and shales of this portion are chiefly of freshwater origin, and many of the beds are formed of the shells of Cyclas. Traces, however, of an influx of estuary waters are seen in the lowest bed but one, in the occurrence of Cerithecæ, Ostreæ, &c.; but the subdivision ends with a purely freshwater deposit called the Downs vein, and composed of Cyclades. This is the vein from which most of the fish are obtained. These beds, in all about forty-four feet, rest upon a band of shale, covered with the Cypris fasciculata, and full of Cyclas on the under side.

The Archæoniscus, which for many months eluded all observation at Durlstone bay, was eventually found in a band of shale, No. 58 of Mr Austen's section, in the brackish portion of these Turtle beds.

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[*] I am informed by Mr Bristow that these marine shells of the middle Purbecks are of decidedly Oolitic types.

[†] A very fine collection of these fossils is in the Museum at Dorchester. It formerly belonged to Mr Wilcox of Swanage, and is referred to by Professor Forbes in his paper.

We now have arrived at one of the most striking deposits among these strata—the oyster bed—called locally the "Cinder." At Durlstone bay it is as much as twelve feet thick, and consists of three portions, in the upper of which are scarcely any shells, except the Ostrea distorta; in the middle portion is a band not above a foot thick, which contains the Hemicidaris Purbeckensis and its spines, with Serpulæ mingled with the oysters. The lower portion affords another species of oyster, broader than the O. distorta, and a Cardium. The Cinder is a rock which, although it crumbles in the weather, nevertheless, from its great thickness and the enormous masses of its blocks, has had more effect upon the contour of the country than any other of the beds of the Purbecks, and may often be found crowning the escarpments of the hills.

Leaving the Cinder bed we descend to beds which are again of freshwater origin, and are lithologically characterized by the presence of much silica, which forms occasionally thick beds of chert. There is also much vegetable matter, mixed with muddy deposit, full of broken shells, forming what may be termed dirt beds, these being interstratified with Cypris-shales and indurated marls. The shells of this portion of the series are Cyclas, Physa Bristowii (Forbes), Planorbis, Valvata, Neritina, Melania Chilina; but they are not found so well preserved at Durlstone bay as at Kingston in Purbeck, at Osmington, and Ridgway. Stems and seed-vessels of the Chara also abound at the latter localities. The thickness of these middle beds is about thirty feet[*].

As we proceed westward from Durlstone bay the "middle Purbecks" lose their pre-eminence in the system. At Worbarrow they retain something of it, but at Lulworth they are much diminished in thickness, and still more so at Osmington and Ridgway. They are not of sufficient importance at either of these places to be generally quarried, though at Osmington the broken-shell-limestone, on account of its durability, is sometimes used for foundations. The Cyrenæ in this bed are often in a much better state of preservation there than they are ever found at Swanage; and at Upwey, close to Ridgway, very good specimens may be obtained. The Archæoniscus has never been found in the middle Purbecks in the Dorsetshire district anywhere but at Swanage; but I am persuaded that the chief deposit of these crustaceans in the Vale of Wardour, as well as of the insects figured in Mr Brodie's "Fossil Insects," are from these estuary beds of the middle Purbecks[†].

The marine bands forming the "Pecten beds" at Ridgway-hill consist of a series of sandy beds, composed of the casts of Mytilus Spatula, Thracia Fisheri (Forbes), Modiola Fittoniana, Perna, Pecten, Avicula Dorsetensis (Forbes), &c.; and towards the bottom two feet of ripple-marked sandstone, splitting into slabs, and used for fencing and rough paving. The ripple marks appear to range N.N.E. and S.S.W., as if the stream had come from the W.N.W. I examined the stones in a fence by the road-side, and found that nine slabs which bore clear ripple marks on both sides, had the two sets parallel to each other, tending to shew that the marks were due to the permanent action of a current, not to the set of the wind. These slabs rest on a bed of brown sand.

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[*] It is amongst these dirt beds, No. 93 of Mr Austen's Section, that Mr W. Brodie has found the Macellodus Brodeii, and that interesting mammalian fossil the Spalacotherium Tricuspidens, described by Professor Owen in the Geological Journal for November, 1854.

[†] Since this paper was read the November number of the Geological Journal has appeared, in which Mr P. Brodie has printed some sections of the Wiltshire Purbecks which I communicated to him for his paper. The middle and lower Purbecks of Wiltshire are very similar to those of Ridgway-hill, but the upper division is absent.

The freshwater beds which succeed next in order, and contain so many remains of turtles and fish at Durlstone bay, are comparatively unimportant at Ridgway, and their chief organic remains are Cyclades and Cyprides. There is, however, a thin band at the top of No. 73 of the Ridgway section, which contains similar fossils to the chert beds below. The "cinder" is very thin at Osmington, but swells out again at Ridgway into two beds, the upper of which is composed of oysters, and the lower contains many Cardia, but I have found in it no traces of the Hemicidaris Purbeckensis.

There is a layer of fibrous carbonate of lime beneath the cinder bed, which displays upon its lower surface the impressions of cracks in the clay beneath, evidently due to a desiccation of the mud before the cinder was formed. This shews that the bed of the ancient river was laid dry, and then deeply depressed again, not under fresh water as before, but now beneath the sea, while the occurrence of the same species of freshwater shells above the cinder as below, seems to point out that some portions of the area occupied by these deposits was not only not subject to this subaerial condition, but also was free from that incursion of the sea-water which produced the cinder bed.

The cherty bands become more siliceous as we proceed westward, and in the same proportion their fossils are better preserved. At Lulworth we have a thick bed of hard white cherty stone, with a few Valvatæ and other shells, but at Osmington, beneath the chief bed of chert, there occurs a layer of tabular flints, resting on carbonaceous earth, whose under surface is covered with shells in a fine state of preservation. At Ridgway, in the same position, some very beautiful specimens may be procured, with Physa, Paludina (two species), Planorbis, Valvata, turtle-bones, fish-teeth, multitudes of Chara seeds, &c. The layer of purplish dirt which at Ridgway forms the upper portion of these bands, contains an admixture of brackish shells, Ostrea, Avicula, Modiola; nodules of black flint occur in it which present silicified Chara stems (?) upon their surface.

Next we come upon a band of shaly stone, ten inches thick, chiefly composed of Cyprides, but full of carbonized impressions of a branching aquatic plant, which must have lived upon the spot. I have placed a fine specimen in the Museum of Practical Geology in Jermyn Street. My impression is, that this is the equivalent of the Zostera-band mentioned by Professor Forbes in his paper, as separating the middle and lower Purbecks at Mupes bay.

The last member of these middle Purbecks, as developed at Ridgway, is a band of hard stone, eight inches thick, almost composed of Cyprides, and containing a small univalve (probably Hydrobia). It contains the Cypris fasciculata (Forbes), which is characteristic of the middle division of the Purbecks.

The following is a comparative list of the principal divisions of the Middle Purbecks, as developed at the extreme points of our district.

	Durlstone Bay. Ft.	Ridgway. Ft.
Broken shell limestone	9	10
Beef	24	3
Corbula beds	8	13
Pecten beds	34	6
Turtle beds	44	7
Cinder	12	7
Cherty and marly beds	30	11
	161	57

We now return to Durlstone bay, and describe the Lower Division of the Purbecks. The upper portion of these beds has been deposited in salt water, but evidently near the coast. The lower part of them is fresh water. They commence in descending order, with a series of marly and shaly bands, after about eleven feet of which we come upon a blue marl, containing a small species of Serpula in great profusion. The insect beds succeed to this, and demand a more particular description.

The insect beds consist of a series of rather soft marly limestone bands, alternating with shaly clays. They are evidently sedimentary, and scarcely contain any vestiges of shells. I have indeed never seen more than one (Hydrobia?) in them myself. This is remarkable among the Purbecks, where so many of the beds are entirely made up of shells. The stone abounds in cavities of a cubical form, with the sides somewhat bulged inwards*, formed apparently by crystals of common salt, which have been dissolved out, and their places sometimes partially filled by carbonate of lime, with a trace of sulphate of barytes†. They lie in horizontal layers, and seem to point to the stone having formed the muddy surface of a saltern, subject at times to be dried slowly, so as to admit of the crystallization of the salt with which it was imbued. I dissolved a portion of the bed from Ridgway in nitric acid, and found the glass in which it stood so much corroded, that fluoride of calcium must have been present. The following is a section of the upper insect beds of the lower Purbecks at the most productive point in Durlstone bay.

	Ft.	In.	
Grey marls with clay interstratified.			
Blue marl with serpula	1	6	
A parting of yellow clay	0	$0\frac{1}{2}$	
Grey marly limestone in three beds	0	9	
——————	0	6	
——————	0	9	Insects (*f*)‡
Thinner bands of marl	2	0	

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* See Ormerod's paper on pseudomorphs of salt in the new red sandstone of Cheshire.—Geol. Journ.—1853, p. 187.

† I am indebted to the kindness of Professor Miller for this account of the crystalline cavities.

‡ The letters refer to the labels on the specimens.

	Ft.	In.	
Cream-coloured marly limestone in three beds ...	1	3	Insects (*e*)
——————————— ...	0	8	Insects (*d*)
——————————— ...	1	0	Insects (*c*)
Softer and more clayey	0	4	Insects
Blue shaly marl	1	0	
Hard cream-coloured marlstone	1	2	
Rubbly marls	3	0	
Laminated sandy shale with chert	0	8	Cerithea
Rubbly marl..	3	0	
Grey marly limestone in two beds	2	0	Insects (*b*)

This section shews a succession of conditions of the ancient surface, several times returning to a state favourable to preserving these structures. In some of the bands they are much more sharply preserved than in others,—the most delicate markings being distinct, and the smallest wings perfect. In others they have a sodden appearance, as if they had been washed about on the surface of the ground before a fresh layer of mud entombed them. The clubs of the antennæ of large beetles occur, with segments and other portions of the bodies, but the most abundant remains are elytra and wings. All the orders are represented*. The insects are mixed with fragments of carbonized matter†; and it is very unusual to find the one without the other. They lie in laminæ throughout the blocks of marl, sometimes one, sometimes several in a block; but wings, especially the larger ones, are met with dispersed throughout the stone. It is evident that the wings and other fragments have been washed together into patches; and, coming upon such a patch, the collector has a rich treat, until it is exhausted, when he may search long comparatively in vain. In short, the fewer veins in a given thickness of the marl the richer they generally are—representing a longer period of deposition (but still, as I conceive, a very brief one for every vein). If the particles of mud were kept in a state of commotion the insect and vegetable remains would continue at the top until a quiescent moment allowed of their being buried. As might be expected, the elytra almost invariably lie on their convex surfaces, and the blocks which contain the specimens are from the upper side of the vein. The elytra at Durlstone do not shew any parallelism of arrangement, from which it may be inferred that the current was imperceptible. It is otherwise at Ridgway.

The deposit in the bed marked (*f*) is rather peculiar. The carbonized fragments are coarser, and the elytra retain their coloured markings in a singular degree. This is probably due to the stone being more aluminous than usual.

Since it is probable that the insects were blown or washed into the estuary only during the latter part of the summer and the autumn, the veins in which they occur assume a curious chronological value, and an interesting question of the rate of deposition of the beds arises, and this is none the less interesting from the fact of different sets of wings being found in the different veins. The scattered wings which are found throughout the blocks were probably casually blown or washed into the stream, but the abundant deposits in the veins seem to indicate periods of great destruction of insect life, such as would be occasioned by heavy storms or cold weather setting in. If, as might be expected, each autumn were represented by its consignment of these beings to their burial, the number of years indicated by the upper insect bed of the lower Purbecks would be very small—about four. But if the different tribes found in the different beds were those which perished at different seasons of the year, and each season were represented, then the time becomes incredibly short—not more than two years for the particular beds now under discussion, one for each set of beds. There is however nothing difficult of conception in a purely sedimentary series of beds like these being rapidly deposited, as there would be in the case of strata made up of the exuviæ of animal life. The portion of the insect beds which is the richest is at a point where they are thicker than usual, and lie somewhat hollow. I believe this to be their original form, and not to be, at any rate not wholly, due to a subsequent flexure. Such a form of the estuary bottom would be favourable to the collection of the remains into patches, and would account for the same locality being rich in several veins in the same vertical. The quarrymen maintain that they usually find fish in similar hollows in the strata.

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* I found a minute but very perfect lepidopterous insect in the upper lias at Ilminster, Somerset; it is now in the Woodwardian collection.

† Professor Henslow thinks that these are for the most part portions of insects in different stages of decomposition.

Below these upper insect beds of the lower Purbecks there follow about fifty-five feet of shaly clays and marls containing estuary shells, and in this part of the series there occur blocks of gypsum, which are quarried for plaster. The strata are much disturbed where the gypsum occurs: but the disturbance appears connected with the chemical change which produced the gypsum, the limestone beds above the gypsum retaining their regular arrangement. We then meet with another bed of cream-coloured marly limestone (*a*), containing insects, associated with layers of small Cardia and Hydrobiæ. It was from this bed (*116* of Mr Austen's section) that the very beautifully preserved hemipterous insect in the Woodwardian Museum was obtained. This is the lower insect bed of the lower Purbecks, but elytra and small wings associated with vegetable matter continue to occur occasionally whenever the strata are of a texture favourable to their preservation. The Archæoniscus has been found in this last-mentioned stratum.

About sixty feet of marls, Cypris-shales, and, at the bottom, more calcareous and sandy shales, complete the series as displayed at Durlstone bay. There is a bed of olive-brown and blue indurated marl about the middle of these, in which the Archæoniscus is tolerably abundant. It also occurs in the Cypris-shales. The junction of the Purbecks with the Portland beds is not well shewn in this locality. In the middle of the bay a fault cuts off the series before the bottom of the Purbecks is quite reached. The beds are then repeated from the middle Purbecks to the Portland, but at the southern end of the bay, where the Portland oolite appears, there is so much disturbance that it is impossible to carry the section through satisfactorily. The disturbance is a general one affecting the Portland as well as the Purbeck beds, and is not solely an example of the peculiar condition in which the lower beds of the Purbecks appear at other places in our district. But that broken state of the strata is very well seen in the Cliff beneath Round down, about a mile to the west. The well-known Portland dirt-bed is there also very distinguishable, but is not carbonaceous, and I could perceive no trees or Cycadeæ. The freshwater strata beneath it are thin—not exceeding about a foot. There is here and at Worbarrow a singular bed of dark slate-coloured Cypris-shale, very thin, and contrasting remarkably with the grey limestones of this part of the series.

We will now again pass to the western extremity of the district, and examine the lower division of the Purbecks at Ridgway-hill. The characteristic difference of their development from that at Durlstone bay consists in their being here much more calcareous and light-coloured. They also afford many beds of good workable limestone; and the dark shaly clays which constitute the chief thickness of the lower division of the series at Durlstone are entirely absent. After about three feet of marly limestones with clay partings, we come upon the insect beds. These are much harder and thinner than at Durlstone bay, and contain a much smaller variety of species*. In some spots, however, the individual clytra are fully as abundant as at Durlstone. A prevailing parallelism of the wing-cases, which lie N.N.W. and S.S.E., indicates a particular direction in the current. In a specimen in which there are two layers of elytra, about an inch and a half apart, the direction is the same. The direction indicated by these remains is therefore 45° more to the north than that indicated by the ripple-marked sandstone of the middle Purbecks. At Worbarrow there are two beds with ripple marks, nearly at the bottom of the lower Purbecks, indicating a north and south current. If the depth of water and distance from the shore were sufficiently great for any conclusion to be drawn from these observations, it would be that the stream of the Purbeck estuary trended more from the north towards the west as time rolled on. Perhaps the lithological structure of the beds may be thought to be in accordance with this supposition.

The association of a minute Serpula with these beds is in accordance with what has been before observed at Durlstone, and shews that the water was probably brackish. On descending lower we come upon a series of softer beds, still of an estuary character. At about ten feet lower down a Corbula and Cardium make their appearance, and six feet lower still are many delicate shells of Leda, Modiola, Ostrea, Cardium, &c.

We have now about ten feet of clays and sands in layers, the fossils being still the same, with an abundance of Cytheræ (for such they must be, associated with Cardium and Serpula,) and then we come to the most valuable bed of stone in the hill, which is a semicrystalline mass of Cardia about four feet thick. This is a beautiful white stone, and is much used for quoins and other purposes. It will stand well even when set against the bed.

Three feet beneath this is what I have called the lower insect bed of the lower Purbecks. It contains a great variety of wings, perhaps more than an average number of them being orthopterous. In this bed too is found not unfrequently a small chrysalis, and also a small aquatic insect allied to Notonecta. It appears to accord with the insect bed 116 of Mr Austen's section at Durlstone, where Cardia occur: but there is no actual cardium-rock there.

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* These beds at Teffont "lias" limekiln, in the Vale of Wardour, are lithologically very similar, but contain Cyprides and no insects.

Beneath this insect bed we have four feet of clays and sands, and with them the estuary condition seems to give place to one of freshwater beds.

In this freshwater portion of the lower Purbecks the beds at Ridgway-hill are almost entirely calcareous, with several beds of useful stone, whereas at Durlstone bay they are chiefly argillaceous, without any beds of stone until we reach the (so called) "broken bands" which cover the Portland dirt-bed. Of these "broken bands" I must soon speak more particularly. At Ridgway-hill the beds are not broken, but on the other hand wholly undisturbed, and consist of rather soft calcareous shales splitting into thin laminæ like biscuit, and containing impressions of wings of insects and the scales of fish; but as far as I know no shells. There is a somewhat remarkable bed not seen in the Ridgway section, but occurring in Mr Roper's quarry, a few yards off, which is very variable in its thickness, and answers to a singular bed at Lulworth, to which I shall refer in speaking of the broken bands. Its place in the section would be below 158.

The Portland dirt-bed is seen at Ridgway, but is thin. Yet it contains the remains of the trunks of large trees, as large as I have seen at any of the numerous points where this curious subaerial bed is shewn. The trees at this locality have the appearance of being much decayed externally, and I have not seen any with their bark preserved.

The surface seems to have been covered with fresh water before it was a forest, and just beneath the ancient mould is a band crowded with the scales of fish (Histionotus breviceps.) Its thickness is only four inches. The fish may perhaps have died suddenly when their habitation became dry. Then we have another four inch band, equally abounding in the remains of a very large species of Archæoniscus, but few tolerable specimens can be got. This bed at a spot six miles to the east (Upton) is full of casts of a small Paludina.

Our section ends with six inches of hard stone, invariably attached to the blocks of Portland, and thus affording remarkable hand specimens, one side of which is freshwater Purbeck limestone, with Cyprides and Cyclas, the other Portland oolite, containing Pectens or other marine shells.

A comparison of the principal divisions of the lower Purbecks at Durlstone and at Ridgway will be somewhat thus:—

Durlstone.	Ridgway.
Insect beds, to fault...... 123 *ft.*	Ditto, to bottom of section...... 95 *ft.*

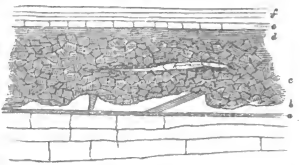
Before concluding I must not omit to describe the "broken bands" to which I have several times already alluded.

The freshwater calcareous shales which cover the Portland dirt-bed, and occupy at Lulworth the places of the beds numbered in the Ridgway section from 135 to 158, but at other points do not extend quite so high up, are in the eastern parts of our district invariably broken into fragments, and recemented (one cannot say consolidated) by a sort of stalactitic deposit.

The cause of this appearance has not, I believe, been explained. To find in a series of sedimentary rocks a thickness of some thirty feet or more in a state of ruin, while the beds both above and below are undisturbed, is certainly very singular.

At Portland, Ridgway, Osmington, Upton, and Ringstead, this broken condition of the beds in question is not found, but at Lulworth it is very remarkable, and also at Round down and Durlstone bay.

Fig. (2).*



The result of an examination I made of these beds at Lulworth was as follows:—

The stumps of the trees sometimes remain in a silicified state, but more frequently they have disappeared. The spot where each stood is surrounded by a dome-shaped swelling of the bed of porous limestone which covers the dirt-bed. These domes have generally a pit in the centre where the trunk protruded, but some have not. Such perhaps may have covered Zamias. Other swellings of the bed are long and trough-shaped, having accumulated about fallen trees. The surface of this bed of stone is very irregular, and it has much the character of the deposit from a petrifying spring on moss. No carbonaceous matter is present, except in the ancient soil beneath. Above this uneven bed all the strata are broken up until we reach the cardium-bed, No. 134 of the Ridgway section, which is undisturbed. The disturbance appears to diminish in intensity as we approach this limit.

Among the broken strata occasional horizontal bands of chert occur, apparently due to a subsequent solidification of some sandy beds amongst them.

It is clear, therefore, that the fracture of these beds took place before the cardium-bed was deposited, and that sufficient time elapsed for them to become solidified before their fracture. I would suggest the following explanation of the phenomena.

The ground, continuing to sink after the subaerial condition in which the forest grew, became swampy, and mosses grew over it, which crept up the stems of the trees, and formed hollow cones around them, and were encrusted with carbonate of lime. The trees then died, and, their branches falling, formed a layer of wood and twigs over the ground. The surface, gradually saturated with wet, became a lake or river-bed, and calcareous mud and sand were spread out in layers over the vegetable stratum. The latter not becoming bituminized, but continuing to decay slowly, gave way, but not equally, the stems of the trees perhaps standing longest. Hence, as the upper beds fell in upon the vacuity, they did not settle regularly, but were broken up. The upper beds are more disturbed than the lower ones, which would agree with this hypothesis. A stratum of clay immediately beneath the cardium-bed dips down among the inequalities of the broken bands, and affords a level surface for the cardium-bed to repose upon.

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* (a) Portland "dirt" bed.
(b) Porous bed (soft cap); shewing the hollows left by the trunks of trees.
(c) Broken bands, with a lenticular layer of chert in sand.
(d) Clay on which reposes
(e) Cardium bed undisturbed.
(f) Calcareous shales.

Since the above paper was read, Mr Westwood's most interesting paper on the Insects of the Dorsetshire Purbecks has appeared in the Quarterly Journal of the Geological Society, Vol. X. p. 378. The collection, from which the figures were drawn, does not appear to have been quite so good an one as that which is in the Woodwardian Museum. Some of the species figured we have not, but there are one or two of interest which we possess that differ from any described, especially a fine wing of Myrmeleon (ant lion), a wing of Termes (white ant), certain large elytra, apparently of the same species as a specimen nearly two inches long which I deposited in the British Museum, and which is now in one of the table-cases there. There are also some orthopterous wing-cases, which are unlike anything figured in Mr Westwood's paper. I believe some of these remains must have belonged to insects of a more tropical type than, with the exception of one or two, Mr Westwood allows to the climate of the Purbeck country.

SWANAGE BAY.

		Character of Rock.	Thickness.		Organic Remains.
		UPPER PURBECKS.	*ft.*	*in.*	
A	Cypris shales	Thin bedded stone	0	2	Paludina elongata
		Clay and thin shale	1	6	The shale is composed of Candone
		Calcareous shale	1	10	Perished Paludinæ
B & C		Coarse fissile stone	1	0	Unios
		Shales	1	6	
		Marble rag	0	8	Paludina
		Shaly clay	6	0	Cypris
		Coarse stone	0	3	
		Shales (covered) say	2	0	
		Rag	0	4	Unio
		Shales	1	6	
		Marble rag	0	6	
		Clay covered by beach	?		
		MIDDLE PURBECKS.			
D	Comminuted shell limestone	Thick stone, bottom not seen	?		Cyrena

The above section is taken from east to west on the south side of Swanage bay.

SECTION OF PURBECK STRATA, DURLSTONE BAY.

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
A	Cypris shales	1	Blue shaly clay	1	10		
B	Marble	2	Marble	2	2	Paludina	
			Shaly clay	9	0		
			Marble	2	0		
			Brown shale (say)	8	0		

At this point there is a Fault, and the Series recommences.

A	Cypris shales	1					
B	Marble bands	2	Marble	1	6		
		3	Marl	5	0		
C	Unio beds	4	Cypris-shales	2	4	Unio, Coprolites ?	
		5	Sandy marl	0	4		
		6	Crocodile bed	0	6	Paludina elongata, Unio, Teeth of Crocodile, Scales of Fish	
		7	Laminated marl	1	0		
		8	Unio bed	0	9		
			MIDDLE.				
D	Comminuted-shell Limestone	9	Sandy stone composed chiefly of broken shells —*Cyrena*	9	6	Remains of Turtle, palatal teeth of Fish, Paludina	Marble rag.
E	Beef beds	10	Shales with bands of fibrous carbonate of lime	1	9		
		11	Hard grey limestone	0	7		
		12	Soft shaly band	0	4		
		13	Hard blue shales	1	6		
		14	Soft shales	3	0		
		15	Alum shales	1	0		
		16	Soft shales with bands of fibrous carbonate of lime	3	0		
		17	Hard blue limestone	0	5		
		18	Soft shales with bands of fibrous carbonate of lime	1	2		
		19	Shelly limestone	0	9		
		20	Soft shales	1	0		
		21	Alum shales	0	9		
		22	Shelly limestone	1	6		
		23	Soft shales with bands of comminuted shells, and of fibrous carbonate of lime	About 7	6		

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Beef beds (*continued*)	24	Hard greenish shales: pink and blue bands of Corbula limestone	1	10	Palatal and other teeth of Fish, Corbula	
		25	Soft and hard shales with bands of fibrous carbonate of lime	1	0		
F	Corbula beds	26	Pink Corbula limestone, with band of indurated marl in the centre	0	10	Ceritheum, 3 species. Neritina	Toad's Eye.
		27	Dark shales with a band of comminuted shells	0	16		
		28	Blue indurated marl, rubbly	1	0	Bivalves, Vegetable remains	
		29	Pink Corbula limestone	1	7	Corbula, Cyrena, &c., Univalves, Teeth	
G	Pecten beds	30	Hard greenish rubbly rock, full of shells	1	0	Pecten Purbeckensis, Cardium, &c.	Green Rag.
		31	Blue and cream-coloured indurated marl without shells except at the bottom	1	2	Insects	
		32	Soft shales and comminuted shell limestone	1	3		
		33	Dark shales and band of hard grey limestone	2	0		
		34	Green shelly rock	2	0		
		35	Soft shales with bands of slaty and hard limestone	3	6		
		36	Blue and cream-coloured indurated marly limestone	1	6		
		37	Shales	3	0		
		38	Green shelly rock	2	0		
		39	Soft and hard shales	3	3		
			Grey marly rock, weathers vertically in zig-zag lines	1	0		
		40	Shales with bands of shell limestone	5	0		
		41	Corbula limestone	0	8	Ceritheum	
		42	Green shelly rock with a thin band of Corbula, and a band of hard shales in the centre	5	0	Teeth of Fish, Pecten, Paludina	
		43	Soft shales	0	4		
		44	Corbula limestone with Paludina	1	9	Slabs of Paludina with Cyclas	White Roach.

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	in.		
	Pecten beds (*continued*)	45	Cream-coloured and grey shell limestone	4	0	Slabs of Cyclas	Lancing Vein
				Here is a fault, the beds being let down seven feet.			
		46	Soft shales				
		47					
		48					Royal
		49	These beds are hidden by the beach	About 5	0		
H	Turtle beds	50					
		51					
		52	Rag composed chiefly of shells	2	0	Turtle bones, Coprolites	Red Rag, Rag, & Under Rag.
		53	Black shaly marl	0	11		
		54	Rag composed chiefly of shells	2	4	Paludina	
		55	Black marl with thin bands of comminuted shells and fibrous carbonate of lime	1	4		
		56	Soft blue sandy bed having a bituminous smell	4	6	Teeth & spines of fish	Lead bed & Shed bed.
		57	Blue sandy rock	0	10	Coprolites	Shingle.
		58	Shaly blue and brown marl	0	9	Shells, Vegetable remains, Archæoniscus, small scales of fish	Under-picking.
		59	Rubbly bed	1	0		Grub
		60	Light-coloured compact limestone	3	3		Free-stone Quarry.
		61	Rubbly band, composed of shells	1	2	Paludina, Melania	
		62	Compact limestone	3	0	Ostrea distorta	
		63	Hard, bluish limestone	1	6		Blue bed
		64	Cream-coloured ditto	1	0		Lias
		65	Compact limestone	2	4	Vegetable remains	Lias Rag
		66	Hard and soft shales	1	10	Planorbis	
		67	Blue marl	1	6		Bones.
		68	Bands of shell limestone and shale alternating; the latter having a bituminous smell	3	0	Slabs of Cyclas. Serpulites, Teeth, Cyprides	Laper, Underpicking

			Character of Rocks.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Turtle beds (*continued*)	69	Shell limestone	2	6	Cyclas, Turtle, Fish, Coprolites, Teeth, Spines	Rotten grey bed
		70	Dark-coloured shales	1	0	Melania chilina, Cypris, Univalves, Coprolites	Downs Vein
I	Cinder	71	Upper cinder Urchin band Lower cinder	12	0	Ostrea distorta, Hemicidaris Purbeckensis, and its spines, Cardium	Cinder
J	Chert beds	72	Blue Cypridiferous shales	1	0	Cyprides, Univalves, Bones	
		73	Cream-coloured cherty limestone	0	6	Broken shells	
		74	Shell limestone with thin soft cherty bands of comminuted shells	2	0		
		75	Blue marlstone with vegetable veins	0	6		
		76	Comminuted shell limestone	0	6	Cyprides, Bivalves	
		77	Hard sandy shales with vegetable remains and shells	0	9		
		78	Blue marl with shells	0	6	Cyprides, Vegetable remains, Bivalves	
		79	Hard shelly limestone	1	0		
		80	Soft shales full of shells	1	10	Univalves, Cyprides	
		81	Cream-coloured shelly limestone	4	0	Bones, Bivalves	Feather Quarry.
		82	Shales with a band of chert	0	6	Unio, Planorbis, Cyclas, &c.	
		83	Cream-coloured limestone with nodules of chert	2	10	Turtle	
		84	Cream-coloured and blue marly limestone, with a band of chert at the bottom	2	8	Turtle, Cyprides Turtle, Cyprides	
K	Marly beds	85	Soft shales	1	0		
		86	Dirt-bed, containing shells	0	9		
		87	Band of chert full of shells	0	4	Unio, Cyclas, Paludina, &c., Teeth	
		88	Bands of sandy, shaly, and indurated marlstones and shales containing green spots	4	0	Paludina, Bivalves, Cyprides, Univalves, Coprolites	
		89	Hard grey limestone	0	9		
		90	Indurated marls	2	4		
		91	Hard crystalline band	0	3		

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	in.		
	Marly beds (*continued*)	92	Shaly marl, comminuted shells	0	6		
		93	Dirt-bed	0	11	Planorbis, Hydrobia, Spalacotherium tricuspidens, Macellodus Brodeii	
			Here the beds for some distance are parallel with the beach but much contorted				
			LOWER.				
	Insect beds	94	Marly limestone with thin band of shells	1	0		
		95	Rubbly marlstone	2	0		
		96	Cream-coloured marly limestone	2	0		
		97	Soft shaly band	0	4		
		98	Cherty band	0	4	Serpula	
		99	Rubbly marl	6	0		
		100	Limestone shale	0	10	Cypris and abundant scales of Nothosomus	
		101	Grey rubbly marl	1	6		
		102	Ditto harder	1	4		
		103	Blue marls	1	6	Serpula	
			Yellow clay	0	0½		
			Grey marly limestone	0	9		
		104	Ditto	0	6	Insects retaining their markings, large carbonaceous specks	
		f	Ditto	0	9		
		105	Thinner bands of marl	2	0		
		e 106	Cream-coloured marly limestone	1	3	Insects (wing bed)	
		d	Ditto	0	8	Insects (beetle bed)	
		c	Ditto	1	0	Insects	
			Softer marl	0	4	Insects	
		107	Blue shaly marl	1	0		
		108	Hard cream-col. marlstone	1	2		
		109	Rubbly marls	3	0		
		110	Laminated sandy shale with chert	0	8	Ceritheum	
		111 112	Rubbly marl	3	0		
		113 *b*	Grey marly limestone in three bands	2	0	Insects, carbonaceous specks	
		cxiv	Shaly marl	11	0	Cypris	
		cxv	Hard crystalline marly limestone	1	6		

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Insect beds (*continued*)	cxvi	Olive shales	2	6		
		cxvii	Hard shaly band	2	0	Serpula	
		cxviii	Rubbly bed	1	6		
		cxix	Laminated green clay with some sharp white sand, say	0	8		
		cxx	Rubbly calcareous marl, sandy at bottom	5	0		
		cxxi	Brown marly stone, conchoidal fracture	2	0		
		cxxii	Blue and brown shaly clay, some sand and cherty nodules	4	6		Plaster
		cxxiii	Cream-coloured marly stone streaked with sand	2	0	Carbonaceous specks	
		cxxiv	Blue and brown shaly clays	3	6		
		cxxv	Brownish rubbly, sand marl and clay with hard bands	2	8		
		cxxvi	Blue clay, streaked with sand and marls (inaccessible, say)	18	0		
	Lower Insect bed	cxxvii (116) *a*	Cream-coloured marly limestone	3	0	Small Cardium, vegetables and insects	
		cxxviii	Blue shales, with occasional hard bands	22	0		
		cxxix	Ditto harder	8	0		
		cxxx	Blue cypris shales	13	0	Cypris, Archæoniscus	
		cxxxi (128)	Olive brown marlstone	4	0	Archæoniscus	
	Broken bands	cxxxii	Hard yellowish calcareous shales, about	16	0	Cypris, Elytra	

NOTE.—The Arabic Numerals stand before the beds as distinguished by Mr Austen, the Roman where the section is mine. From 99 to 113 the beds are divided according to Mr Austen, but described somewhat differently, partly to suit the labels upon the specimens in the Museum. I am obliged to Mr Austen for the use of his section.

SECTION OF PURBECK STRATA, RIDGWAY HILL.

UPPER.

	Paludina clays		Passage of Hastings sands into Purbeck strata, band with lignite				
		1	Rubbly rock with iron stain	1	6		
		2	Pale green clay	1	0	Paludina	
		3	Paludina marl	1	6		
		4	Alum shale parting	0	6	Fish-teeth	
		5	Pale green paludina marl	0	10		
		6	Sandy shale	0	6	Cypris	

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Upper Cypris shales & clays	7	Dark blue clay	4	0		
		8	Band of coarse brown stone	0	4	Turtle, Fish-scales	
		9	Clay	0	10		
		10	Band of coarse stone	0	6	Turtle, Fish-scales, Unio, Paludina	
		11	Clay parting	0	3		
		12	Same stone as the last	0	4		
		13	Marl	0	4	Unio	
		14	Stone	0	3	Turtle	
		15	Paludina marl	0	7	Paludina	
		16	Blue clay based on sand	1	4		
		17	Stone	0	2		
		18	Marl and sandstone	0	10	Paludina	
		19	Blue clay and sand laminated	2	0		
		20	Greenish hard sandy rock	0	4		
		21	Hard greenish marl with green specks	1	2	Paludina, Cypris, scales, &c., in a band at the bottom	
		22	Clay and sands with much drift wood	1	3		
		23	Hard sand rock full of remains	1	10	Wood, Cypris, Paludina, Cyclas?, Fish-scales, Palates, &c.	
		24	Alum clay	2	6		
		25	Hard brown sandy rock	0	4	Unio, Fish-scales, many Coprolites	
		26	Clay	0	10		
		27	Similar rock to last	0	4	Ditto	
	Unio beds	28	Nodular brown earthy limestone embedded in blue clay, at times composed of Paludina	3	0	This is the equivalent of the "marble" of Purbeck, Paludina	
		29	Laminated clays and sands, much drift at bottom	1	0		
		30	Hard sandy stone "Unio bed"	1	0	Unio, two species, some excellent specimens may be obtained	
		31	Laminated clay	0	3		
		32	Soft brown sandy rock	0	8	Unio	
		33	Alum clays	1	0		
		34	Hard brown sandy rock	0	6		
		35	Olive clays and sands laminated (say)	2	0		
		36	A sandy dirt-bed full of perished shells, resting on a line of clay	1	0	Unio, Paludina, Fish-teeth	

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Unio beds (*continued*)	37	Sand full of Unios	0	3	Unio	
		38	Yellowish brown sandy rock	0	9	Unio	
		39	Ditto finer and harder	1	0	Unio and small bivalve	
			Middle.				
	Upper broken shell limestone	40	Hard grey marlstone conchoidal fracture in brown sand, with narrow lines of "beef," (fibrous carbonate of lime)	2	6		
		(41)	(Drift with flint gravel)	0	1		
		42	Extremely hard grey marly rock	1	0		
		43	Coarse drift	0	6		
		44	Extremely hard grey marly rock	0	8		
			Sandy parting				
		45	Hard limestone composed of bivalves (like Swanage paving stone)	0	4		
		46	Ditto	0	10		
		47	Fine hard limestone of comminuted shells, blue towards centre	1	0		
		48	Shelly parting 1 inch beef	0	3		
		49	Comminuted shell limestone	0	4		
		50	Rotten shelly earth	1	0		
		51	Hard comminuted shell limestone	2	0		
	Chief beef beds	52	Beef	0	2		
		53	Compressed bivalves in ribbony layers	0	10		
		54	Irregular bed of extremely hard great crystalline nodules of marl, with beef, clay, shells, and limestone	2	0		
		55	Limestone of comminuted shell	1	6		
	Corbula beds	56	Reddish brown sand and earth	1	0		
		57	Brown sandy rock of Cyrena	2	0	Crystalline bands with shells	
		58	Sand with rotten bivalve shells	2	0		

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Corbula beds (*continued*)	59	Rotten dark greenish sandy rock, with small Ostreæ and bivalves	1	0	Ostrea	
		60	Irregular clay and rock	2	0		
		61	Extremely hard crystalline nodular rock in alum shale with shells	2	0		
		62	Hard shelly rock of irregular thickness, beef	1	0	Teeth, Turtle ?	
		63	Coarse shelly shale	1	0		
		64	Band of stone composed of bivalves	0	6		
	Intermarine	65	Coarse brown sandy rock composed of casts of Thracia	1	2	Thracia	
		66	Ditto harder, fewer shells	1	6	Thracia, Modiola, Perna, Pecten, &c.	
		67	Ripple-marked sandstone, splitting into slabs	2	0		
		68	Brown sand	1	4		
	Freshwater	69	Marly white rock	1	6	Cypris, shells	
		70	Shaly whitish cypris stone, parting of beef, sand, &c.	0	10		
		(71)	(Gravel)	0	6		
		72	Clay, rotten shell and beef in layers, olive-green and white	1	6		
		73	Hard grey shelly limestone, with green specks, the shells not bedded	1	6		
		74	Hard white marl with yellow specks, conch. fract.	0	6		
		75	Crushed bivalves	1	0		
		76	White line with broken oysters, washed out of cinder	0	1		
	Cinder bed	77	Cinder, composed of oysters and marl		0	Ostrea distorta, Cardium at the bottom	
		78	Lower cinder of comminuted shell	2	0		
		79	Beef with impressions of cracks from drying of the next bed	0	2		
	Cherty middle freshwater	80	Laminated clays, brown and yellow, compressed Cyclas	0	10	Cyclas	
		81	Purplish dirt full of shells	1	6	Cypris, Cyclas, Physa, Paludina, Planorbis, Valvata, Melanopsis, Pinna, Bones of saurians, Turtle, Chara	

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Cherty middle freshwater (*continued*)	82	Chert, containing the same shells, better preserved	1	4		
		83	Lenticular lumps of dark chert, containing the same shells, with Chara seeds, & resting upon a layer of carbonaceous matter	0	4	Excellent for specimens of the smaller shells	
		84	Sand	0	3		
		85	Laminated light-coloured clays and sands, somewhat indurated (shaly)	0	10	Large impressions of vegetables, branching	
		86	Brown clay parting	0	1		
		87	Band with Cyclas	0	4	Cyclas, Physa	
	Marly middle freshwater	88	Coarse marly limestone with yellow stains	1	6	Cypris	
		89	Beef and clay alternating	1	0		
		90	Olive-green clay	0	10	Cypris, Hydrobia	
		91	Marly limestone, white and fine-grained conch. fracture	1	10		
		92	Laminated yellow and brown clay	0	10	Cypris	
		93	Yellowish Cypris-stone	0	8	Cypris, Hydrobia	Pitching stone
			Lower.				
		94	Olive-green clay	0	6		
		95	Marly limestone weathering into vertical splinters, with vertical stripes of yellow stain	0	10	Fucoids	
		96	Clay parting	0	1		
		97	Pale salmon-coloured limestone of comminuted shell and Cypris	1	0		
		98	Clay parting	0	6		
		99	Shaly limestone	0	10		
		100	Hard blueish grey marly limestone, conch. fracture	0	6		
	Upper Insect beds	101	Laminated green and brown clay parting, with carbonaceous specks	0	1		
		102	Laminated calcareous marls, carbonaceous specks	0	6	Cypris Purbeckensis	

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	in.		
	Upper Insect beds (*continued*)	103	Hard grey marls, conch. fracture	0	9	Elytra and bodies of beetles	
		104	Hard grey marls with carbonaceous specks occasionally large	0	6		
		105	Parting of yellowish cream-coloured marl	0	2		
		106	Hard cream-coloured marl with carbonaceous specks and elytra	0	6	Elytra of beetles, occasionally very abundant	
		107	Ditto, with no remains	0	5		
		108	Ditto, softer	0	8		
		109	Yellow clay passing into marlstone, carbonaceous specks and elytra	1	0	Elytra of beetles	
		110	Cream-coloured marlstone, carb. specks, & elytra	0	4	Elytra of beetles	
	Soft cockle beds	111	Olive brown laminated clay with bivalves and carbonaceous specks	0	4	Cypris	
		112	Yellow Cypris-stone shewing the bedding	1	4	Minute Serpula	
		113	Dark clay	0	10		
		114	Greenish grey marlstone	1	0		
		115	Ochreous clay	0	4		
		116	Soft oolitic-looking Cypris-stone with sandy veins	6	6		
		117	Marly rock	1	6	Corbula, Cardium, small Serpula at the bottom	
		118	Ribbony clays and sands	0	10	A bivalve, Cypris	
		119	Marly stone	0	10	Small Cardium, Serpula, Modiola, Cypris,	
		120	Hard crystalline rock of comminuted shells, marly at the bottom	1	4	Serpula	Blue Stone
		121	Green and brown laminated clays, polished surfaces as in slicken-slide	0	6	Cypris, carb. specks	
		122	Marly stone veined with Cypris, and comminuted shells	2	6	Serpula, Cypris	
		123	Brown laminated clay	0	4	Cypris, carb. specks	
		124	Hard marly rock	2	0	Leda, Modiola, &c., near the top	

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	in.		
	Soft cockle beds (*continued*)	125	Dark laminated clay, containing a band of comminuted shells	1	6		
		126	Slightly indurated Cypris-clay not laminated (grey)	0	6	Cypris, small Cardium, Serpula	
		127	Ditto, laminated	0	6		
		128	Ditto, sandy	0	3		
		129	Ditto, passing into calcareous sandstone	0	8		
		130	Ribbony clays and white sandstones	4	0		
	Hard cockle beds	131	Rotten limestone, composed of casts of Cardium, &c.	0	8	Cardium, Rissoa, Cypris	
		132	Laminated brown clay	0	3		
		133	Hard marlstone, conch. fracture	1	0	Cypris	
		134	Hard crystalline rock of shells (quarried for)	3	6	Cardium, Cypris	Hard rock
		135	Laminated calc. sand and soft sandy rock	3	6		
	Lower Insect bed	136	Hard marly rock with streaky bedding	1	9	A chrysalis common, wings of libellula, dipterous, hemipterous, and orthopterous insects, Elytra of Beetles, carb. specks	
		137	Ribbon clays and sands, sometimes indurated	4	0		
		138	Dense Cypris-stone (quarried)	1	6	Fish-scales	Top rock
	Cypris freestone	139	Ribbony clays and sands	1	6	Carbonaceous specks	
		140	Soft Cypris-stone (quarried)	3	0		Middle rock
		141	Sandy limestones shale (say)	2	0		
		142	Cypris-stone (quarried)	2	0		Bottom rock
		143	Harder ditto (quarried)	2	0		
		144	Fissile white limestone shales having a bituminous smell when struck	5	0	Carbonaceous specks, Fish-scales, Cypris	
		145	Hardish Cypris-rock	2	6		
		146	Brown clay parting	0	1		
		147	Sandy limestone shales	4	0		

			Character of Rock.	Thickness.		Organic Remains.	Quarry List.
				ft.	*in.*		
	Cypris free-stone (*continued*)	148	Soft white sandy rock	2	0		
		149	Ditto, more friable (some coarse sand occasionally)	1	0		
	Broken bands	150	Cherty marl with flint at the bottom	1	3	Valvata helicoides, Lymneus physoides	
		151	Shaly limestone	3	0		
		152	Argillaceous shale of variable thickness	1	6		
		153	Band of indurated rubbly shale	0	4		
		154	Clay parting with thin chert	0	2		
		155	Indurated calcareous mud	3	0	Cypris, Valvata helicoides, Vegetable specks disseminated	
	Hard and soft cap	156	Rubbly calcareous shales with some chert	1	6		
		157	Chert, saccharine limestone	1	0		
		158	Calcareous shales	2	0		
		159	"Portland dirt-bed"	1	0	Fossil wood	
		160	Band with Fish	0	4	Histionotus breviceps	
		161	Band with Archæoniscus	0	4	Fragments of a large Archæoniscus very abundant	
		162	Hard cap	0	6	Cypris, Paludina	
			TOTAL	190	9		
			Portland Stone.				

XXIV. *On Plato's Survey of the Sciences.* BY W. WHEWELL, D.D. *Master of Trinity College.*

[Read *April* 23, 1855.]

A SURVEY by Plato of the state of the Sciences, as existing in his time, may be regarded as hardly less interesting than Francis Bacon's Review of the condition of the Sciences of *his* time, contained in the *Advancement of Learning.* Such a survey we have, in the seventh Book of Plato's *Republic;* and it will be instructive to examine what the Sciences then were, and what Plato aspired to have them become; aiding ourselves by the light afforded by the subsequent history of Science.

In the first place, it is interesting to note, in the two writers, Plato and Bacon, the same deep conviction that the large and profound philosophy which they recommended, had not, in their judgement, been pursued in an adequate and worthy manner, by those who had pursued it at all. The reader of Bacon will recollect the passage in the *Novum Organon* (Lib. I. Aphorism 80) where he speaks with indignation of the way in which philosophy had been degraded and perverted, by being applied as a mere instrument of utility or of early education: "So that the great mother of the Sciences is thrust down with indignity to the offices of a handmaid;—is made to minister to the labours of medicine or mathematics; or again, to give the first preparatory tinge to the immature minds of youth[*]."

In the like spirit, Plato says (*Rep.* VI. § 11, Bekker's ed.)

"Observe how boldly and fearlessly I set about my explanation of my assertion that philosophers ought to rule the world. For I begin by saying, that the State must begin to treat the study of philosophy in a way opposite to that now practised. Now, those who meddle at all with this study are put upon it when they are children, between the lessons which they receive in the farm-yard and in the shop[†]; and as soon as they have been introduced to the hardest part of the subject, are taken off from it, even those who get the most of philosophy. By the hardest part, I mean, the discussion of principles—Dialectic[‡]. And in their succeeding years, if they are willing to listen to a few lectures of those who make philosophy their business, they think they have done great things, as if it were something foreign to the business of life. And as they advance towards old age, with a very few exceptions, philosophy in them is extinguished: extinguished far more completely than the Heraclitean sun, for theirs is not lighted up again, as that is every morning:" alluding to the opinion which was propounded, by way of carrying the doctrine of the *unfixity* of sensible objects to an extreme; that the Sun is extinguished every night and lighted again in the morning. In opposition to this practice, Plato holds that philosophy should be the especial employment of men's minds when their bodily strength fails.

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[*] Accedit et illud quod naturalis philosophia in iis ipsis viris, qui ei incubuerunt, vacantem et integrum hominem, præsertim his recentioribus temporibus, vix nacta sit; nisi forte quis monachi alicujus in cellula, aut nobilis in villula lucubrantis, exemplum adduxerit; sed facta est demum naturalis philosophia instar transitus cujusdam et pontisternii ad alia. Atque magna ista scientiarum mater ad officia ancillæ detrusa est; quæ medicinæ aut mathematicis operibus ministrat, et rursus quæ adolescentium immatura ingenia lavat et imbuat velut tinctura quadam prima, ut aliam postea felicius et commodius excipiant.

[†] *μεταξὺ οἰκονομίας καὶ χρηματισμοῦ*, between housekeeping and money-getting.

[‡] *τὸ περὶ τοὺς λόγους*

What Plato means by *Dialectic*, which he, in the next Book, calls the highest part of philosophy, and which is, I think, what he here means by the hardest part of philosophy, I may hereafter consider: but at present I wish to pass in review the Sciences which he speaks of, as leading the way to that highest study. These Sciences are Arithmetic, Plane Geometry, Solid Geometry, Astronomy and Harmonics.

The view in which Plato here regards the Sciences is, as the instruments of that culture of the philosophical spirit which is to make the philosopher the fit and natural ruler of the perfect State—the Platonic Polity. It is held that to answer this purpose, the mind must be instructed in something more stable than the knowledge supplied by the senses;—a knowledge of objects which are constantly changing, and which therefore can be no real permanent Knowledge, but only Opinion. The real and permanent Knowledge which we thus require is to be found in certain sciences, which deal with *truths necessary and universal*, as we should now describe them: and which therefore are, in Plato's language, a knowledge of that which really *is*[*].

This is the object of the Sciences of which Plato speaks. And hence, when he introduces Arithmetic, as the first of the Sciences which are to be employed in this mental discipline, he adds (VII. § 8) that it must be not mere common Arithmetic, but a science which leads to speculative truths[†], seen by Intuition[‡]; not an Arithmetic which is studied for the sake of buying and selling, as among tradesmen and shopkeepers, but for the sake of pure and real Science[§].

I shall not dwell upon the details with which he illustrates this view, but proceed to the other Sciences which he mentions.

Geometry is then spoken of, as obviously the next Science in order; and it is asserted that it really does answer the required condition of drawing the mind from visible, mutable phenomena to a permanent reality. Geometers indeed speak of their visible diagrams, as if their problems were certain practical processes; to erect a perpendicular; to construct a square: and the like. But this language, though necessary, is really absurd. The figures are mere aids to their reasonings. Their knowledge is really a knowledge not of visible objects, but of permanent realities: and thus, Geometry is one of the helps by which the mind may be drawn to Truth; by which the philosophical spirit may be formed, which looks upwards instead of downwards.

Astronomy is suggested as the Science next in order, but Socrates, the leader of the dialogue, remarks that there is an intermediate Science first to be considered. Geometry, treats of plane figures; Astronomy treats of solids in motion, that is, of spheres in motion; for the astronomy of Plato's time was mainly the doctrine of the sphere. But before treating of solids in motion, we must have a science which treats of solids simply. After taking space of two dimensions, we must take space of three dimensions, length, breadth and depth, as in cubes and the like[*]. But such a Science, it is remarked, has not yet been discovered. Plato "notes as deficient" this branch of knowledge; to use the expression employed by Bacon on the like occasions in his Review. Plato goes on to say, that the cultivators of such a science have not received due encouragement; and that though scorned and starved by the public, and not recommended by any obvious utility, it has still made great progress, in virtue of its own attractiveness.

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[*] The sciences are to draw the mind from that which grows and perishes to that which really is: μάθημα ψυχῆς ὁλκὸν ἀπὸ τοῦ γιγνομένου ἐπὶ τὸ ὄν.

[†] ἐπὶ θέαν τῆς τῶν ἀριθμῶν φύσεως.

[‡] τῇ νοήσει αὐτῇ.

[§] He adds "and for the sake of war;" this point I have passed by. Plato does not really ascribe much weight to this use of Science, as we see in what he says of Geometry and Astronomy.

In fact, researches in Solid Geometry had been pursued with great zeal by Plato and his friends, and with remarkable success. The five Regular Solids, the Tetrahedron or Pyramid, Cube, Octahedron, Dodecahedron and Icosahedron, had been discovered; and the curious theorem, that of Regular Solids there can be just so many, these and no others, was known. The doctrine of these Solids was already applied in a way, fanciful and arbitrary, no doubt, but ingenious and lively, to the theory of the Universe. In the *Timæus*, the elements have these forms assigned to them respectively. Earth has the Cube: Fire has the Pyramid: Water has the Octahedron: Air has the Icosahedron: and the Dodecahedron is the plan of the Universe itself. This application of the doctrine of the Regular Solids shews that the knowledge of those figures was already established; and that Plato had a right to speak of Solid Geometry as a real and interesting Science. And that this subject was so recondite and profound,—that these five Regular Solids had so little application in the geometry which has a bearing on man's ordinary thoughts and actions,—made it all the more natural for Plato to suppose that these solids had a bearing on the constitution of the Universe; and we shall find that such a belief in later times found a ready acceptance in the minds of mathematicians who followed in the Platonic line of speculation.

Plato next proceeds to consider Astronomy; and here we have an amusing touch of philosophical drama. Glaucon, the hearer and pupil in the Dialogue, is desirous of shewing that he has profited by what his instructor had said about the real uses of Science. He says Astronomy is a very good branch of education. It is such a very useful science for seamen and husbandmen and the like. Socrates says, with a smile, as we may suppose: "You are very amusing with your zeal for utility. I suppose you are afraid of being condemned by the good people of Athens for diffusing Useless Knowledge." A little afterwards Glaucon tries to do better, but still with no great success. He says, "you blamed me for praising Astronomy awkwardly: but now I will follow your lead. Astronomy is one of the sciences which you require, because it makes men's minds look upwards, and study things above. Any one can see that." "Well," says Socrates, "perhaps any one can see it except me—I cannot see it." Glaucon is surprized, but Socrates goes on: "Your notion of 'the study of things above' is certainly a very magnificent one. You seem to think that if a man bends his head back and looks at the ceiling he 'looks upwards' with his mind as well as his eyes. You may be right and I may be wrong: but I have no notion of any science which makes the *mind* look upwards, except a science which is about the permanent and the invisible. It makes no difference, as to that matter, whether a man gapes and looks up or shuts his mouth and looks down. If a man merely look up and stare at sensible objects, his mind does not look upwards, even if he were to pursue his studies swimming on his back in the sea."

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[*] ὀρθῶς ἔχει ἑξῆς μετὰ δευτέραν αὔξην τρίτην λαμβάνειν, ἔστι δέ που τοῦτο περὶ τὴν τῶν κύβων αὔξην καὶ τὸ βάθους μέτεχον.

The Astronomy, then, which merely looks at phenomena does not satisfy Plato. He wants something more. What is it? as Glaucon very naturally asks.

Plato then describes Astronomy as a real science (§ 11). "The variegated adornments which appear in the sky, the visible luminaries, we must judge to be the most beautiful and the most perfect things of their kind: but since they are mere visible figures, we must suppose them to be far inferior to the true objects; namely, those spheres which, with their real proportions of quickness and slowness, their real number, their real figures, revolve and carry luminaries in their revolutions. These objects are to be apprehended by reason and mental conception, not by vision." And he then goes on to say that the varied figures which the skies present to the eye are to be used as *diagrams* to assist the study of that higher truth; just as if any one were to study geometry by means of beautiful diagrams constructed by Dædalus or any other consummate artist.

Here then, Plato points to a kind of astronomical science which goes beyond the mere arrangement of phenomena: an astronomy which, it would seem, did not exist at the time when he wrote. It is natural to inquire, whether we can determine more precisely what kind of astronomical science he meant, and whether such science has been brought into existence since his time.

He gives us some further features of the philosophical astronomy which he requires. "As you do not expect to find in the most exquisite geometrical diagrams the true evidence of quantities being equal, or double, or in any other relation: so the true astronomer will not think that the proportion of the day to the month, or the month to the year, and the like, are real and immutable things. He will seek a deeper truth than these. We must treat Astronomy, like Geometry, as a series of problems suggested by visible things. We must apply the intelligent portion of our mind to the subject."

Here we really come in view of a class of problems which astronomical speculators at certain periods have proposed to themselves. What is the real ground of the proportion of the day to the month, and of the month to the year, I do not know that any writer of great name has tried to determine: but to ask the reason of these proportions, namely, that of the revolution of the earth on its axis, of the moon in its orbit, and of the earth in its orbit, are questions just of the same kind as to ask the reason of the proportion of the revolutions of the planets in their orbits, and of the proportion of the orbits themselves. Now who has attempted to assign such reasons?

Of course we shall answer, Kepler: not so much in the Laws of the Planetary motions which bear his name, as in the Law which at an earlier period he thought he had discovered, determining the proportion of the distances of the several Planets from the Sun. And, curiously enough, this solution of a problem which we may conceive Plato to have had in his mind, Kepler gave by means of the Five Regular Solids which Plato had brought into notice, and had employed in his theory of the Universe given in the *Timæus*.

Kepler's speculations on the subject just mentioned were given to the world in the *Mysterium Cosmographicum* published in 1596. In his Preface, he says "In the beginning of the year 1595 I brooded with the whole energy of my mind on the subject of the Copernican system. There were three things in particular of which I pertinaciously sought the causes; why they are not other than they are: the number, the size, and the motion of the orbits." We see how strongly he had his mind impressed with the same thought which Plato had so confidently uttered: that there must be some reason for those proportions in the scheme of the Universe which appear casual and vague. He was confident at this period that he had solved two of the three questions which haunted him;—that he could account for the number and the size of the planetary orbits. His account was given in this way.—"The orbit of the Earth is a circle; round the sphere to which this circle belongs describe a dodecahedron; the sphere including this will give the orbit of Mars. Round Mars inscribe a tetrahedron; the circle including this will be the orbit of Jupiter. Describe a cube round Jupiter's orbit; the circle including this will be the orbit of Saturn. Now inscribe in the Earth's orbit an icosahedron: the circle inscribed in it will be the orbit of Venus. Inscribe an octahedron in the orbit of Venus; the circle inscribed in it will be Mercury's orbit. This is the reason of the number of the planets;" and also of the magnitudes of their orbits.

These proportions were only approximations; and the Rule thus asserted has been shewn to be unfounded, by the discovery of new Planets. This Law of Kepler has been repudiated by succeeding Astronomers. So far, then, the Astronomy which Plato requires as a part of true philosophy has not been brought into being. But are we thence to conclude that the demand for such a kind of Astronomy was a mere Platonic imagination?—was a mistake which more recent and sounder views have corrected? We can hardly venture to say that. For the questions which Kepler thus asked, and which he answered by the assertion of this erroneous Law, are questions of exactly the same kind as those which he asked and answered by means of the true Laws which still fasten his name upon one of the epochs of astronomical history. If he was wrong in assigning reasons for the number and size of the planetary orbits, he was right in assigning a reason for the proportion of the motions. This he did in the *Harmonice Mundi*, published in 1619: where he established that the squares of the periodic times of the different Planets are as the cubes of their mean distances from the central Sun. Of this discovery he speaks with a natural exultation, which succeeding astronomers have thought well founded. He says: "What I prophesied two and twenty years ago as soon as I had discovered the five solids among the heavenly bodies; what I firmly believed before I had seen the *Harmonics* of Ptolemy; what I promised my friends in the title of this book (*On the perfect Harmony of the celestial motions*), which I named before I was sure of my discovery; what sixteen years ago I regarded as a thing to be sought; that for which I joined Tycho Brahe, for which I settled in Prague, for which I devoted the best part of my life to astronomical contemplations; at length I have brought to light, and have recognized its truth beyond my most sanguine expectations." (*Harm. Mundi*, Lib. V.)

Thus the Platonic notion, of an Astronomy which deals with doctrines of a more exact and determinate kind than the obvious relations of phenomena, may be found to tend either to error or to truth. Such aspirations point equally to the five regular solids which Kepler imagined as determining the planetary orbits, and to the Laws of Kepler in which Newton detected the effect of universal gravitation. The realities which Plato looked for, as something incomparably more real than the visible luminaries, are found, when we find geometrical figures, epicycles and eccentrics, laws of motion and laws of force, which explain the appearances. His Realities are Theories which account for the Phenomena, Ideas which connect the Facts.

But, is Plato right in holding that such Realities as these are *more real* than the Phenomena, and constitute an Astronomy of a higher kind than that of mere Appearances? To this we shall, of course, reply that Theories and Facts have each their reality, but that these are realities of different kinds. Kepler's Laws are as real as day and night; the force of gravity tending to the Sun is as real as the Sun; but not more so. True Theories and Facts are equally real, for true Theories *are* Facts, and Facts are familiar Theories. Astronomy is, as Plato says, a series of Problems suggested by visible Things; and the Thoughts in our own minds which bring the solutions of these Problems, have a reality in the Things which suggest them.

But if we try, as Plato does, to separate and oppose to each other the Astronomy of Appearances and the Astronomy of Theories, we attempt that which is impossible. There are no Phenomena which do not exhibit some Law; no Law can be conceived without Phenomena. The heavens offer a series of Problems; but however many of these Problems we solve, there remain still innumerable of them unsolved; and these unsolved Problems have solutions, and are not different in kind from those of which the extant solution is most complete.

Nor can we justly distinguish, with Plato, Astronomy into transient appearances and permanent truths. The theories of Astronomy are permanent, and are manifested in a series of changes: but the change is perpetual just *because* the theory is permanent. The perpetual change *is* the permanent theory. The perpetual changes in the positions and movements of the planets, for instance, manifest the permanent machinery: the machinery of cycles and epicycles, as Plato would have said, and as Copernicus would have agreed; while Kepler, with a profound admiration for both, would have asserted that the motions might be represented by ellipses, more exactly, if not more truly. The cycles and epicycles, or the ellipses, are as real as space and time, *in* which the motions take place. But we cannot justly say that space and time and motion are more real than the bodies which move in space and time, or than the appearances which these bodies present.

Thus Plato, with his tendency to exalt Ideas above Facts,—to find a Reality which is more real than Phenomena,—to take hold of a permanent Truth which is more true than truths of observation,—attempts what is impossible. He tries to separate the poles of the Fundamental Antithesis, which, however antithetical, are inseparable.

At the same time, we must recollect that this tendency to find a Reality which is something beyond appearance, a permanence which is involved in the changes, is the genuine spring of scientific discovery. Such a tendency has been the cause of all the astronomical science which we possess. It appeared in Plato himself, in Hipparchus, in Ptolemy, in Copernicus, and most eminently in Kepler; and in him perhaps in a manner more accordant with Plato's aspirations when he found the five Regular Solids in the Universe, than when he found there the Conic Sections which determine the form of the planetary orbits. The pursuit of this tendency has been the source of the mighty and successful labours of succeeding astronomers:

and the anticipations of Plato on this head were more true than he himself could have conceived.

When the above view of the nature of true astronomy has been proposed, Glaucon says:

"That would be a task much more laborious than the astronomy now cultivated." Socrates replies: "I believe so: and such tasks must be undertaken, if our researches are to be good for anything."

After Astronomy, there comes under review another Science, which is treated in the same manner. It is presented as one of the Sciences which deal with real abstract truth; and which are therefore suited to that developement of the philosophic insight into the highest truth, which is here Plato's main object. This Science is ***Harmonics***, the doctrine of the mathematical relations of musical sounds. Perhaps it may be more difficult to explain to a general audience, Plato's views on this than on the previous subjects: for though Harmonics is still acknowledged as a Science including the mathematical truths to which Plato here refers, these truths are less generally known than those of geometry or astronomy. Pythagoras is reported to have been the discoverer of the cardinal proposition in this Mathematics of Music:—namely, that the musical notes which the ear recognizes as having that definite and harmonious relation which we call an *octave*, a *fifth*, a *fourth*, a *third*, have also, in some way or other, the numerical relation of 2 to 1, 3 to 2, 4 to 3, 5 to 4. I say "some way or other," because the statements of ancient writers on this subject are physically inexact, but are right in the essential point, that those simple numerical ratios are characteristic of the most marked harmonic relations. The numerical ratios really represent the rate of vibration of the air when those harmonics are produced. This perhaps Plato did not know: but he knew or assumed that those numerical ratios were cardinal truths in harmony: and he conceived that the exactness of the ratios rested on grounds deeper and more intellectual than any testimony which the ear could give. This is the main point in his mode of applying the subject, which will be best understood by translating (with some abridgement) what he says. Socrates proceeds:

(§ 11 near the end.) "Motion appears in many aspects. It would take a very wise man to enumerate them all: but there are two obvious kinds. One which appears in astronomy, (the revolutions of the heavenly bodies,) and another which is the echo of that*. As the eyes are made for Astronomy, so are the ears made for the motion which produces Harmony†: and thus we have two sister sciences, as the Pythagoreans teach, and we assent.

(§ 12.) "To avoid unnecessary labour, let us first learn what *they* can tell us, and see whether anything is to be added to it; retaining our own view on such subjects: namely this:—that those whose education we are to superintend—real philosophers—are never to learn any imperfect truths:—anything which does not tend to that point (exact and permanent truth) to which all our knowledge ought to tend, as we said concerning astronomy. Now those who cultivate music take a very different course from this. You may see them taking immense pains in measuring musical notes and intervals by the ear, as the astronomers measure the heavenly motions by the eye.

"Yes, says Glaucon, they apply their ears close to the instrument, as if they could catch the note by getting near to it, and talk of some kind of recurrences‡. Some say they can distinguish an interval, and that this is the smallest possible interval, by which others are to be measured; while others say that the two notes are identical: both parties alike judging by the ear, not by the intellect.

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* ἀντίστροφον αὐτοῦ.

† πρὸς ἐναρμόνιον φορὰν ὦτα παγῆναι.

‡ πυκνώματα ἄττα.

"You mean, says Socrates, those fine musicians who torture their notes, and screw their pegs, and pinch their strings, and speak of the resulting sounds in grand terms of art. We will leave them, and address our inquiries to our other teachers, the Pythagoreans."

The expressions about the small interval in Glaucon's speech appear to me to refer to a curious question, which we know was discussed among the Greek mathematicians. If we take a keyed instrument, and ascend from a key note by two *octaves* and a *third*, (say from A_1 to C_3) we arrive at the *same nominal note*, as if we ascend four times by a *fifth* (A_1 to E_1, E_1 to B_2, B_2 to F_2, F_2 to C_3). Hence one party might call this the *same* note. But if the Octaves, Fifths, and Third be perfectly true intervals, the notes arrived at in the two ways will not be really the same. (In the one case, the note is $\frac{1}{2} \times \frac{1}{2} \times \frac{4}{5}$; in the other $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$; which are $\frac{1}{5}$ and $\frac{16}{81}$, or in the ratio of 81 to 80). This small interval by which the two notes really differ, the Greeks called a *Comma*, and it was the smallest musical interval which they recognized. Plato disdains to see anything important in this controversy; though the controversy itsef is really a curious proof of his doctrine, that there is a mathematical truth in Harmony, higher than instrumental exactness can reach. He goes on to say:

"The musical teachers are defective in the same way as the astronomical. They do indeed seek numbers in the harmonic notes, which the ear perceives: but they do not ascend from them to the Problem, What are harmonic numbers and what are not, and what is the reason of each"?[*] "That, says Glaucon, would be a sublime inquiry."

Have we in Harmonics, as in Astronomy, anything in the succeeding history of the Science which illustrates the tendency of Plato's thoughts, and the value of such a tendency?

It is plain that the tendency was of the same nature as that which induced Kepler to call his work on Astronomy *Harmonice Mundi;* and which led to many of the speculations of that work, in which harmonical are mixed with geometrical doctrines. And if we are disposed to judge severely of such speculations, as too fanciful for sound philosophy, we may recollect that Newton himself seems to have been willing to find an analogy between harmonic numbers and the different coloured spaces in the spectrum.

But I will say frankly, that I do not believe there really exists any harmonical relation in either of these cases. Nor can the problem proposed by Plato be considered as having been solved since his time, any further than that the recurrence of vibrations, when their ratios are so simple, may be easily conceived as affecting the ear in a peculiar manner. The imperfection of musical scales, which the *comma* indicates, has not been removed; but we may say that, in the case of this problem, as in the other ultimate Platonic problems, the duplication of the cube and the quadrature of the circle, the impossibility of a solution has been already established. The problem of a perfect musical scale is impossible, because no power of 2 can be equal to a power of 3; and if we further take the multiplier 5, of course it also cannot bring about an exact equality. This impossibility of a perfect scale being recognized, the practical problem is what is the system of *temperament* which will make the scale best suited for musical purposes; and this problem has been very fully discussed by modern writers.

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[*] *τίνες ξύμφωνοι ἀριθμοί*, &c.

XXV. *On Plato's Notion of Dialectic. By* W. Whewell, D.D. *Master of Trinity College.*

[Read *May* 7, 1855.]

The survey of the sciences, arithmetic, plane geometry, solid geometry, astronomy and harmonics—which is contained in the seventh Book of the Republic (§ 6—12), and which has been discussed in the preceding paper, represents them as instruments in an education, of which the end is something much higher—as steps in a progression which is to go further. "Do you not know," says Socrates, (§ 12) "that all this is merely a prelude to the strain which we have to learn?" And what that strain is, he forthwith proceeds to indicate. "That these sciences do not suffice, you must be aware: for—those who are masters of such sciences—do they seem to you to be good in dialectic? (*δεινοὶ διαλεκτικοὶ εἶναι ;*)

"In truth, says Glaucon, they are not, with very few exceptions, so far as I have fallen in with them."

"And yet, said I, if persons cannot give and receive a reason, they cannot attain that knowledge which, as we have said, men ought to have."

Here it is evident that "to give and to receive a reason," is a phrase employed as coinciding, in a general way at least, with being "good in dialectic;" and accordingly, this is soon after asserted in another form, the verb being now used instead of the adjective. "It is dialectic discussion (*τὸ διαλέγεσθαι,*) which executes the strain which we have been preparing". It is further said that it is a progress to clear intellectual light, which corresponds to the progress of bodily vision in proceeding from the darkened cave described in the beginning of the Book to the light of day. This progress, it is added, of course you call *Dialectic* (*διαλεκτικήν*).

Plato further says, that other sciences cannot properly be called sciences. They begin from certain assumptions, and give us only the consequences which follow from reasoning on such assumptions. But these assumptions they cannot prove. To do so is not in the province of each science. It belongs to a higher science: to the science of Real Existences. You call the man Dialectical, who requires a reason of the essence of each thing*.

And as Dialectic gives an account of other real existences, so does it of that most important reality, the true guide of Life and of Philosophy, the Real Good. He who cannot follow this through all the windings of the battle of Life, knows nothing to any purpose. And thus Dialectic is the pinnacle, the top stone of the edifice of the sciences†.

Dialectic is here defined or described by Plato according to the *subject* which it treats, and the *object* with which it is to be pursued: but in other parts of the Platonic Dialogues,

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* *Ἦ καὶ διαλεκτικὸν καλεῖς τὸν λόγον ἑκάστου λαμβάνοντα τῆς οὐσίας;* (§ 14)

† *ὥσπερ θριγγὸς τοῖς μαθήμασιν ἡ διαλεκτικὴ ἡμῖν ἐπάνω κεῖσθαι.* (§ 14)

Dialectic appears rather to imply a certain *method* of investigation;—to describe the *form* rather than the *matter* of discussion; and it will perhaps be worth while to compare these different accounts of Dialectic.

(*Phædrus.*) One of the cardinal passages on this point is in the Phædrus, and may be briefly quoted. Phædrus, in the Dialogue which bears his name, appears at first as an admirer of Lysias, a celebrated writer of orations, the contemporary of Plato. In order to expose this writer's style of composition as frigid and shallow, a specimen of it is given, and Socrates not only criticizes this, but delivers, as rival compositions, two discourses on the same subject. Of these discourses, given as the inspiration of the moment, the first is animated and vigorous; the second goes still further, and clothes its meaning in a gorgeous dress of poetical and mythical images. Phædrus acknowledges that his favourite is outshone; and Socrates then proceeds to point out that the real superiority of his own discourse consists in its having a dialectical structure, beneath its outward aspect of imagery and enthusiasm. He says: (§ 109, Bekker. It is to be remembered that the subject of all the discourses was *Love*, under certain supposed conditions.)

"The rest of the performance may be taken as play: but there were, in what was thus thrown out by a random impulse, two features, of which, if any one could reduce the effect to an art, it would be a very agreeable and useful task.

"What are they? Phædrus asks.

"In the first place, Socrates replies, the taking a connected view of the scattered elements of a subject, so as to bring them into one Idea; and thus to give a definition of the subject, so as to make it clear what we are speaking of; as was then done in regard to *Love*. A definition was given of it, what it is: whether the definition was good or bad, at any rate there was a definition. And hence, in what followed, we were able to say what was clear and consistent with itself.

"And what, Phædrus asks, was the other feature?

"The dividing the subject into kinds or elements, according to the nature of the thing itself:—not breaking its natural members, like a bad carver who cannot hit the joint. So the two discourses which we have delivered, took the irrational part of the mind, as their common subject; and as the body has two different sides, the right and the left. with the same names for its parts; so the two discourses took the irrational portion of man; and the one took the left-hand portion, and divided this again, and again subdivided it, till, among the subdivisions, it found a left-handed kind of Love, of which nothing but ill was to be said. While the discourse that followed out the right-hand side of phrenzy, (the irrational portion of man's nature,) was led to something which bore the name of *Love* like the other, but which is divine, and was praised as the source of the greatest blessing."

"Now I," Socrates goes on to say, "am a great admirer of these processes of division and comprehension, by which I endeavour to speak and to think correctly. And if I can find any one who is able to see clearly what is by nature reducible to one and manifested in many elements, I follow his footsteps as a divine guide. Those who can do this, I call—whether rightly or not, God knows—but I have hitherto been in the habit of calling them *dialectical* men."

It is of no consequence to our present purpose whether either of the discourses of Socrates in the Phædrus, or the two together, as is here assumed, do contain a just division and subdivision of that part of the human soul which is distinguishable from Reason, and do thus exhibit, in its true relations, the affection of Love. It is evident that division and subdivision of this kind is here presented as, in Plato's opinion, a most valuable method; and those who could successfully practise this method are those whom he admires as dialectical men. This is here his *Dialectic.*

(*Sophistes*) We are naturally led to ask whether this method of dividing a subject as the best way of examining it, be in any other part of the Platonic Dialogues more fully explained than it is in the Phædrus; or whether any rules are given for this kind of Dialectic.

To this we may reply, that in the Dialogue entitled *The Sophist*, a method of dividing a subject, in order to examine it, is explained and exemplified with extraordinary copiousness and ingenuity. The object proposed in that Dialogue is, to define what a Sophist is; and with that view, the principal speaker, (who is represented as an Eleatic stranger,) begins by first exemplifying what is his method of framing a definition, and by applying it to define an *Angler*. The course followed, though it now reads like a burlesque of philosophical methods, appears to have been at that time a *bona fide* attempt to be philosophical and methodical. It proceeds thus:

"We have to inquire concerning *Angling*. Is it an Art? It is. Now what kind of art? All art is an art of making or an art of getting: (*Poietic* or *Ktetic*.) It is Ktetic. Now the art of getting, is the art of getting by exchange or by capture: (*Metabletic* or *Chirotic*.) Getting by capture is by contest or by chase: (*Agonistic* or *Thereutic*.) Getting by chase is a chase of lifeless or of living things: (the first has no name, the second is *Zootheric*.) The chase of living things is the chase of land animals or of water animals. (*Pezotheric* or *Enygrotheric*.) Chase of water animals is of birds or of fish: (*Ornithothereutic* and *Halieutic*.) Chase of fish is by inclosing or by striking them: (*Hercotheric* or *Plectic*.) We strike them by day with pointed instruments, or by night, using torches: (hence the division *Ankistreutic* and *Pyreutic*.) Of Ankistreutic, one kind consists in spearing the fish downwards from above, the other in twitching them upwards from below: (these two arts are *Triodontic* and *Aspalieutic*.) And thus we have, what we sought, the notion and the description of angling: namely that it is a Ktetic, Chirotic, Thereutic, Zootheric, Enygrotheric, Halieutic, Plectic, Ankistreutic, Aspalieutic Art."

Several other examples are given of this ingenious mode of definition, but they are all introduced with reference to the definition of the Sophist. And it will further illustrate this method to shew how, according to it, the Sophist is related to the Angler.

The Sophistical Art is an art of getting, by capture, living things, namely men. It is thus a Ktetic, Chirotic, Thereutic art, and so far agrees with that of the Angler. But here the two arts diverge, since that of the Sophist is Pezotheric, that of the Angler Enygrotheric. To determine the Sophist still more exactly, observe that the chase of land animals is either of tame animals (including man) or of wild animals: (*Hemerotheric* and *Agriotheric*.) The chase of tame animals is either by violence, (as kidnapping, tyranny, and war in general,) or by persuasion, (as by the arts of speech;) that is, it is *Biaiotheric* or *Pithanurgic*. The art of persuasion is a private or a public proceeding: (*Idiothereutic* or *Demosiothereutic*.)

The art of private persuasion is accompanied with the giving of presents, (as lovers do,) or with the receiving of pay: (thus it is *Dorophoric* or *Mistharneutic*.) To receive pay as the result of persuasion, is the course, either of those who merely earn their bread by supplying pleasure, namely flatterers, whose art is *Hedyntic*; or of those who profess for pay to teach virtue. And who are they? Plainly the Sophists. And thus *Sophistic* is that kind of Ktetic, Chirotic, Thereutic, Zootheric, Pezotheric, Hemerotheric, Pithanurgic, Idiothereutic, Mistharneutic art, which professes to teach virtue, and takes money on that account.

The same process is pursued along several other lines of inquiry: and at the end of each of them the Sophist is detected, involved in a number of somewhat obnoxious characteristics. This process of division it will be observed, is at every step bifurcate, or as it is called, *dichotomous*. Applied as it is in these examples, it is rather the vehicle of satire than of philosophy. Yet, I have no doubt that this bifurcate method was admired by some of the philosophers of Plato's time, as a clever and effective philosophical invention. We may the more readily believe this, inasmuch as one of the most acute persons of our own time, who has come nearer than any other to the ancient heads of sects in the submission with which his followers have accepted his doctrines, has taken up this Dichotomous Method, and praised it as the only philosophical mode of dividing a subject. I refer to Mr Jeremy Bentham's *Chrestomathia* (published originally in 1816,) in which this exhaustive bifurcate method, as he calls it, was applied to classify sciences and arts, with a view to a scheme of education. How exactly the method, as recommended by him, agrees with the method illustrated in the Sophist, an examination of any of his examples will shew. Thus to take Mineralogy as an example: according to Bentham, Ontology is Cœnoscopic or Idioscopic: the Idioscopic is Somatoscopic or Pneumatoscopic; the Somatoscopic is Pososcopic or Poioscopic: Poioscopic is Physiurgoscopic or Anthropurgoscopic: Physiurgoscopic is Uranoscopic or Epigeoscopic: Epigeoscopic is Abioscopic or Embioscopic. And thus Mineralogy is the Science Idioscopic, Somatoscopic, Poioscopic, Physiurgoscopic, Epigeoscopic, Abioscopic: inasmuch as it is the science which regards bodies, with reference to their qualities, bodies, namely, the works of nature, terrestrial, lifeless.

I conceive that this bifurcate method is not really philosophical or valuable: but that is not our business here. What we have to consider is whether this is what Plato meant by the term *Dialectic*.

The general description of Dialectic in the *Sophistes* agrees very closely with that quoted from the Phædrus, that it is the separation of a subject according to its natural divisions.

Thus, see in the Sophist the passage § 83: "To divide a subject according to the kinds of things, so as neither to make the same kind different nor different kinds identical, is the office of the Dialectical Science." And this is illustrated by observing that it is the office of the science of Grammar to determine what letters may be combined and what may not; it is the office of the science of Music to determine what sounds differing as acute and grave, may be combined, and what may not: and in like manner it is the office of the science of Dialectic to determine what *kinds* may be combined in one subject and what may not. And the proof is still further explained.

In many of the Platonic Dialogues, the Dialectic which Socrates is thus represented as approving, appears to include the form of Dialogue, as well as the subdivision of the subject into its various branches. Socrates is presented as attaching so much importance to this form, that in the Protagoras (§ 65) he rises to depart, because his opponent will not conform to this practice. And generally in Plato, Dialectic is opposed to Rhetoric, as a string of short questions and answers to a continuous dissertation.

Xenophon also seems to imply (Mem. IV. 5, 11) that Socrates included in his notion of Dialectic the form of Dialogue as well as the division of the subject.

But that the method of close Dialogue was not called *Dialectic* by the author of the *Sophist*, we have good evidence in the work itself. Among other notions which are analysed by the bifurcate division here exhibited, is that of getting by contest (*Agonistic*, previously given as a division of *Ktetic*.) Now getting by contest may be by peaceful trial of superiority, or by fight: (*Hamilletic* or *Machetic*.) The fight may be of body against body, or of words against words: these may be called *Biastic* and *Amphisbetic*. The fight of words about right and wrong, may be by long discourses opposed to each other, as in judicial cases; or by short questions and answers: the former may be called *Dicanic*, the latter *Antilogic*. Of these colloquies, about right and wrong, some are natural and spontaneous, others artificial and studied: the former need no special name; the latter are commonly called *Eristic*. Of Eristic colloquies, some are a source of expense to those who hold them, some of gain: that is, they are *Chrematophthoric* or *Chrematistic:* the former, the occupation of those who talk for pleasure's and for company's sake, is *Adoleschic*, wasteful garrulity; the latter, that of those who talk for the sake of gain, is *Sophistic*. And thus Sophistic is an art Eristic, which is part of Antilogic, which is part of Amphisbetic, which is part of Agonistic, which is part of Chirotic, which is a part of Ktetic. (§ 23.)

We may notice here an indication that satire rather than exact reason directs these analyses; in that Sophistic, which was before a part of the *thereutic* branch of *chirotic* and *ktetic*, is here a part of the other branch, *agonistic*.

But the remark which I especially wish to make here is, that the art of discussing points of right and wrong by short questions and answers, being here brought into view, is not called *Dialectic*, which we might have expected; but *Antilogic*. It would seem therefore that the Author of the Sophist did not understand by *Dialectic* such a process as Socrates describes in Xenophon; (Mem. IV. 5, 11, 12;) where he says it was called *Dialectic*, because it was followed by persons *dividing things into their kinds in conversation:* (*κοινῇ βουλεύεσθαι διαλέγοντας*:) or such as the Socrates of Plato insisted upon in the Protagoras and the Gorgias. Of the two elements which the Dialectical Process of Socrates implied, Division of the subject and Dialogue, the author of the *Sophistes* does not claim the name of *Dialectic* for either, and seems to reject it for the second.

But without insisting upon the name, are we to suppose that the Dichotomous Method of the *Sophistes* Dialogue, (I may add of the *Politicus*, for the method is the same in this Dialogue also,) is the method of division of a subject according to its natural members, of which Plato speaks in the Phædrus?

If the *Sophistes* be the work of Plato, the answer is difficult either way. If this method be Plato's *Dialectic*, how came he to omit to say so there? how came he even to seem to deny it? But on the other hand, if this dichotomous division be a different process from the division called *Dialectic* in the Phædrus, had Plato two methods of division of a subject? and yet has he never spoken of them as two, or marked their distinction?

This difficulty would be removed if we were to adopt the opinion, to which others, on other grounds, have been led that the Sophistes, though of Plato's time, is not Plato's work. The grounds of this opinion are,—that the doctrines of the Sophistes are not Platonic: (the doctrine of Ideas is strongly impugned and weakly defended:) Socrates is not the principal speaker, but an Eleatic stranger: and there is, in the Dialogue, none of the dramatic character which we generally have in Plato. The Dialogue seems to be the work of some Eleatic opponent of Plato, rather than his.

(*Rep.* B. vii.) But we can have no doubt that the Phædrus contains Plato's real view of the nature of Dialectic, as to its form; let us see how this agrees with the view of Dialectic, as to its matter and object, given in the seventh Book of the Republic.

According to Plato, Real Existences are the objects of the exact sciences (as number and figure, of Arithmetic and Geometry). The things which are the objects of sense are transitory phenomena, which have no reality, because no permanence. Dialectic deals with Realities in a more general manner. This doctrine is everywhere inculcated by Plato, and particularly in this part of the *Republic*. He does not tell us how we are to obtain a view of the higher realities, which are the objects of Dialectic: only he here assumes that it will result from the education which he enjoins. He says (§ 13) that the Dialectic Process (ἡ διαλεκτικὴ μέθοδος) alone leads to true science: it makes no assumptions, but goes to First Principles, that its doctrines may be firmly grounded: and thus it purges the eye of the soul, which was immersed in barbaric mud, and turns it upward; using for this purpose the aid of the sciences which have been mentioned. But when Glaucon inquires about the details of this Dialectic, Socrates says he will not then answer the inquiry. We may venture to say, that it does not appear that he had any answer ready.

Let us consider for a moment what is said about a philosophy rendering a reason for the First Principles of each Science, which the Science itself cannot do. That there is room for such a branch of philosophy in some sciences, we easily see. Geometry, for instance, proceeds from Axioms, Definitions and Postulates; but by the very nature of these terms, does not prove these First Principles. These—the Axioms, Definitions and Postulates,—are, I conceive, what Plato here calls the *Hypotheses* upon which Geometry proceeds, and for which it is not the business of Geometry to render a reason. According to him, it is the business of "Dialectic" to give a just account of these "Hypotheses." What then is *Dialectic?*

(*Aristotle.*) It is, I think, well worthy of remark, that Aristotle, giving an account in many respects different from that of Plato, of the nature of Dialectic, is still led in the same manner to consider Dialectic as the branch of philosophy which renders a reason for First Principles. In the *Topics*, we have a distinction drawn between reasoning demonstrative, and reasoning dialectical: and the distinction is this:—(*Top.* i. 1) that Demonstration is by syllogisms from true first principles, or from true deductions from such principles: and that the Dialectical Syllogism is that which syllogizes from probable propositions (ἐξ ἐνδόξων). And he adds that probable propositions are those which are accepted by all, or by the greatest part, or by the wise. In the next chapter, he speaks of the uses of Dialectic, which, he says, are three, mental discipline, debates, and philosophical science. And he adds (*Top.* I. 2, 6) that it is also useful with reference to the First Principles in each Science: for from the appropriate Principles of each science we cannot deduce anything concerning First Principles, since these principles are the beginning of reasoning. But from the probable principles in each province of science we must reason concerning First Principles: and this is either the peculiar office of Dialectic, or the office most appropriate to it; for it is a process of investigation, and must lead to the Principles of all methods.

That a demonstrative science, as such, does not explain the origin of its own First Principles, is undoubtedly true. Geometry does not undertake to give a reason for the Axioms, Definitions, and Postulates. This has been attempted, both in ancient and in modern times, by the Metaphysicians. But the Metaphysics employed on such subjects has not commonly been called Dialectic. The term has certainly been usually employed rather as describing a Method, than as determining the subject of investigation. Of the Faculty which apprehends First Principles, both according to Plato and to Aristotle, I will hereafter say a few words.

The object of the dichotomous process pursued in the Sophistes, and its result in each case, is a Definition. Definition also was one of the main features of the inquiries pursued by Socrates, Induction being the other; and indeed in many cases Induction was a series of steps which ended in Definition. And Aristotle also taught a peculiar method, the object and result of which was the construction of Definitions:—namely his *Categories*. This method is one of division, but very different from the divisions of the Sophistes. His method begins by dividing the whole subject of possible inquiry into ten heads or *Categories*—Substance, Quantity, Quality, Relation, Place, Time, Position, Habit, Action, Passion. These again are subdivided: thus Quality is Habit or Disposition, Power, Affection, Form. And we have an example of the application of this method to the construction of a Definition in the Ethics; where he determines Virtue to be a Habit with certain additional limitations.

Thus the Induction of Socrates, the Dichotomy of the Eleatics, the Categories of Aristotle, may all be considered as methods by which we proceed to the construction of Definitions. If, by any method, Plato could proceed to the construction of a Definition, or rather of an Idea, of the Absolute Realities on which First Principles depend, such a method would correspond with the notion of Dialectic in the Republic. And if it was a method of division like the Eleatic or Aristotelic, it would correspond with the notion of Dialectic in the Phædrus.

That Plato's notion, however, cannot have been exactly either of these is, I think, plain. The colloquial method of stimulating and testing the progress of the student in Dialectic is implied, in the sequel of this discussion of the effect of scientific study. And the method of Dialogue, as the instrument of instruction, being thus supposed, the continuation of the account in the *Republic*, implies that Plato expected persons to be made dialectical by the study of the exact sciences in a comprehensive spirit. After insisting on Geometry and other sciences, he says (*Rep.* VII. § 16): "The synoptical man is dialectical; and he who is not the one, is not the other."

But, we may ask, does a knowledge of sciences lead naturally to a knowledge of Ideas, as absolute realities from which First Principles flow? And supposing this to be true, as the Platonic Philosophy supposes, is the Idea of the Good, as the source of moral truths, to be thus attained to? That it is, is the teaching of Plato, here and elsewhere; but have the speculations of subsequent philosophers in the same direction given any confirmation of this lofty assumption?

In reply to this inquiry, I should venture to say, that this assumption appears to be a remnant of the Socratic doctrine from which Plato began his speculations, that Virtue is a kind of knowledge; and that all attempts to verify the assumption have failed. What Plato added to the Socratic notion was, that the inquiry after The Good, the Supreme Good, was to be aided by the analogy or suggestions of those sciences which deal with necessary and eternal truths; the supreme good being of the nature of those necessary and eternal truths. This notion is a striking one, as a suggestion, but it has always failed, I think, in the attempts to work it out. Those who in modern times, as Cudworth and Samuel Clarke, have supposed an analogy between the necessary truths of Geometry and the truths of Morality, though they have used the like expressions concerning the one and the other class of truths, have failed to convey clear doctrines and steady convictions to their readers; and have now, I believe, few or no followers.

The result of our investigation appears to be, that though Plato added much to the matter by means of which the mind was to be improved and disciplined in its research after Principles and Definitions, he did not establish any form of Method according to which the inquiry must be conducted, and by which it might be aided. The most definite notion of Dialectic still remained the same with the original informal view which Socrates had taken of it, as Xenophon tells us, (Mem. iv. 5, 11) when he says: "He said that Dialectic (τὸ διαλέγεσθαι) was so called because it is an inquiry pursued by persons who take counsel together, separating the subjects considered according to their kinds (διαλέγοντας). He held accordingly that men should try to be well prepared for such a process, and should pursue it with diligence: by this means, he thought, they would become good men, fitted for responsible offices of command, and truly dialectical" (διαλεκτικωτάτους). And this is I conceive the answer to Mr Grote's interrogatory exclamation (Vol. viii. p. 577): "Surely the Etymology here given by Xenophon or Socrates of the word (διαλέγεσθαι) cannot be considered as satisfactory." The two notions, of investigatory Dialogue, and Distribution of notions according to their kinds, which are thus asserted to be connected in etymology, were, among the followers of Socrates, connected in fact; the dialectic dialogue was supposed to involve of course the dialectic division of the subject.

Trinity Lodge, *April* 4, 1855.

XXVI. *Of the Intellectual Powers according to Plato.*

BY THE REV. W. WHEWELL, D.D. *Master of Trinity College.*

[Read *November* 12, 1855.]

IN the Seventh Book of Plato's *Republic*, we have certain sciences described as the instruments of a philosophical and intellectual education; and we have a certain other intellectual employment spoken of, namely, Dialectic, as the means of carrying the mind beyond these sciences, and of enabling it to see the sources of those truths which the sciences assume as their first principles. These points have been discussed in the two preceding papers. But this scheme of the highest kind of philosophical education proceeds upon a certain view of the nature and degrees of knowledge, and of the powers by which we know; which view had been presented in a great measure in the Sixth Book; this view I shall now attempt to illustrate.

To analyse the knowing powers of man is a task so difficult, that we need not be surprized if there is much obscurity in this portion of Plato's writings. But as a reason for examining what he has said, we must recollect that if there be in it anything on this subject which was true then, it is true still; and also, that if we know any truth on that subject now, we shall find something corresponding to that truth in the best speculations of sagacious ancient writers, like Plato. It may therefore be worth while to discuss the Platonic doctrines on this matter, and to inquire how they are to be expressed in modern phraseology.

Plato's doctrine will perhaps be most clearly understood, if we begin by considering the *diagram* by which he illustrates the different degrees of knowledge*. He sets out from the distinction of *visible* and *intelligible* things. There are visible objects, squares and triangles, for instance; but these are not the squares and triangles about which the geometer reasons. The exactness of his reasoning does not depend on the exactness of his diagrams. He reasons from certain mental squares and triangles, as he conceives and understands them. "Thus there are visible and there are intelligible things. There is a visible and an intelligible world†: and there are two different regions about which our knowledge is concerned. Now take a line divided into two unequal segments to represent these two regions: and again, divide each segment in the same ratio. The parts of each segment are to represent differences of clearness and distinctness, and in the visible world these parts are *things* and *images*. By *images* I mean shadows, and reflections in water, and in polished bodies; and by *things*, I mean that of which these images are the resemblances; as animals, plants, things made by man. This difference corresponds to the difference of *Knowledge* and mere *Opinion;* and the *Opinable* is to the *Knowable* as the Image to the Reality."

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* Pol. VI. § 19.

† He adds, "This *oraton*, this visible world, I will not say has any connexion with *ouranon*, heaven, that I may not be accused of playing upon words."

This analogy is assented to by Glaucon; and thus there is assumed a ground for a further construction of the diagram.

"Now," he says, "we have to divide the segment which represents Intelligible Things in the same way in which we have divided that which represents Visible Things. The one part must represent the knowledge which the mind gets by dealing as it were with images, and by reasoning downwards *from* Principles; the other that which it has by dealing with the Ideas themselves, and going *to* First Principles.

"The one part depends upon assumptions or hypotheses*, the other is unhypothetical or absolute truth.

"One kind of Intelligible Things, then, is Conceptions; for instance, geometrical conceptions of figures, by means of which we reason downwards, assuming certain First Principles.

"Now the other kind of Intelligible Things is this:—that which the Reason includes in virtue of its power of reasoning, when it regards the assumptions of the Sciences as, what they are, assumptions only; and uses them as occasions and starting points, that from these it may ascend to the *absolute*. (ἀνυπόθετον, unhypothetical) which does not depend upon assumption, but is the origin of scientific truth. The Reason takes hold of this first principle of truth; and availing itself of all the connections and relations of this principle, it proceeds to the conclusion; using no sensible image in doing this, but contemplating the Ideas alone; and with these Ideas the process begins, goes on, and terminates."

This account of the matter will probably seem to require at least further explanation; and that accordingly is acknowledged in the Dialogue itself. Glaucon says:

"I apprehend your meaning in a certain degree, but not very clearly, for the matter is somewhat abstruse. You wish to prove that the knowledge which, by the Reason, we acquire, of Real Existence and Intelligible Things, is of a higher degree of certainty than the knowledge which belongs to what are commonly called Sciences. Such sciences, you say, have certain assumptions for their bases; and these assumptions are, by the students of such sciences, apprehended, not by Sense (that is, the Bodily Senses), but by a Mental Operation,—by Conception. But inasmuch as such students ascend no higher than the assumptions, and do not go to the First Principles of Truth, they do not seem to you to have true knowledge—intuitive insight—*Nous*—on the subject of their reasonings, though the subjects are intelligible, along with their principle. And you call this habit and practice of the Geometers and others by the name *Conception*, not *Intuition* †; taking Conception to be something between Opinion on the one side, and Intuitive Insight on the other."

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* It is plain that Plato, by *Hypotheses*, in this place, means the usual foundations of Arithmetic and Geometry; namely, Definitions and Postulates. He says that "the arithmeticians and geometers take as hypotheses (ὑποθέμενοι) odd and even, and the three kinds of angles, (right, acute, and obtuse;) and figures, (as a triangle, a square,) and the like." I say his "hypotheses" are the Definitions and Postulates, not the Axioms: for the Axioms of Arithmetic and Geometry belong to the Higher Faculty, which ascends to First Principles. But this Faculty operates rather in using these axioms than in enunciating them. It knows them implicitly rather than expresses them explicitly.

† διάνοιαν ἀλλ' οὐ νοῦν.

"You have explained it well, said I. And now consider the four sections (of the line) of which we have spoken, as corresponding to four affections in the mind. Intuition, the highest; Conception, the next; the third, Belief; and the fourth, Conjecture (from likenesses); and arrange them in order, so that they may have more or less of certainty, as their objects have more or less of truth[*].

"I understand, said he. I agree to what you say, and I arrange them as you direct."

And so the Sixth Book ends: and the Seventh Book opens with the celebrated image of the Cave, in which men are confined, and see all external objects only by the shadows which they cast on the walls of their prison. And this imperfect knowledge of things is to the true vision of them, which is attained by those who ascend to the light of day, as the ordinary knowledge of men is to the knowledge attainable by those whose minds are purged and illuminated by a true philosophy.

Confining ourselves at present to the part of Plato's speculations which we have mentioned, namely, the degrees of knowledge, and the division of our knowing faculties, we may understand, and may in a great degree accept, Plato's scheme. We have already (in the preceding papers) seen that, by the knowledge of real things, he means, in the first place, the knowledge of universal and necessary truths, such as Geometry and the other exact sciences deal with. These *we* call sciences of Demonstration; and we are in the habit of contrasting the knowledge which constitutes such sciences with the knowledge obtained by the Senses, by Experience or mere Observation. This distinction of Demonstrative and Empirical knowledge is a cardinal point in Plato's scheme also; the former alone being allowed to deserve the name of *Knowledge*, and the latter being only *Opinion*. The Objects with which Demonstration deals may be termed *Conceptions*, and the objects with which Observation or Sense has to do, however much speculation may reduce them to mere Sensations, are commonly described as *Things*. Of these Things, there may be Shadows or Images, as Plato says; and as we may obtain a certain kind of knowledge, namely Opinion or Belief, by seeing the Things themselves, we may obtain an inferior kind of Opinion or Belief by seeing their Images, which kind of opinion we may for the moment call *Conjecture*. Whether then we regard the distinctions of knowledge itself or of the objects of it, we have three terms before us.

If we consider the kinds of Knowledge, they are

Demonstration: Belief: Conjecture:

If the objects of this knowledge, they are

Conceptions: Things: Images.

But in each of these Series, the first term is evidently wanting: for Demonstration supposes

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[*] The Diagram, as here described, would be this ;

Intelligible World.		*Visible World.*	
Intuition.	Conception.	Things.	Images.

Plato supposes the whole, and each of the two parts, to be divided in the same ratio, in order that the *analogy* of the division in each case may be represented.

Principles to reason from. Conceptions suppose some basis in the mind which gives them their evidence. What then is the first term in each of these two Series?

The Principles of Demonstration must be seen by *Intuition*.

Conceptions derive their properties from certain powers or attributes of the mind which we may term *Ideas*.

Therefore the two series are

Intuition : Demonstration : Belief : Conjecture.
Ideas : Conceptions : Things : Images.

Plato further teaches that the two former terms in each Series belong to the Intelligible, the two latter to the Visible World: and he supposes that the ratio of these two primary segments of the line is the same as the ratio in which each segment is divided [*].

In using the term *Ideas* to describe the mental sources from which Conceptions derive their validity in demonstration, I am employing a phraseology which I have already introduced in the *Philosophy of the Inductive Sciences*. But independently altogether of this, I do not see what other term could be employed to denote the mental objects, attributes, or powers, whatever they be, from which Conceptions derive their evidence, as Demonstrative Truths derive their evidence from Intuitive Truths.

That the Scheme just presented is Plato's doctrine on this subject, I do not conceive there can be any doubt. There is a little want of precision in his phraseology, arising from his mixing together the two series. In fact, his final series

Noësis: Dianoia: Pistis: Eikasia;

is made by putting in the second place, instead of *Demonstration*, which is the *process* pursued, or *Science*, which is the *knowledge* obtained, *Conception*, which is the *object* with which the mind deals. Such deviations from exact symmetry and correlation in speaking of the faculties of the mind, are almost unavoidable in every language. And there is yet another source of such inaccuracies of language; for we have to speak, not only of the process of acquiring knowledge, and of the objects with which the mind deals, but of the Faculties of the mind which are thus employed. Thus *Intuition* is the Process; *Ideas* are the Object, in the first term of our series. The Faculty also we may call *Intuition;* but the Greek offers a distinction. *Noësis* is the *Process* of Intuition; but the *Faculty* is *Nous*. If we wish to preserve this distinction in English, what must we call the Faculty? I conceive we must call it *the Intuitive Reason*, a term well known to our older philosophical writers [†]. Again: taking the second term of the series, *Demonstration* is the process, *Science*, the result; and *Conceptions* are the objects with which the mind deals. But what is the *Faculty* thus employed? What is the Faculty employed in Demonstration? The same philosophical writers of whom I spoke would have answered at once, *the Discursive Reason;* and I do not know that, even now, we can suggest any better term. The Faculty employed in acquiring the two lower kinds of knowledge, the Faculty which deals with Things and their Images is, of course, *Sense* or *Sensation*.

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[*] The four segments might be as 4 : 2 : 2 : 1; or as 9 : 6 : 6 : 4; or generally, as $a : ar : ar : ar^2$.

[†] Hence the mind Reason receives
Intuitive or Discursive.

MILTON.

The assertion of a Faculty of the mind by which it apprehends Truth, which Faculty is higher than the Discursive Reason, as the Truth apprehended by it is higher than mere Demonstrative Truth, agrees (as it will at once occur to several of my readers) with the doctrine taught and insisted upon by the late Samuel Taylor Coleridge. And so far as he was the means of inculcating this doctrine, which, as we see, is the doctrine of Plato, and I might add, of Aristotle, and of many other philosophers, let him have due honour. But in his desire to impress the doctrine upon men's minds, he combined it with several other tenets, which will not bear examination. He held that the two Faculties by which these two kinds of truth are apprehended, and which, as I have said, our philosophical writers call *the Intuitive Reason* and *the Discursive Reason*, may be called, and ought to be called, respectively, *The Reason* and *The Understanding;* and that the second of these is of the nature of the *Instinct* of animals, so as to be something intermediate between Reason and Instinct. These opinions, I may venture to say, are altogether erroneous. The Intuitive Reason and the Discursive Reason are not, by any English writers, called the Reason and the Understanding; and accordingly, Coleridge has had to alter all the passages, namely, those taken from Leighton, Harrington, and Bacon, from which his exposition proceeds. The Understanding is so far from being especially the Discursive or Reasoning Faculty, that it is, in universal usage, and by our best writers, *opposed* to the Discursive or Reasoning Faculty. Thus this is expressly declared by Sir John Davis in his poem *On the Immortality of the Soul.* He says, of the soul.

> When she *rates* things, and moves from ground to ground,
>   The name of *Reason* (*Ratio*) she acquires from this:
> But when by reason she truth hath found,
>   And standeth fixt, she *Understanding* is.

Instead of the Reason being fixed, and the Understanding discursive, as Mr Coleridge says, the Reason is distinctively discursive; that is, it obtains conclusions by running from one point to another. This is what is meant by *Discursus;* or, taking the full term, *Discursus Rationis, Discourse of Reason.* Understanding is fixed, that is, it dwells upon one view of a subject, and not upon the steps by which that view is obtained. The verb *to reason*, implies the substantive, *the Reason*, though it is not coextensive with it: for as I have said, there is the Intuitive Reason as well as the Discursive Reason. But it is by the Faculty of Reason that we are capable of reasoning; though undoubtedly the practice or the pretence of reasoning may be carried so far as to seem at variance with reason in the more familiar sense of the term; as is the case also in French. Molière's Crisale says (in the *Femmes Savantes*)

> Raisonner est l'emploi de toute ma maison,
> Et le raisonnement en bannit la Raison.

If Mr Coleridge's assertion were true, that the Understanding is the discursive and the Reason the fixed faculty, we should be justified in saying that *The Understanding is the faculty by which we reason, and the Reason is the faculty by which we understand.* But this is not so.

Nor is the Understanding of the nature of Instinct, nor does it approach nearer than the Reason to the nature of Instinct, but the contrary. The Instincts of animals bear a very obscure resemblance to any of man's speculative Faculties: but so far as there is any such resemblance, Instinct is an obscure image of Reason, not of Understanding. Animals are said to act as if they reasoned, rather than as if they understood. The verb *understand* is especially applied to man as distinguished from animals. Mr Coleridge tells a tale from Huber, of certain bees which, to prevent a piece of honey from falling, balanced it by their weight, while they built a pillar to support it. They did this by Instinct, not *understanding* what they did; men, doing the same, would have *understood* what they were doing. Our Translation of the Scriptures, in making it the special distinction of man and animals, that *he has Understanding* and they have not, speaks quite consistently with good philosophy and good English.

Mr Coleridge's object in his speculations is nearly the same as Plato's; namely, to declare that there is a truth of a higher kind than can be obtained by mere reasoning; and also to claim, as portions of this higher truth, certain fundamental doctrines of Morality. Among these, Mr Coleridge places the Authority of Conscience, and Plato, the Supreme Good. Mr Coleridge also holds, as Plato held, that the Reason of man, in its highest and most comprehensive form, is a portion of a Supreme and Universal Reason; and leads to Truth, not in virtue of its special attributes in each person, but by its own nature.

Many of the opinions which are combined with these doctrines, both in Plato and in Coleridge, are such as we should, I think, find it impossible to accept, upon a careful philosophical examination of them; but on these I shall not here dwell.

I will only further observe, that if any one were to doubt whether the term Νοῦς is rightly rendered *Intuitive Reason*, we may find proof of the propriety of such a rendering in the remarkable discussion concerning the Intellectual Virtues, which we have in the Sixth Book of the Nicomachean Ethics. It can hardly be questioned that Aristotle had in his mind, in writing that passage, the doctrines of Plato, as expounded in the passage just examined, and similar passages. Aristotle there says that there are five Intellectual Virtues, or Faculties by which the Mind aims at Truth in asserting or denying:—namely, *Art*, *Science*, *Prudence*, *Wisdom*, *Nous*. In this enumeration, passing over Art, Prudence, and Wisdom, as virtues which are mainly concerned with practical life, we have, in the region of speculative Truth, a distinction propounded between *Science* and *Nous*: and this distinction is further explained (c. 6) by the remarks that Science reasons from Principles; and that these Principles cannot be given *by* Science, because Science reasons *from* them; nor by Art, nor Prudence, for these are conversant with matters contingent, not with matters demonstrable; nor can the First Principles of the Reasonings of Science be given by Wisdom, for Wisdom herself has often to reason from Principles. Therefore the First Principles of Demonstrative Reasoning must be given by a peculiar Faculty, *Nous*. As we have said, *Intuitive Reason* is the most appropriate English term for this Faculty.

The view thus given of that higher kind of Knowledge which Plato and Aristotle place above ordinary Science, as being the Knowledge of and Faculty of learning First Principles, will enable us to explain some expressions which might otherwise be misunderstood. Socrates, in the concluding part of this Sixth Book of the *Republic*, says, that this kind of knowledge is "that of which the Reason (λόγος) takes hold, *in virtue of its power of reasoning**." Here we are plainly not to understand that we arrive at First Principles *by reasoning*: for the very opposite is true, and is here taught;—namely, that First Principles are not what we reason *to*, but what we reason

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* τῇ τοῦ διαλέγεσθαι δυνάμει.

from. The meaning of this passage plainly is, that First Principles are those of which the Reason takes hold *in virtue of its power of reasoning*;—they are the conditions which must exist in order to make any reasoning possible:—they are the propositions which the Reason must involve implicitly, in order that we may reason explicitly;—they are the intuitive roots of the dialectical power.

In accordance with the views now explained, Plato's Diagram may be thus further expanded. The term ἰδέα is not used in this part of the *Republic*; but, as is well known, occurs in its peculiar Platonic sense in the Tenth Book.

	Intelligible World. νοητόν.		Visible World. ὁρατόν.	
Object ...	Ideas ἰδέαι	Conceptions διάνοια	Things ζῶα κ.τ.λ.	Images εἰκόνες
Process ..	Intuition νόησις	Demonstration ἐπιστήμη	Belief πίστις	Conjecture εἰκασία
Faculty...	Intuitive Reason νοῦς	Discursive Reason λόγος	Sensation αἴσθησις	

TRINITY LODGE, *May* 22, 1855.

XXVII. *Remarks on the Fundamental Principle of the Theory of Probabilities.*

By R. L. Ellis, M.A. *late Fellow of Trinity College.*

[Read *Nov.* 13, 1854.]

I wish to make an addition to the remarks on the foundation of the theory of probabilities which were offered some years since to the notice of the Society*. My intention in doing so is to consider, in what way the proposition, which I conceive to be the fundamental principle of the theory, may be the most clearly and conveniently expressed. This principle may for the moment be thus stated: "On a long run of similar trials, every possible event tends ultimately to recur in a definite ratio of frequency." Our conviction of the truth of this proposition is, I think, intuitive,—the word being used, as in all similar cases, with reference to the intuitions of a mind, which has fully and clearly apprehended the subject before it, and to which therefore to have arrived at the truth and to perceive that it has done so are inseparable elements of the same act of thought. If we endeavour to translate the proposition just stated into ordinary philosophical language, we may in the first place remark that the phrase "similar trials," expresses the notion of a group or genus of phenomena to which the different results are subordinated as distinct species. If the trial is the throwing of a die, this may be regarded as the generic character; the occurrence of ace, deuce, &c. constituting different species. Thus much is clear; but it is less obvious how the idea expressed by a "long run of trials" in "definite series of experiments," and the like, is to be expressed, so as to make the analogy between the fundamental principle of the theory of probabilities and those of other sciences more obvious than it has hitherto been. The idea in question is not readily expressed in any way, because in its own nature it is negative and indefinite. The phrases I have just quoted imply merely the absence of the limitations inseparable from individual cases, or from any finite number of such cases, whether contemplated as actually existent or as about to be developed within definite limits of space and time.

When individual cases are considered, we have no conviction that the ratios of frequency of occurrence depend on the circumstances common to all the trials. On the contrary, we recognise in the determining circumstances of their occurrence an extraneous element, an element, that is, extraneous to the idea of the genus and its species. Contingency and limitation come in (so to speak) together; and both alike disappear when we consider the genus in its entirety, or (which is the same thing), in what may be called an ideal and practically impossible realization of all which it potentially contains. If this be granted, it seems to follow that the fundamental principle of the theory of probabilities may be regarded as included in the following statement;—"The conception of a genus implies that of numerical relations among the species subordinated to it."

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* *Transactions of the Cambridge Philosophical Society*, Vol. VIII. page 1.

(2) But in what relation, it may be asked, do these conceptions stand to outward realities? How can they be made the foundation of a real science, that is, of a science relating to things as they really exist? We are by such questions led back to what was long the great controversy of philosophy;—I mean the contest between the realists and the nominalists. The former in asserting the reality of universals did not maintain that what we think of when we use a general term is an actually existing thing. Like every one else they admitted, that in one sense nothing can exist but the individual, nevertheless they held that universals are not mere figments of the mind, but that they have a reality of their own which is the foundation of the truth of general propositions. To assert therefore that the theory of probabilities has for its foundation a statement touching genera and their species, and is at the same time a real science, is to take a realistic view of its nature. And this I believe is what, on consideration, we cannot avoid doing.

If it be said that the grouping phenomena together is merely a mental act wholly disconnected from outward reality and altogether arbitrary, it may be replied that no mental act can be so. Why and how facts and ideas correspond is no doubt one of the great questions of philosophy; but the answer to it is surely to be developed from the consideration, that man in relation to the universe is not *spectator ab extra*, but in some sort a part of that which he contemplates, and that the *rebus avolsa ratio*, which is in truth the fundamental postulate of nominalism, is therefore inconcessible. The thoughts we think are, it is true, ours, but so far as they are not mere error and confusion, so far as they have anything of truth and soundness, they are something and much more. The *veritas essendi* (to recur to the language of the schoolmen) is the fountain from whence the *veritas cognoscendi* is derived. The meaning which these phrases were intended to convey is expressed in more modern language by Leibnitz in the passage which I have cited in the note[*]. In every science the fact and the idea correspond because the former is the realization of the latter, but as this realization is of necessity partial and incomplete—or rather because in the same fact are simultaneously realized a variety of separate ideas, separate, that is, as we conceive them—this correspondence is but imperfect and approximate. It is only when in thought we remove the action of disturbing causes to an indefinite distance, that we can conceive the absolute verification of any *à priori* law. Only on the horizon of our mental prospect earth and sky, the fact and the idea, are seen to meet, though in reality the atmosphere is everywhere present. Everywhere it surrounds and interpenetrates the γῆ μέλαινα on which we stand;—making it put forth and sustain all the numberless forms of organization and of life. The indefinitely prolonged series of trials, which enters into the ordinary statement of the fundamental principle of the theory of probabilities, is analogous to the infinite and infinitely smooth horizontal plane, which would enable us to verify the first law of motion.

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[*] C'est Dieu qui est la dernière raison des choses, et la connaissance de Dieu n'est pas moins le principe des sciences, que son essence et sa volonté sont les principes des êtres. [Erdmann, p. 106]. A little further on he adds: C'est sanctifier la philosophie, que de faire couler ses ruisseaux de la fontaine des attributs de Dieu.

(3) The simple negative notion of the absence of disturbing forces is perpetually confounded with that of a tendency inherent in a series of successively developed results to restore the balance of frequency of occurrence, when this has been by accidental circumstances temporarily deranged. It is commonly thought that this notion, which, as we know, is the foundation of many unsuccessful attempts to circumvent fortune, is sufficiently refuted by saying, that what is past can exert no influence on what is yet to come. But in reality the past influences the future in a thousand different ways; and it is only in idea that we can secure the possibility of an indefinite series of trials, of which those which we regard as the permanent circumstances are not progressively, however slowly, undergoing alteration. The dice box for example wears smooth, and the edges of the die are rounded; and though, in this example, we cannot say what result is facilitated by the change, yet this is not always the case. Such progressive alterations may tend so to alter the ratio of frequency of occurrence, as to restore the balance which the result of past trials has disturbed. There is thus nothing absurd in the notion of a restorative and balancing tendency, though the grounds on which it is commonly assumed indicate much confusion of thought. It would for instance be perfectly reasonable to inquire, whether in the succession of seasons hot years are not oftener followed by cold and cold by hot, than *vice versâ*. Such questions indicate a branch of the theory of methods of observation to which hitherto but little attention has been paid.

XXVIII. *On the Singular Points of Curves, and on Newton's Method of Co-ordinated Exponents. By* AUGUSTUS DE MORGAN, *of Trinity College, Vice-President of the Royal Astronomical Society, and Professor of Mathematics in University College, London.*

[Read *May* 21, 1855.]

MANY of the critical points of Algebra have first been brought to notice in the treatment of curves: and all have been usefully illustrated. The subject also has its controversies. Some of these might have been avoided, if writers had distinctly stated what they meant by a *curve*, and by a *singular point*. In algebraic *Geometry*, or geometry investigated algebraically, a curve should be either the intersection of two surfaces, or a continuous line described by a point moving under one permanent law. But in geometrical *Algebra*, or algebra represented in space, a curve ought to be defined as the collection of all points whatsoever, and howsoever connected, whose co-ordinates satisfy a given equation. The present paper is on algebra, and on real values only, so far as interpretation is concerned.

According to Cramer, singular points are those "qui ont quelque chose qui les distingue des autres." This is somewhat too general: the point at which $x = 3$, $y = 4\frac{1}{2}$, has something which distinguishes it from all others; namely, that $x = 3$, $y = 4\frac{1}{2}$. Dr Peacock, in the heading of a chapter, speaks of "singular or remarkable" points, in which the disjunctive particle appears to have a defining force. According to M. Cauchy, singular points are those which present some remarkable peculiarity inherent in the curve, and independent of the position of the axes. I should prefer to divide singular points with reference to *position* and to *curvature*. A point of singular *position* is one which has a *notable* property, such as no continuous arc of the curve[*], however small, can have at all its points, and which has reference to the axes: a point of singular *curvature* has such a property independently of all reference to the axes. But since the point is especially taken into consideration as a point in a curve, we must understand that reference to adjacent points is intended in both cases: that is, the singular property must either involve some use of differential coefficients, or some other reference to adjacent points. In truth, we are rather concerned with singular *elements*, infinitely small arcs, than with singular *points*.

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[*] Of *the* curve: thus $y' = 0$ gives a point of singular position on all curves except $y = \text{const.}$ I cannot find a definition free from the idea of notability, which contains matter of opinion, varying according to circumstances. When we think of a curve as the tabulation of y in terms of x, maxima, contrary flexures, and points of regression, are single and notable. But let a curve be used to supply directing data for the formation of other curves, and a point of the most ordinary character may be perfectly single, and especially notable. Use a curve $y = \phi x$ for the determination of $\phi\phi x$, $\phi\phi\phi x$, &c., and the point at which $y = x$ is the most notable of any.

The old words *univocal* and *equivocal* may be revived, though not precisely in their old meanings, to signify functions of one value only, and functions giving choice of more than one value. Thus e^x is univocal or equivocal, according to the value of x. A univocal function which is always continuous and finite has a differential coefficient of the same qualities.

Let y' strictly denote the *limit* of $dy : dx$. When dx and dy are infinitely small, I do not say that $dy = y'dx$, but that $dy = pdx$, where p is infinitely near to y'. There are cases in which y' is real, and p imaginary, of the form $y' + adx^m + bdx^n \sqrt{-1}$.

Let the words *before* and *after* relate to value progressing from $-\infty$ to $+\infty$: thus x is -8 before it is -7. Let the phrases 'just before' and 'just after' relate to infinitely small differences of value.

Let the axes of x and y be called *horizontal* and *vertical*. And, $\phi(x, y) = 0$ being the equation of a curve, let $d\phi : dx = 0$ or $\phi_x = 0$, and $\phi_y = 0$, be called the horizontal and vertical *subordinates* of $\phi = 0$. When a curve meets one only of its *subordinates*, its tangent is of the same name with that subordinate, either horizontal or vertical.

Let $\phi(x, y)$ be a function which, for all real and finite values of x and y, is real, finite, and univocal, and such that $\phi = 0$ does not always give $\phi_x = 0$, $\phi_y = 0$, as would happen, for instance, if ϕ were a positive power of another such function. The curve $\phi = 0$ then divides the whole plane of co-ordinates into regions in which ϕ is always positive, and regions in which ϕ is always negative. And every passage over $\phi = 0$ is a change of sign in ϕ, unless it be also a passage over one of the subordinates. Thus, if (x, y) be a point of $\phi = 0$, $\phi(x, y - dy)$ and $\phi_y(x, y - dy)$ differ in sign, and $\phi(x, y + dy)$ and $\phi_y(x, y + dy)$ agree in sign, if dy be positive: whence either ϕ or ϕ_y changes sign in passing through (x, y). Again, x being constant, y cannot change from one point to the next nearest of $\phi = 0$, without passing over $\phi_y = 0$ in the interval. Accordingly, when $\phi_y = 0$ is not a separator of positive and negative regions, as when $\phi_y = (x + y)^2$, it follows that one value of x gives to $\phi = 0$ only one real value of y, at most. A curve which is not a separator must be regarded as having its branches repeated an even number of times: and then we may say that y (or rather the curve $x = \text{const.}$) does not pass from one point of $\phi = 0$ to the next before or after without crossing $\phi_y = 0$ an odd number of times.

When (x, y) and $(x + dx, y + dy)$ are both on $\phi = 0$, we know from $\phi_x dx + \phi_y dy + \frac{1}{2}(\phi_{xx} dx^2 + \ldots) + \ldots = 0$ that $\phi_x dx + \phi_y dy$ is of the second order, at lowest. Hence the closest straight line to the curve is that on which, $(x + dx, y + dy)$ being on the straight line, $\phi_x dx + \phi_y dy$ is absolutely zero: and thus we get the tangent and the usual forms connected with it. Of the two regions divided by this tangent, $dx = 0$ and $dy > 0$ point out the one in which $\phi_x dx + \phi_y dy$ has the sign of ϕ_y, or, just after (x, y), the sign of ϕ. Proceed to $(x + dx, y + dy)$ on the tangent. Then ϕ, at $(x + dx, y + dy)$ has the sign of

$$\phi_{xx} + 2\phi_{xy} y' + \phi_{yy} y'^2, \text{ or of } -\phi_y \cdot y'',$$

unless this last vanish. That is, dy being > 0, ϕ agrees with ϕ_y at $(x, y + dy)$, and with $-\phi_y \cdot y''$ at $(x + dx, y + dy)$ on the tangent. If then ϕ_y and $-\phi_y \cdot y''$ agree in sign, and if (x, y) be not a multiple point, the curve $\phi = 0$ does not fall between the positive continuation (or *retracement*, if y be negative) of y and the tangent: if ϕ_y and $-\phi_y y''$ differ, the contrary. That is, y being measured positively upwards, the concavity is upwards or downwards, according as y'' is positive or negative. Again, a point on the tangent, just after contact, *abuts* on convexity. Hence, a point close to the curve abuts on convexity or concavity according as ϕ at the point and $\phi_y . y''$ at the contiguous points of the curve differ or agree in sign. Hence another proof of the criterion of flexure: for if a point be a little nearer the axis of x than (x, y), ϕ and ϕ_y agree or differ, according as y is negative or positive; if the point abut on convexity, $\phi . \phi_y . y''$ is negative, so that y'' agrees in sign with y.

The point of contrary flexure, at which y'' changes sign, stands apart from other points of singular curvature, and seems to form a class by itself: but only because it belongs to a class in which, though all the cases are *singular*, it is the only one *remarkable* to sight. Attempts have been made to give name and character to the points at which y''', y^{iv}, &c. change sign, but, though $\phi = 0$ may from the existence of such points be properly called a singular *case* of $\phi =$ const., the points themselves are not remarkable points of $\phi = 0$. It seems that our senses are not capable of feeling anything beyond second differential coefficients. In geometry, we recognise direction and flexure, and change of direction simultaneously with flexure: in mechanics, velocity and pressure, and change of velocity simultaneously with pressure. But change in the quantity of flexure, or of pressure, presents nothing which the common sense of mankind embodies in a separate notion; so that the third differential coefficient is not presented to us in the form, as it were, of a separate[*] idea.

Let two points of $\phi = 0$ be called *similar*, when a line drawn from one to the other cuts $\phi = 0$ an even or an odd number of times, according as it begins and ends with the same or different abutments: otherwise, let the points be called *dissimilar*. It follows from the preceding connexion between $\phi_y . y''$ at (x, y) and ϕ just after leaving (x, y), that points are similar or dissimilar according as their values of $\phi_y . y''$ agree or differ in sign. Also points are similar or dissimilar, according as, in passing from one to the other, we pass over the curve $\phi_y^2\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy} = 0$ an even (0 included) or an odd number of times. A change on the curve to points of a different character takes place when there is a change of vertical flexure (flexure with respect to the axis of x, or which depends on the sign of y'') or passage over the vertical subordinate with change of sign, but not both. And the equation $\phi_y^3 y'' - \phi_x^3 x'' = 0$ connects the criteria with respect to the two axes, and justifies one from the other.

Hence no curve in which $\phi(x, y)$ is strictly real, univocal, and finite, can have an abrupt termination: for in this case, the same two points would be pronounced similar or dissimilar, according as the line of progress fell just on one side or the other of the abrupt termination.

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[*] Voltaire's Saturnian, who had only seventy-two senses, lamented the poverty of his human nature to the Sirian, who had a thousand: but the Sirian replied that the same complaint was made in his own planet. Both had reason, for the train of differential coefficients is interminable. Voltaire could not make these worthies talk their own minds. If we had another mode of perception, of which neither touch nor sight would give any notion, and which would associate with pressure the idea of its cause, in the sense in which we say that pressure causes acceleration, we should probably, as now, look upon the cause of a cause, and that cause, in the light of cause and effect: but we should, in associating the cause and effect of an intermediate cause, pass over the third side of an intellectual triangle of which at present we can only pass over the other two.

The multiple point, the cusp, and the isolated point, must be looked for only in places where $\phi_x = 0$, $\phi_y = 0$, or where both the subordinates meet the curve and each other. The following proof applies with ease to a multiple point of real branches, a cusp which is not in any other branch, and an isolated point which is not also on a real branch of the curve. At a multiple point, either co-ordinate being constant, the other may pass from $\phi = 0$ to $\phi = 0$, infinitely near the multiple point, and during this passage ϕ_x and ϕ_y change sign; hence $\phi_x = 0$ and $\phi_y = 0$ at the multiple point. The same of any cusp near which are small deflexions parallel to both axes. But where such deflexion can only be parallel to one axis, as where y is a maximum or a minimum at a cusp* of contrary flexures, we see that points on different sides of the cusp are similar, while y'' has one sign for both, whence ϕ_y has one sign. Nevertheless, ϕ_y vanishes at the cusp, since y'' is infinite, and $\phi_y{}^2\phi_{xx} - \dots$ is not. As to the isolated†, or conjugate point, we know that $\phi\phi_y$ changes sign in passing *through* the point, while ϕ does not change sign, as may be proved by going *round* it: hence ϕ_y changes sign at the point, or $\phi_y = 0$ passes through it. And similarly for ϕ_x.

A cusp is an evanescent loop; and an isolated point is an evanescent oval, or bounded portion of the curve. In one view, it is so, or is not so, at our pleasure. It is always possible to make $\phi(x, y)$ a particular case of $\psi(x, y, a)$, say when $a = 0$, in such manner that $\psi(x, y, 0)$ and $\psi(x, y, da)$ shall both have cusps, or both isolated points. But, looking at the fact that in our present subject we are only concerned with the relations between $\phi = 0$ and its subordinates $\phi_x = 0$ $\phi_y = 0$, it seems obvious that we are only concerned with $\phi = 0$ as being one of the family $\phi(x, y) = c$, belonging to the pair of subordinates $\phi_x = 0$ $\phi_y = 0$, which are common to all. This being granted, the assertion follows. Let $\phi = 0$ have an isolated point (x, y). Let c be infinitely small: then, since $\phi(x, y)$ is always real, we cannot proceed in any one direction through points $(x + dx, y + dy)$ without $\phi(x + dx, y + dy)$ passing through c in value. If the isolated point be not on any real branch, so that ϕ has one sign in all parts of the neighbourhood, it is clear that the isolated point marks the passage from a real oval to an imaginary branch, as c changes sign through 0. But if the isolated point be on a real branch, so that ϕ has different signs in different regions of the neighbourhood, it is indicated that $\phi = c$ has one or more ovals (according to the branches which pass through the point) when c has either sign. And similar reasoning for the evanescent loop of the cusp.

It has been asserted that if the isolated point were always an evanescent oval, y' would admit an infinite number of values. Now first, this does not happen explicitly when we contrive a conjugate or isolated point which *is* an evanescent oval; as when we obtain $y = x\sqrt{(x^2 - 1)}$ from $a = 0$ in $y = \sqrt{\{(x - a)(x - 2a)(x^2 - 1)\}}$. Next, I have no doubt that y' *has* an infinite number of values; not *equivocal* values, between which there is choice, but *simultaneous* values, between which there is no choice: the sense being that in which $1 - 1 + 1 - \ldots$ is 0 and 1. And here the principle of mean values, on which my reliance increases with every new case in which I try it, may place the meaning of y' at an isolated point in satisfactory connexion with other results, with which it must stand or fall. I may perhaps be able to meet the powerful objection which can be urged against any application of this principle taken alone, in a future communication. From examples, it seems that $y' = a + b\sqrt{-1}$, at an isolated point, means that $\int \tan\theta d\theta : \int d\theta = a + b\sqrt{-1}$, where θ is the angle made by the tangent of the infinitely small oval with the axis of x. The integrals are taken through all the revolution of the tangent, with different signs for the contrary revolutions, if any. And it seems further that when $b\sqrt{-1} = 0$, either (perhaps always both) of two things may happen. Either the passage of $\tan\theta$ through ∞ may be made the same number of times in each direction of revolution, so that the discontinuous constants put together give $k(m\pi\sqrt{-1} - m\pi\sqrt{-1})$, or 0: or else axes perpendicular to some line ($y = ax$) are infinitely small, so that y' is always infinitely near to a, except at points which, compared with the extent of the oval, are infinitely near those at which $y' = -a^{-1}$. That an isolated point may lie on a real branch, was noticed by D'Alembert: but the remark is seldom repeated in modern writings: D. F. Gregory alludes to it in his *Examples*.

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* A cusp may be said to be of similar or contrary flexures, according as the tangent does not or does divide the branches. I have long ceased trying to keep in memory which is the cusp of the *first* kind, and which of the *second* kind. Nor do I admit the uncouth terms *ceratoid* and *ramphoid*; for the cusp called ramphoid is as much like the point of a horn as that of a beak, and the cusp called ceratoid is not the point of a horn at all, but that of a spear.

† The idea implied in the word *conjugate* is that the point is algebraically conjoined to the curve, but not geometrically part of it. In geometrical algebra, the word *isolated* is preferable, since, by definition, the point is really on, or part of, the curve. Nor is it any objection that an isolated point may be on a real branch of the curve; we mean that a point on a real branch may be the only real point of an imaginary branch.

Two methods have been adopted of treating the inquiry into the singular points of curves. The first, of which the fullest development ever given is in the well-known and highly-valued, but (as I shall show) little read work of Cramer '... *Analyse des Lignes Courbes Algébriques*,' Geneva, 1750, 4to, reduces the question to dependence on the theory of equations, using differential coefficients, if at all, only as convenient aids to development. The second endeavours to present canonical criteria, expressed in terms of differential coefficients. This second method never was successful, and never will be, in the full determination of points of singular curvature. No given number of differentiations can in all cases discriminate, for instance, a cusp from a multiple point.

The real dependence of the second method upon the theory of equations, when the method is carried far enough, lies hid under an attempt to make the investigation one of *order*, examining first the dependence on ϕ_x and ϕ_y, then that on ϕ_{xx}, ϕ_{xy}, and ϕ_{yy}, and so on. When the whole progression is looked into at once, the reduction to the theory of equations is easily seen.

Let (x, y) and $(x + dx, y + dy)$ be points on $\phi(x, y) = 0$, and let $dy = pdx$. We have then

$$\phi(x + dx, y) + \phi_y(x + dx, y) . pdx + \phi_{yy}(x + dx, y)\frac{p^2dx^2}{2} + \ldots = 0.$$

Expand in powers of dx: in each expansion preserve the lowest power of dx whose coefficient has value, to determine the limit of p, or y' at (x, y); and also the lowest but one, to aid in examining the neighbourhood of (x, y). Suppose we thus obtain,

$$A\,dx^a + A'dx^{a'} + (B\,dx^b + B'dx^{b'})\,p + (C\,dx^c + C'dx^{c'})\,p^2 + \ldots = 0,$$

where a is not < 1, since $\phi(x, y) = 0$; b is not < 1; c is not < 2; and so on. Let the lowest power of dx in the whole expansion occur for the first time with p^k, and for the last time with p^l: this lowest power must have one of the unaccented exponents. We then determine the *finite* values of y', which are the values of p in $Kp^k + \ldots + Lp^l = 0$. But p may have *infinite* values. Suppose, for instance, that the lowest power of dx is dx^7, occurring for the last time in the coefficient of p^4. It is only by casual vanishing of coefficients that p^5, p^6, and p^7 are wanting in the final equation: the powers of dx are not necessarily above the 7th, and their effects do not necessarily vanish in the final result, until we have passed p^7. The final equation, then, is $Kp^k + \ldots + Lp^4 + 0\,.\,p^5 + 0\,.\,p^6 + 0\,.\,p^7 = 0$. In confirmation, observe that if the lowest terms be of the seventh order, and Ldx^3dy^4 be the last of them, then, dx being qdy, Lq^3 will be the *lowest* term of the equation for determining y'^{-1}. Hence, the lowest power of dx determines the number of values of y': the final equation determines the number which are not infinite.

If the final equation, which write $Kp^k + \ldots + Lp^l + \Sigma\,.\,0p^m = 0$, have no equal roots whatsoever—and infinites here count as equals, so that Σ must not have more than one term—or if all equal roots be sets of imaginary pairs, the problem is solved. If μ of the roots be real, and μ' imaginary, the point (x, y) is in μ ways on a real branch of the curve, and in $\frac{1}{2}\mu'$ ways an isolated point. There is no need to introduce the terms $A'dx^{a'}$, &c.: for infinitely small alterations in the coefficients of an equation do not affect the character of its *single* roots. But if two or more roots be equal, the effect of the second terms may be of either of a threefold kind. First, a pair of roots may become real and unequal, for both signs of dx: this gives a pair of branches with contact passing through (x, y). Secondly, a pair of roots may become real and unequal for one sign of dx, and imaginary for the other sign: this gives a cusp. Thirdly, a pair of roots may become imaginary for both signs of dx: this gives an isolated point with a real tangent.

The curve $\phi(x, y) = c$, c being infinitely small, adjacent to the point (x, y), is defined by $Kdx^{h-k}dy^k + \ldots + Ldx^{h-l}dy^l = c$, dx^h being the lowest power of dx already mentioned. Let $c : dx^h = H$, where H may be any quantity whatever. Then p, derived from $Kp^k + \ldots + Lp^l = H$, indicates all the directions in which $\phi(x, y) = c$ has branches cutting the vertical at a distance dx from (x, y). Let $Kp^k + \ldots + Lp^l = fp$. Then for every real root of $f'p = 0$ for which $fp\,.\,f''p$ is positive, there are values of H limited or unlimited as the case may be, between which imaginary roots of $fp = 0$, have corresponding real roots in $fp - H = 0$. And hence may be detected the character of the parts of $\phi(x, y) = c$, which vanish in the isolated points of $\phi(x, y) = 0$. The case in which $fp = 0$ has equal roots may be similarly reasoned on.

It thus appears that the theory of equations is the proper instrument for considering the singular points of curves. A few months ago, I saw in the notes to the second edition of M. Serret's excellent *Algébre Supérieure*, an account of a process for eliminating between two equations, attributed to Mr Minding, with a reference to the sixth volume of Liouville's journal, into which it was translated from Crelle's journal. This process of elimination depends upon an elegant and powerful theorem, which is demonstrated at full length by M. Serret, without the slightest hint of any other author except Mr Minding. Perceiving the importance of this theorem, and also some simplification of the demonstration, I wrote a short account of it for the *Quarterly Journal of Mathematics*, giving the authorship to Mr Minding. Happening to fall in with the volume of Liouville's journal above cited, I was surprised to find that no such theorem is either enunciated or proved, but that, on the point of which it treats, reference is made to Lacroix, Vol I. p. 223. This reference shews nothing to the purpose; but taking it as a hint to examine the whole volume, I found at page 102 of the introduction, a somewhat prolix and uninviting account of the very theorem, which it appears was given by Lagrange in the Berlin Memoirs for 1776. How completely it has dropped* out of sight will appear from the uses which can be made of it, and which, it seems to me, must have been most obvious to any writer on curves, or on the theory of equations, who had really obtained possession of it. The theorem determines the initial and terminal branches of a curve, and the character of its singular points, with much more ease and power than any method given by elementary writers. It also determines the effect of infinitely small changes in the coefficients of an equation upon its equal roots. But there is something still more remarkable about its history: it was suggested to Lagrange by a method of Newton which is now totally lost sight of, though, as will presently appear, it is difficult to see how any reader of Taylor, Stirling, or Cramer, could have passed it over.

Mr Serret has only adverted to the mode of finding descending series for the roots; but the theorem applies equally to ascending series. In both the forms, it is as follows;—Given functions of x, such as A, of the form $x^a(\mathrm{a} + A')$, where A' vanishes with x, or vanishes with x^{-1}, and a is independent of x, and finite† both ways: this is the definition of a function of the lower degree a, or of the higher degree a. Let there be an equation

$$Ay^\alpha + By^\beta + \ldots + Ky^\kappa + \ldots + Ty^\tau + \ldots + Zy^\zeta = 0$$

where A, B, &c. are functions of x of the lower degrees a, b, &c., and α, β, &c. ascend; or else A, B, &c. are of the higher degrees a, b, &c., and α, β, &c. descend. It is required to determine the degrees of the values of y in terms of x. The degrees a, b, &c., α, β, &c. may be integer or fractional, positive or negative: the only difference made is that if α, β, &c. differ by fractions, it will require further process to determine the number of roots of each kind. Let $y = x^v(u + U)$, and write $x^a(\mathrm{a} + A')$, &c. for A, &c. which gives

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* I dare venture the supposition that Lagrange himself had forgotten it before the *Prairial* of *An* V. It must be remembered that every direct use of the *théorie des suites* was desirable in the differential calculus presented as the *Théorie des Fonctions*: the direct treatment of singular points by series alone would have been no mean triumph of the principle. The reader who refers to this work (first edition, pp. 130—133; second edition, pp. 181—185) will see that Lagrange reasons only on the form $y = fx$, omitting all consideration of $\phi(x, y) = 0$, and then abandons the singular points under reference to Cramer: he himself, twenty years before, having given the means of finding a mode of development from the implicit equation of a far more easy character than that which he had just applied to the explicit equation.

Lagrange's paper of 1776 is entitled *Sur l'usage des fractions continues dans le calcul intégral*. The theorem is constructed upon an application to differential equations only, as a means of obtaining continued fractions. Lacroix presents it applied to the implicit form $\phi(x, y) = 0$.

† This short phrase, which is much wanted, serves to indicate that the quantity is neither infinitely small nor infinitely great. When writers speak of a finite quantity, some mean only that it is not infinite, others that it is neither infinite nor nothing; and I think the first meaning is the more common. Thus 'a point at a finite distance from the origin' usually includes the origin itself. And it ought to be so. For *finite* and *infinite* ought to be contraries: and *infinite*, used alone, always means infinitely *great*.

$$(\mathrm{a} + A)(u + U)^{\alpha}x^{a+\alpha r} + \ldots + (\mathrm{k} + K)(u + U)^{\kappa}x^{k+\kappa r} + \ldots + (\mathrm{t} + T)(u + U)^{\tau}x^{t+\tau r} + \ldots = 0.$$

We have then to find in how many and what ways r can be so taken that two or more of the set $a + \alpha r$, $b + \beta r$, &c. shall be equal, and lower than all the rest, or higher than all the rest, according as we are seeking lower dimensions or higher dimensions. These cases both come under one rule, the difference of arrangement making all the difference required. For the demonstration I refer to the paper in the *Quarterly Journal* (No. I. p. 1) already cited. Take the succession $\begin{matrix} a & b \\ \alpha & \beta \end{matrix}$ ...... and from it form the fractions

$$\frac{b-a}{\alpha-\beta}\ \frac{c-a}{\alpha-\gamma} \ldots\ldots \frac{k-a}{\alpha-\kappa} \ldots\ldots \frac{t-a}{\alpha-\tau} \ldots\ldots \frac{z-a}{\alpha-\zeta},$$

take the greatest (the furthest from $-\infty$) or if several be equal and greater than all the rest, take the last of the equal maxima. Let it be $\rho = (k - a) : (\alpha - \kappa)$, and, to shew the case of equal maxima, say that $\rho = (e - a) : (\alpha - \epsilon)$ also. Then there are roots of the form $y = x^{\rho}(u + U)$, and u is derived from $\mathrm{a}u^{\alpha} + \mathrm{e}u^{\epsilon} + \mathrm{k}u^{\kappa} = 0$, every value of u finite both ways giving one form, and the final equation being always derived from the first term of the succession, with all the terms which contribute to the equal maxima. The next step is, to treat the succession $\begin{matrix} k \ldots t \ldots \\ \kappa \ldots \tau \ldots \end{matrix}$ in the same way; and so on until the last maximum contains ζ. If all the set α, β, &c. be integer, the number of solutions is $\alpha - \zeta$ or $\zeta - \alpha$, the whole number of roots which are not in $y = 0$: the forms obtained having their multiple character duly augmented, if a, b, &c., or any of them, be fractional. And the successive values obtained for r are always descending in algebraic magnitude.

Having thus obtained $y = x^r(u + U)$, substitution gives an equation for determining U, which may thus be found as $x^{r_{,}}(u_{,} + U_{,})$, and so on. Confining ourselves to integer data, a, α, &c. I may remark that, when r is fractional, x^r must be interpreted univocally, the equivocality of u supplying the requisite multiplicity of forms. For if, μ being a proper root of 1, we use $\mu^r x^r$ for x^r, this is equivalent to assuming $y = (\mu x)^r (u + U)$, which substitutes $\mu^r u$ for u in the final equation, and all the combinations are but repetitions of those which we obtain from $y = x^r(u + U)$. Next, since there can only be $\zeta - \alpha$ or $\alpha - \zeta$ values of y, the process for determining $U_{,}$, $U_{,,}$, &c. is bound to shew this: and the mode in which it succeeds is an instructive example of the use of the theorem.

Having obtained a value of r with, say $\mathrm{a}u^{\alpha} + \ldots + \mathrm{k}u^{\kappa} = 0$ for determining u, write $x^r u$ in place of y, and collecting powers of x arranged according to the problem, let $Ay^a + \ldots$ become $\mu u \,.\, x^m + \nu u \,.\, x^n + \varpi u \,.\, x^p + \ldots$, where $\mu u = 0$ determines u.

The equation for U is found by substituting $u + U$ for u in the last. And m is $k + \kappa r$, where k and κ are integers from the data, while n, p, &c. are of the same form, with integers usually not found among the data. We have then, since $\mu u = 0$,

$$(\nu u \,.\, x^n + \varpi u \,.\, x^p + \ldots) + (\mu' u \,.\, x^m + \nu' u \,.\, x^n + \varpi' u \,.\, x^p + \ldots)\, U + (\mu'' u \,.\, x^m + \nu'' u \,.\, x^n + \varpi'' u \,.\, x^p + \ldots)\frac{u^2}{2}$$
$$+ \ldots = 0.$$

To fix our thoughts, let us suppose lower degree is in question. Let $m+$ signify 'm or more': thus the dimension of the first term is $n+$, which is n unless $\nu u = 0$, and is then p unless ϖu also $= 0$, &c. And first let $\mu u = 0$ have no other roots equal to the one selected for trial; that is, let $\mu' u$ have value. Remark that we must exclude negative and zero dimensions of U as irrelevant; preceding processes having obtained u only as derived from $U = 0$ when $x = 0$. Using the general theorem, we examine

$$\frac{(n+)-m}{1-0}\quad \frac{(n+)-(m+)}{2-0}\quad \frac{(n+)-(m+)}{3-0}\ \ldots\ldots$$

where $n+$ is of one meaning throughout. The greatest of these must be the first; there is then *one* root of the positive dimension $(n+)-m$ which, if fractional, is of the form $k+\kappa r$, so that its denominator is that of r or a factor of it. Beginning with m and 1, we find in the next succession of fractions nothing but zeros or negatives, hence the next form of U (and all which follow) must be rejected. The value of $r_{,}$ is then $(n+)-m$; $u_{,}$ is determined from

$$(\text{the first having value of } \nu u,\ \varpi u, \ \&\text{c.}) + \mu' u \,.\, u_{,} = 0.$$

And in the subsequent processes, that is, in writing $x^{r_{,}} u_{,}$ for $U_{,}$, and then $u_{,} + U_{,}$ for $u_{,}$, &c. it will easily appear that because $\mu' u$ has value, and the first term of the first term vanishes, the dimension of the second term is always, throughout the processes, lower than that of the first, and at least as low as any one which follows. The results are then univocal throughout, if we remember that x^{r} and its integer powers are taken univocally, and we have a univocal series for y, belonging to the value of u chosen. Moreover, if u be real, so in this case are $u_{,}, u_{,,}, \ldots$, being obtained from equations of the first degree.

Next, let there be θ roots of equal value in $\mu u = 0$, one of which is taken. Striking out the terms which certainly vanish, we have, to determine U, denoting $1\,.\,2\,.\,3 \ldots a$ by $[a]$,

$$(\nu u \,.\, x^{n} + \ldots) + (\nu' u \,.\, x^{n} + \ldots)\, U + \ldots + (\nu^{(\theta-1)} u \,.\, x^{n} + \ldots)\frac{U^{\theta-1}}{[\theta-1]} + (\mu^{(\theta)} u \,.\, x^{m} + \ldots)\frac{U^{\theta}}{[\theta]}$$
$$+ (\mu^{(\theta+1)} u \,.\, x^{m} + \ldots)\frac{U^{\theta+1}}{[\theta+1]} + \ldots = 0,$$

where $\mu^{(\theta)} u$ does not vanish. We proceed to examine the fractions

$$\frac{(n+)-(n+)}{1-0},\ \frac{(n+)-(n+)}{2-0} \ldots \frac{(n+)-m}{\theta-0},\ \frac{(n+)-(m+)}{(\theta+1)-0},\ \ldots\ldots$$

in which the first $(n+)$ in the numerators means the same throughout.

Here, since $(n+)-(m+)$ is not greater than $(n+)-m$, and since the denominators increase, there must be a maximum at or before $\{(n+)-m\} : \theta$. Should it happen before, a repetition of the process produces the same result, until a maximum at last has $(n+)-m$ in the numerator, though not the same $(n+)$ with which we began. And by this time we have obtained θ solutions, all of positive degrees: for, in the general theorem, by the time the maximum involves the use of k in the numerator, the number of solutions, after giving each result due multiplicity, amounts to $\kappa - a$.

In the next process, the numerators are all of the form $m - (m+)$, zero or negative: the next solution, and all which follow, are therefore inadmissible. We have then θ forms of U, and no more. The values of $u_{,}$ real or imaginary, are obtained from one or more equations, whose united degrees amount to θ. And the fact of $\mu^{(\theta)}u$ having value, as before, determines $U_{,}$, &c. univocally, unless indeed some of the values of $u_{,}$ should be equal, in which case the phenomena just described are repeated. This appearance of equal roots may continue to any extent: thus, having only two values of u equal, the two values of $u_{,}$ may be equal, and so on, say up to $u_{,\nu}$, the two values of which are different; giving for y a double form such as

$$y = x^{r}\left\{u + x^{r_{,}}\left\{u_{,} + x^{r_{,,}}\left\{u_{,,} + x^{r_{,,,}}\left\{u_{,,,} + \left\{\begin{matrix} x^{r_{,\nu}}\left\{u_{,\nu} + x^{r_{\nu}}\left\{u_{\nu} + \dots\right\}\right\} \\ x^{r_{,\nu}}\left\{u_{,\nu} + x^{r_{\nu}}\left\{u_{\nu} + \dots\right\}\right\} \end{matrix}\right\}\right\}\right\}\right\}\right\}$$

in which the illustration hardly requires different symbols for the two values of $u_{,\nu}$, u_{ν}, &c.

Before applying this to an example, for the determination of singular points, I take the explicit problem of infinitely small alteration in the coefficients of an equation having equal roots. Let $A_n y^n + \dots + A_0 = 0$ be an equation having θ equal roots, with a for their common value; and let us consider

$$(A_n + B_n\,dx^{b_n})\,y^n + (A_{n-1} + B_{n-1}\,dx^{b_{n-1}})\,y^{n-1} + \dots + (A_0 + B_0\,dx^{b_0}) = 0.$$

Substitution of $y + a$ for y will convert this into $C_n y^n + \dots + C_0 = 0$ where $C_{\theta-1}, \dots C_0$ are infinitesimals, and possibly some of the others, but not C_θ. Lagrange's theorem of 1776 gives an account of the roots which replace the equal roots, and an approximation to their values, by expression of y in ascending powers of dx. For example, suppose the only change made is that of A_0 into $A_0 + dx$. The transformed equation is $C_n x^n + \dots + C_\theta x^\theta + dx = 0$; where $C_n = A_n$, C_θ does not vanish, though intermediates may vanish. The θ roots under inquiry are of the form $y = (dx)^{\frac{1}{\theta}}(u + U)$, where $C_\theta u^\theta + 1 = 0$: and not more than two can be real. As another case, let the two last terms be changed into $A_1 + B_1\,dx^{b_1}$ and $A_0 + B_0\,dx^{b_0}$, so that the transformed equation is

$$C_n y^n + C_{n-1} y^{n-1} + \dots + C_\theta y^\theta + B_1\,dx^{b_1} y + (a\,B_1\,dx^{b_1} + B_0\,dx^{b_0}) = 0.$$

The results are as follows:—

(1) $b_0 < b_1$, $\theta(b_0 - b_1) > b_0$. One real root $y = dx^{b_0 - b_1}(u + U)$, where $B_0 + B_1 u = 0$. And $\theta - 1$ roots of the form $y = dx^{b_1 : (\theta-1)}(u + U)$, where $B_1 + C_\theta u^{\theta-1} = 0$. Not more than three of the altered roots can be real.

(2) $b_0 < b_1$, $\theta(b_0 - b_1) = b_0$. There are θ roots of the form $y = dx^{b_0 - b_1}(u + U)$, where $B_0 + B_1 u + C_\theta u^\theta = 0$. Not more than two can be real.

(3) $b_0 < b_1$, $\theta(b_0 - b_1) < b_0$. There are θ roots of the form $dx^{b_0 : \theta}(u + U)$, where $B_0 + C_\theta u^\theta = 0$. Not more than two can be real.

(4) $b_0 =$ or $> b_1$. There are θ roots of the form $y = dx^{b_1 : \theta}(u + U)$, where $aB_1 + B_0 + C_\theta u^\theta = 0$, or $a B_1 + C_\theta u^\theta = 0$. Not more than two can be real.

When singular points are to be determined, we first examine $\phi(x, y) = 0$, $\phi_x = 0$, $\phi_y = 0$, and having ascertained the simultaneous solutions, if any, of the three, it will generally be convenient to remove the origin to the point under examination. Suppose the resulting equation to be

$$x^{12} + x^{14} + x^{11}y - x^8y^2 + 2x^7y^3 - x^4y^4 + y^6 - 3xy^9 + x^{14}y^{13} = 0,$$

the character of which is to be determined both for infinitely small and infinitely great values of x. Beginning with the former case, we write down the lower degrees as to x and y, and form the first succession of fractions, as follows:—

$$\begin{array}{cccccccc} 12 & 11 & 8 & 7 & 4 & 0 & 1 & 14 \\ 0 & 1 & 2 & 3 & 4 & 6 & 9 & 13 \\ & \frac{1}{1} & \frac{4}{2} & \frac{5}{3} & \frac{8}{4} & \frac{12}{6} & \frac{11}{9} & \frac{-2}{13}. \end{array}$$

Three equal maxima, 12 : 6 the last. Six roots of the form $y = x^2(u + U)$, where u is to be determined from $1 - u^2 - u^4 + u^6 = 0$, of which the roots are $+1$, $+1$, -1, -1, $\pm\sqrt{-1}$. Proceeding from 0 and 6, we have $-1 : 3$ and $-14 : 7$; so that there are three solutions of the form $y = x^{-\frac{1}{3}}(u + U)$, where u is determined by $1 - 3u^3 = 0$.

Lastly, beginning with 1 and 9, we have $-13 : 4$, giving four solutions of the form $y = x^{-\frac{13}{4}}(u + U)$, where $-3 + u^4 = 0$.

To examine the origin as a sextuple point, write x^2u for y, and reject the factor x^{12}: we have then

$$1 + x^2 + xu - u^2 + 2xu^3 - u^4 + u^6 - 3x^7u^9 + x^{28}u^{13}.$$

For u write $u + U$, preserve only the coefficients of U^0, U^1, U^2, and only the lower dimensions of these, which gives

$$(3x + \dots) + (7x + \dots)\,U + (16 + \dots)\,\frac{U^2}{2} + \dots = 0,$$

when $u = +1$. The fractions are 0 : 1 and 1 : 2, and there are two solutions of the form $U = x^{\frac{1}{2}}(u_1 + U_1)$, where $3 + 8u_1^2 = 0$. This *pair of branches* gives only an isolated point, the vertex of a common parabola. But $u = -1$ writes $(-3x + \dots)$ for $(3x + \dots)$, determines u_1 from $-3 + 8u_1^2$, and gives two real branches of the form $U = x^{\frac{1}{2}}(u_1 + U_1)$, or $y = x^2\{1 + x^{\frac{1}{2}}(u_1 + U_1)\}$. These two branches form a cusp with similar flexures, and the axis of x for a tangent. Lastly, $y = \pm\sqrt{-1}\,.\,x^2$ gives an isolated point. The sextuple point consists of two isolated points and a cusp. The axis of y is an asymptote to three branches, two on the positive side, and one on the negative.

To find the ultimate character of the branches, we proceed with higher dimensions, and examine

$$\begin{array}{cccccccc} 14 & 1 & 0 & 4 & 7 & 8 & 11 & 14 \\ 13 & 9 & 6 & 4 & 3 & 2 & 1 & 0 \end{array}$$

which shews at once 13 solutions of the form $y = x^0(u + U)$ where $1 + u^{13} = 0$. There is then one real case, approaching to the asymptote $y = -1$, consisting of two infinite branches approaching the asymptote on opposite sides. This will arise from the solution which begins as $y = x^{-\frac{1}{3}}(u + U)$; the other infinite branches will terminate at finite values of x, and the branches which begin at a cusp will give a bounded figure.

The two curves $\phi(x, y) = 0$, $\phi(x^{-1}, y^{-1}) = 0$, the latter equation being cleared of fractions, have the final branches of either connected with the initial branches of the other, in a manner which the preceding theorem readily points out. This connexion, when its study shall be called for, will elucidate a point to which little attention has been given, the relation of 0 to $0 . \sqrt{-1}$, and of ∞ to $\infty \sqrt{-1}$. If $x = 0$ give $y = 0 + 0 . \sqrt{-1}$, we admit the origin as an isolated point. But if $x = \infty$ give $y = \infty + \infty \sqrt{-1}$, we do not admit a real branch at an infinite distance: when an imaginary branch terminates in an isolated point at the origin, we admit the point; but in the reciprocal curve we do not admit that imaginary branches begin to be real at $x = \infty$. For this I hold we have full reason in the true meaning of ∞, as used, which is the reciprocal of an infinitely small quantity, and not of 0. The *infinite among infinites* which is properly represented by $\frac{1}{0}$ is not yet a recognised notion, as distinguished from any one of the successive *status* through which it is approached.

This simple, powerful, and fundamental theorem seems hitherto to have been eliminated between the theory of equations and the differential calculus: it belongs to the first, but elementary writers, without whose help no theorem can obtain due notoriety, have referred the points in which it is most useful to the second. That it has fallen quite out of notice might be shewn[*] in many ways. But, remarkable as this may be, it is still more remarkable that Newton's *parallelogram*, or method of *co-ordinated exponents*, as I shall call it, has still more completely fallen into oblivion. This method has not dropped because it was contained in a part of Newton's writings little known or seldom cited; for it occupies a prominent place in the second epistle to Oldenburgh. Any one who has turned[†] over the pages of that memorable letter, while passing from one of the anagrams to the other, must have been struck by a couple of diagrams, side by side, the one containing terms of the form $x^m y^n$ arranged in squares placed in rank and file, the other exhibiting stars which, had there been but one more, might have passed for a representation of the Great Bear, very badly figured. Had this epistle ever been read by any one of our time whose mind was not full of the controversy on fluxions, the method which I shall proceed to develope would have been seen, briefly but clearly indicated in its leading step. The cognate application to expansion from differential equations appears in those letters to Wallis of 1692, which Wallis incorporated in his second volume of works: and also, but much less clearly, in the *Quadratura Curvarum*. Stirling, in his *Lineæ Tertii Ordinis*, &c. (1617) has given a good and full account of the method, "quæ, quamvis a plurimis qui eam haud intellexerunt, ut Methodus Mechanica diu neglecta jacuit, est omnium quam quis excogitare potest generalissima et elegantissima." In this opinion I fully coincide, even up to *potest*. Taylor, in his *Methodus Incrementorum* (1615) had given a very sufficient account of the application to differential equations. De Gua (as I learn from Cramer and Lacroix) adopted this method in his *Usage de l'Analyse de Descartes*. John Stewart explained it fully* in his commentary on the *Analysis per Æquationes*, &c. And lastly, Cramer has made it the leading method of his well-known work on curve lines, in which those who have searched for examples or classifications have probably passed it over as some fanciful form of a tentative method. But all, from Newton to Cramer, have rather dwelt upon the meaning of the separate sides of the polygon than taken the polygon, as a whole, in its relation to the equation, as the containing form of all its solutions: to this point Stewart comes far the nearest. I cannot doubt that both Newton's method, and Lagrange's arithmetical translation of it, will henceforward find that place in elementary systems which they ought long ago to have occupied.

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[*] For instance, writers developing y in descending powers of x, from $\phi(x, y) = 0$, an algebraical equation, almost universally obtain only the developments which give the rectilinear asymptotes of the curve. M. Serret himself has got no further in his *first* edition: and he is a writer who shews abundant acquaintance with his predecessors. Again, D. F. Gregory, a real reader of the older works, and especially of Lagrange's Berlin memoirs, had no knowledge of this theorem, though well prepared to see its use, if it had ever caught his eye: for he abandons the differential formulæ, when the point is more than double, as "too complicated to be of much use." But the following is still more remarkable. Gregory prefers Stirling's method of finding asymptotes by expanding y in descending powers of x, and cites the *Lineæ*, &c. p. 48. The very first example which Stirling gives obliges him to refer back to p. 18, in which the matter of the reference is a use of co-ordinated exponents. But Gregory makes no allusion to any method of expansion as found in Stirling, though he appears, in this and other places, to be well acquainted with the *Lineæ*, &c. This is the only proof I have found, in no small amount of search, that any writer of our own day has, while thinking of curves, cast his eye upon any case of co-ordinated exponents.

[†] Hardly anything has been done with these letters for more than a century, except as to the passages which bear on the dispute about fluxions. Montucla, in his rapid enumeration of the contents of the second letter, includes "l'usage de son parallélogramme pour la résolution des équations." *Newton's parallelogram* has been mentioned by several writers; but I suspect, with Stirling, that the meaning of it has not been understood. Perhaps many have passed it over with the impression that it was some kind of *arithmetical triangle* which must needs have been superseded by some kind of *binomial theorem*. But, as it happens, the algebraical substitute commenced by several, and completed by Lagrange, has been equally neglected with its predecessor.

Lagrange himself did not refer to Newton, but to Taylor and Stirling, of whom he says "...la méthode du premier demande une espèce de construction géométrique, et...celle du second dépend du parallélogramme de Newton, et par conséquent ne peut être régardée que comme une méthode mécanique." This is not a correct criticism: the *methods* of both, though founded on the geometrical idea, are algebraically applied, insomuch that to Lagrange himself was left only the digestion of the theorem which both Taylor and Stirling had used. I fully admit that the extrication of a theorem which has only existed in combination with other things is a great step, and one for which we have sometimes long to wait: but due acknowledgment must be made to those who furnished the combined material. Stirling (p. 22) does with $n-2$, $n+1$, and 1, and Taylor (p. 32) does with $\vartheta + 1$, $2\vartheta + \frac{1}{2}$, &c. precisely what Lagrange does with $m + na$, $m' + n'a$, &c. To Lagrange belongs that *canonical* exhibition, the want of which would have prevented Taylor or Stirling from treating a very complicated example without reference to Newton's parallelogram for mechanical help.

The method of *co-ordinated exponents* connects every equation of the form $\Sigma A x^m y^n = 0$, or which takes that form, *quam proximé*, when x is infinitely small, or infinitely great, with a convex rectilinear polygon, from one part of the contour of which we read the ultimate relations between y and x when x is infinitely small, and from the other part the ultimate relations when x is infinitely great.

Let the exponent of y in any one term be made an abscissa, and that of x an ordinate, so that every possible case of $x^m y^n$ has what we may call its exponent point (n, m). When none but integer exponents occur, the exponent points will all be found at the corners of squares of integer sides. Lay down all the exponent points belonging to the terms of an equation, and choose from among them the corners of a *convex* polygon which contains all the rest within it: any side of that polygon being produced, all the points, except the two or more which lie in that side, are either all above the line of the side, or all below it. To determine this containing polygon, take the lowest[*] value of n, and the lowest corresponding value of m, and, drawing lines from (n, m) thus taken to all the other points, proceed upon the *lowest* of those lines, that is, upon the line of greatest slope downwards, or of least slope upwards. Thus is determined the first side of the lower part of the polygon. Proceed upon the side determined to its end, that is, to the last of the points, if more than two, which lie upon it. From this last point repeat the process, neglecting all the points which have been passed, that is, through whose values of n we have passed. We thus get a second line, and so on: in time we must arrive at a point having the greatest value of n, and the lowest value of m which goes with it. We have thus completed the lower contour of the polygon. Proceeding from the last point obtained, which has the highest value of n, draw a vertical from the lowest value of m to the highest, if more than one value of m belong to this highest value of n: this vertical is one of the sides of the polygon. Proceed from this highest point backwards, and draw lines to all the others, retaining only the line of greatest slope upwards, or of least slope downwards: reverse, in fact, the preceding process. We thus obtain the higher contour of the polygon, and finally reach the point we began from, reaching it at last by a vertical side downwards, if necessary.

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* 'Sir Isaac Newton's two Treatises...explained....By John Stewart, A.M. Professor of Mathematicks in the Marischal College and University of Aberdeen.' London, 1745, 4to. This was an invasion from Scotland of a very serious character: it contains the *Quadratura Curvarum* and *Analysis per Æquationes*, &c. in English; 55 pages of Newton, and 424 pages of explanation. It was published by a 'Society for the encouragement of Learning' (on which see Nichols's *Anecdotes*, Vol. II. p. 90), to which we owe both Tanner's *Notitia Monastica* and his *Bibliotheca*. In the *Analysis*, &c. Newton shews a decided attempt at what he afterwards succeeded in by his *parallelogram*: of which attempt Stewart takes advantage to introduce a very full account of the subsequent methods. He has given the theorem, which I call Lagrange's, so nearly, that nothing but a repetition is wanted: he obtains only the *first* of the maxima from which dimensions are derived. For the lower dimension, the equation being $x^\mu + \Sigma x^r y^s = 0$, he directs to "run over the several terms, and observe where the value of $\frac{\mu - r}{s}$ is greatest:" but he lays it down that only one dimension of solution of each kind can be found in this way; and when all are wanted, he has recourse to the polygon (pp. 428-436), under express declaration of the necessity of such a step. And from this, looking at the fact that Stewart was well acquainted with the earlier Newtonian school, I derive my final assurance that the *complete* theorem belongs to Lagrange.

The process of Lagrange's theorem will be easily seen in the above. If (n, m) be the first point, and (n', m') another, the downward slope of the joining line is more or less according as $(m' - m) : (n' - n)$ is lower or higher, that is, according as $(m - m') : (n' - n)$ is higher or lower: and the rule for finding the greatest slope downwards, in proceeding forwards, is also the rule for finding the greatest slope upwards, in proceeding backwards.

The external points of the containing polygon, whether they terminate sides or lie within sides, are the exponent points of the *essential terms* of the equation; meaning, the terms on which the characters of the solutions of y in terms of x depend, or on which the initial and terminal characters of the branches depend. The lower contour of the polygon settles the expansions in ascending powers: the upper contour in descending powers. To shew this, let (n, m), (n', m'), (n'', m'') lie on a side of the lower contour: we can then take ρ so that $m + n\rho = m' + n'\rho = m'' + n''\rho$. Every other point is above the line passing through these three points: so that, (n''', m''') being any fourth point, $m''' + n'''\rho$ is higher than $m + n\rho$, &c. Consequently, $y = x^\rho (u + U)$ substituted in the equation, gives three terms of the degree $m + n\rho$, and all the others of higher degree. And similarly for any side of the upper contour.

Cramer (and the same may be said of his predecessors) does not grapple with the whole polygon, but takes one or another side, as wanted: the sides[†] being obtained tentatively by help of a ruler. This method pervades the whole of his book, and constitutes its great peculiarity. He himself feels that it is his main point. I almost suspect* him to have imagined that his analytical triangle and ruler have a power to which algebraical process cannot supply an equivalent: I can in no other way explain the question asked in the passage cited below, as coming from an algebraist accustomed to the *ignotum per ignotum*.

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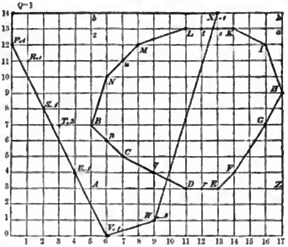
[*] By *lower* and *higher* I mean algebraically less and greater: thus -10 is lower than -1. It would perhaps be convenient if lower and higher were reserved for this purpose, and *greater* and *less* were allowed to retain their arithmetical meaning, which, for the most part, they do. Thus when we say, Let x be infinitely less than y, we never mean, Let $x = -\infty$. The words lower and higher might be used when we proceed from $-\infty$ to $+\infty$, while *previous* and *subsequent* might be used, as convenient, either when we proceed from $-\infty$ to $+\infty$, or from $+\infty$ to $-\infty$.

[†] This ruler was wanted because the squares, or *cases*, of Newton's parallelogram were supposed to be filled up with all



In the diagram, the points *PQRSTUVWX* are the exponent points in the equation previously discussed: the essential points are *PQSUVWX*. In the vertical side *PQ*, *P* counts on the lower contour, and *Q* on the higher.

Given the lower or higher solutions of an equation $\phi(x, y) = 0$, as to the character of their leading terms; that is, given r in $y = x^r(u + U)$, where U vanishes with x, or with x^{-1}; or given both;—required all conditions of compatibility, and the essential terms of the equation. A side of the polygon drawn from (n, m) to (n', m') denotes solutions of the degree $(m' - m) : (n - n')$, higher or lower, according as the side is on the higher or lower contour: this may be denoted by $-\Delta m : \Delta n$. Where integers are in question, the number of solutions is $n - n'$ or $n' - n$, whichever is positive. Hence sides which when produced, would cut a right-angled triangle out of the region in which m and n are positive, denote solutions of positive degree; other sides, negative. The degree must be given so that the denominator expresses the number of roots wanted: thus 12 roots of the degree $\frac{1}{2}$ must be demanded as of the degree $\frac{6}{12}$.

This being premised, let it be required that when x is infinitely small, we shall have 2, 4, 2, 1, 2, 1 branches of the dimensions 1, $\frac{1}{2}$, 0, -1, $-\frac{3}{2}$, -2. Write the dimensions down in descending order of magnitude, with the proper denominators:

$$\frac{2}{2}\quad \frac{2}{4}\quad \frac{0}{2}\quad \frac{-1}{1}\quad \frac{-3}{2}\quad \frac{-2}{1}.$$

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the varieties of $x^m y^n$. I suspect that this conversion of the plane of co-ordinates into a sort of abacus was one great cause of the gradual dismissal of the method. The last notice that I can find of it is in the *Encyclopédie Méthodique*, in which D'Alembert, who is of all writers the one to be relied upon for separating the kernel from the shell, requires the term $x^m y^n$ to be imagined at the centre of each square, and preserves the squares in which the terms are supposed to be written. Under such an arrangement, it would of course not be very easy to say, in all cases, whether the ruler would pass on one side or the other of the centre of a square.

[*] Quel moyen y aura-t-il donc pour discerner les plus grands termes d'une Equation, puisqu'il semble qu'on ne les peut reconnoitre sans savoir de quel Ordre est y par rapport à x, et qu'on ne peut découvrir ce rapport de y à x sans avoir séparé des autres les plus grands termes de l'équation? Quel fil nous conduira dans ce Labyrinthe? (p 152).

Begin from any point *B*, and taking the rule for the signs given above, shew the numerators and denominators in the vertical and horizontal projections of *BC*, *CD*, *DE*, *EF*, *FG*, *GH*. We are at liberty to add the points *p*, *q*, and *r*, which affect no dimension, but only the values of *u*. The essential terms of the equation, for satisfaction of these lower conditions, are the terms whose exponent points are *B*, *p*, *C*, *q*, *D*, *r*, *E*, *F*, *G*, *H*: but in estimating the terms, the origin should be removed to *A*, to avoid branches of the form $x = 0$, $y = 0$. We may add terms to the equation from any or all the points on or within *Bb*, *Hh*, produced upwards *ad infinitum*, without affecting the character of any root. We may now ask, in what manner are we limited as to the assignment of superior conditions. Write the desired dimensions, with properly modified denominators, also in descending order. Then the sums of the numerators, with the signs, must be equal in the two sets, and the sums of the denominators, taken positively, must be equal; and further, there must be ascent from the last of the inferiors to the first of the superiors, and also from the last of the superiors to the first of the inferiors. Thus 1, 2, 3, 3, 2, 1 solutions of the superior dimensions 3, $\frac{1}{2}$, 0, $-\frac{1}{3}$, -1, -3 will do; for, writing down the two sets together

$$\frac{2}{2}\ \ \frac{2}{4}\ \ \frac{0}{2}\ \ \frac{-1}{1}\ \ \frac{-3}{2}\ \ \frac{-2}{1}\ \ \Bigg|\ \ \frac{3}{1}\ \ \frac{1}{2}\ \ \frac{0}{3}\ \ \frac{-1}{3}\ \ \frac{-2}{2}\ \ \frac{-3}{1}$$

$$2+2+0-1-3-2 = 3+1+0-1-2-3,\ \ 2+4+2+1+2+1 = 1+2+3+3+2+1,\ \ \frac{3}{1} > \frac{-2}{1},\ \ \frac{2}{2} > \frac{-3}{1}.$$

Proceeding as before, the essential terms are derived from *H*, *I*, *K*, *L*, *M*, *N*; to which *s*, *t*, *u*, may be added. We may further add any terms derived from points inside the polygon; and the equation, and the fullest equation, which satisfies the conditions, is obtained by supplying any numerical coefficients we please which are finite both ways to the under-marked terms ($x_{,}^{6}$, &c.) of the following, and finite or zero coefficients to the rest:

$$\begin{aligned}&x_{,}^{6}y^{12} + (x_{,}^{4} + \dots + x_{,}^{9})\,y^{11} + (x^{3} + \dots + x^{9})\,y^{10} + (x_{,} + \dots + x_{,}^{10})\,y^{9}\\ &+ (x_{,}^{0} + \dots + x^{10})\,y^{8} + (x^{0} + \dots + x^{10})\,y^{7} + (x_{,}^{0} + \dots + x_{,}^{10})\,y^{6} + (x + \dots + x^{9})\,y^{5}\\ &+ (x_{,} + \dots + x^{9})\,y^{4} + (x^{2} + \dots + x_{,}^{9})\,y^{3} + (x_{,}^{2} + \dots + x^{8})\,y^{2} + (x^{3} + \dots + x_{,}^{7})\,y + x_{,}^{4} = 0.\end{aligned}$$

By inverting the polygon we obtain what we may properly call the *inverted* curve, in which the initial and terminal characters are interchanged: that is, from $\phi(x, y) = 0$, we get $\phi(x^{-1}, y^{-1}) = 0$; and by properly shifting the origin, we clear the second equation of fractions. This may all be done at once, by shifting the origin to *a*, and changing the order of signs on both axes. By taking *Z* for an origin, and reading *ZA* positively, we obtain $\phi(x, y^{-1}) = 0$, in which all the dimensions simply change sign. By taking *z* for an origin, and reading *zA* positively, we interchange initial and terminal characters, and also make all dimensions change sign; as in $\phi(x^{-1}, y) = 0$. Hence, by throwing one of two given singular points to an infinite distance, it might be settled whether the two can coexist in an algebraic curve.

By looking at the polygon from the axis of *m*, we see that if y have β root-functions of x of the degree $\alpha : \beta$, x has α root-functions of y of the degree $\beta : \alpha$. This seems to follow very easily from an obvious mode of algebraical proof, which, however, contains either much assumption on the inversion of expansions, or some prolixity of operation.

The values of the coefficients being written* at the exponent points, we see immediately how to write down the equations for the determination of u. This question remains;—The roots determined by the upper contour being identical, each to each, with those determined by the lower contour, how are we to equate them? Denote the roots by the sides of the polygon; thus we may say there is one root IH and three roots ML. If we can take off a slice from either end by a vertical diagonal joining two essential points, it *seems* clear that the roots thrown away above and below are identical. Bound the polygon by GI, instead of GH, HI: eleven roots are untouched in dimension, and in value of u: the root GH is identical with HI. Carry on this process, and we see that the roots in FG are those in KI, that in EF is that in sK, those in LS are those in DE, those in CD are those in uM and ML, pC is Nu, and BP is BN. This seems clear, and many simple instances confirm it so far as they go: but it is not proved, and certainly is not true in *reducible* functions. If the polygon admit of formation into two or more polygons by use of sides equal and parallel to the several sides (that is, if the system of balancing forces be composed of several such systems) we have the case which may give $\phi(x, y)$ reducible into real factors, and without which it cannot be so. Take forms of $\phi(x, y)$ from different polygons and multiply them together, the sides of the simple polygons will be intermixed in the compound polygon, and, since each simple polygon contains two sets of roots, identical each to each, the theorem cannot be universally true for *reducible* equations. And since an infinitely small alteration in a constant will change a reducible expression into an irreducible one, it seems proved that the theorem cannot be universally true even for irreducible expressions. The mode of pairing the roots of the upper and lower contour seems to depend upon all the roots, at least with exception of certain cases.

When we multiply two expressions together, essential terms in the product are products of essential terms in the factors; but the converse is not true. Having paper ruled both ways, the simplest plan of detecting the essential part of the product will be to write down *dimensions only* of the coefficients in x, in columns devoted to the powers of y, substituting some sign of blankness for missing powers, since 0 is appropriated to the function of no degree. The form of algebraic multiplication may then be used, the component processes being only additions [of exponents]. Essential terms only are retained in the factors, and in the partial multiplications only highest and lowest powers need be retained: while the final process is selection of lowest and highest in the columns. Thus, having to multiply together

$$x^3y^6 + (x + x^5)\,y^5 + (x^4 + x^7)\,y^4 + (x + x^2)\,y^3 + x^4y^2 + (x^2 + x^3)\,y + x + x^2$$

$$(x^2 + x^5)\,y^6 + (x^1 + x^8)\,y^5 + (x^3 + x^4)\,y^4 + (x^5 + x^7)\,y^3 + (x + x^6)\,y^2 + (x^7 + x^3)\,y + x + x^6$$

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* This is done in one of the diagrams. If the coefficients be thus written, the whole equation is written down, and it would seem that, for many purposes, there is no way so good of writing down an equation. Nothing is wanted but a supply of paper ruled both ways. And here it must be remarked that for all matters connected with intersections, angular gradations (not magnitudes) and general figure of curves, the paper need not be ruled in squares: equal rectangles, or even equal oblique parallelograms, will do as well.

we pick out the essential terms by help of the polygons, and proceeding as above directed, we have the following;

3	1+5	7	1+7	–	–	1+2						
2+5	8	–	9	1	–	1+6						
5+8	3+10	9+12	3+12	–	–	3+7						
	11	9+13	15	9+15	–	–	9+10					
			12	10+14	16	10+16	–	–	10+11			
				3	2+6	8	2+8	–	–	2+3		
						4+9	2+11	8+13	2+13	–	–	2+8
5+8	3+11	9+13	3+15	3+15	2+16	3+16	2+11	8+13	2+13	2+3	–	2+8

Reckoning from the lowest left-hand corner, the sides of the first polygon may be signified by *A* (horizontal), *B*, *C* + *D* (in one straight line), *E*, *F*, *G* (vertical): and those of the second by I (horizontal), II, III (vertical), IV, V, VI, VII (vertical). The polygon belonging to the product has then 14 sides as follows;—*A* + I (in one straight line), II, *B*, III, IV, (*C* + *D*), V, *E*, VI, *F*, (VII + *G*). This order is plainly determined by the slopes of the sides. Through one point draw lines equal and parallel to all the sides of both polygons in the directions in which we pass over the several sides in making the circuits: the order of intermixture thus established shews the order of the sides in the compound polygon. This would be, on ruled paper, the easiest way of making the multiplication. Also, by taking note of the projections on both axes, in the *pencil* of the polygon, we may facilitate the inquiry whether a given expression is reducible: if it be so, the pencil can be divided into two systems of rays, in each of which the sum of both projections is zero. It depends on the coefficients whether the reduction can be completed: but in every case in which the polygon is reducible into other polygons, there must be a certain reducibility of form in the function, and a certain divarication of the roots, which does not belong to expressions in general.

Returning to the original question, if in the product we destroy the highest term by introduction of a coefficient 0, we lose one root *B* on the lower contour, and one root IV on the higher: these are certainly not the same.

In treating of the roots of an equation, of a degree higher than the fourth, whatever we may think probable, we have no right absolutely to assume that the roots *formulize.* My meaning is as follows;—Let there be an equation of the second degree, one coefficient of which is *a*, and the roots *m* and *n*. Alter *a* into *A*, which changes the roots, say, into *M* and *N*. We can lay our hand on one of the two, *M* or *N*, and say, this is the root into which change of *a* into *A* has converted *m*. But if we pretend to say as much, or to suppose as much must be predicable, in the case of an equation of the fifth degree, we assume the qualities of formular expression to belong to results of which we know nothing except, at most, that algebraic formulæ do *not* exist.

When an equation has irrational terms, the whole process is in no respect altered. If no exponents enter except definite fractions, the essential polygon of the curve will have some or all of its points, not on the intersections of the squares, but on the intersections of rectangles made by subdividing each horizontal side into some one fixed number of parts, and each vertical side into some other fixed number. Let this polygon be augmented in size, without alteration of form, so as to be the smallest which can have all its points on the intersections of the squares, and we have the essential polygon of the rationalized equation of which the given irrational equation was one of the irrational forms. Or thus:—Let all the fractional exponents be reduced to a common denominator k; divide each side of all the squares into k equal parts, and thus construct other squares. Draw the essential polygon of the irrational equation, and then read each kth part as unity: we have thus all the essential terms of the rationalized equation. And thus the character of all the solutions of an equation can be determined from any one of its irrational forms, and certain *à priori* limitations placed upon the number and character of the terms of the rational equation.

In an inverse question, such as above described, it depends partly, but not entirely, on the numerical coefficients introduced, how many of the required branches are real, and how many imaginary. But the criteria for initial and terminal branches are wholly distinct, no one essential term for the former being necessarily an essential term for the latter; it is even impossible that one term should be essential in both cases, except the highest and lowest powers of y, when they have only one power of x in their coefficients. I do not lengthen this paper by suggestions which arise relative to the classification of curves,—the mode of dependence of x expressed in terms of y upon y expressed in terms of x, on which every square within the polygon must tell something—the manner in which the introduction of a non-essential term peculiarly affects the roots to the square of which the new exponent point is a corner, &c. &c.

To form the polygons for ϕ_y, ϕ_{yy}, &c. we have but to advance the axis of m one interval for each differentiation, rejecting all the exponent points left behind: and similarly for ϕ_x, &c. Thus, taking the first instance in the diagram, ϕ_y has SRX for its boundary instead of $SPQX$, and all the higher forms change dimension at once. The lower forms of the roots are lost or modified in descending order: the higher forms in ascending order.

The application of this method to surfaces is, so far as I have examined it, of very close analogy with the preceding results: but as I can now give nothing which would not easily suggest itself after what precedes, I will here only recommend it to the notice of those who take particular interest in the subject. It will, I think, demand a special inquiry into the theory of polyhedrons, considered as having triangular faces only or else three-angled corners only; a face of more than three sides being looked upon as a casual degeneracy of two or more triangles into the same plane.

It remains to notice the *pointed branch*, or *branche pointillée*. Writers are much divided on the question of admitting this part of a curve: but their controversy is one of definition of terms. In *geometry*, it may be excluded: in *algebra*, it must be admitted. Fully to represent $y = \epsilon^x$, we must admit every point contained in $x = (2p + 1) : 2q$, $y =$ the negative value of $(\epsilon^{(2p+1):q})^{\frac{1}{2}}$. In fact, $\epsilon_0^{\,x}$ being the real and positive value of ϵ^x, we see that $\epsilon^x = \epsilon_0^{\,x} \{\cos 2\pi m x + \sin 2\pi m x \,.\, \sqrt{-1}\}$ m being any integer, positive or negative. The condition $\sin 2\pi m x = 0$ includes a branch all points of which are not admissible, but which, nevertheless, contains an infinite number of points of either character between any two of the other.

To avoid the pointed branch, we are at perfect liberty to invent a symbol: we may say, let ϵ_0 be understood as $\epsilon\ (\cos 0 + \sin 0 . \sqrt{-1})$, a perfectly distinct thing from ϵ_1 or $\epsilon\ (\cos 2\pi + \sin 2\pi . \sqrt{-1})$, &c. Such a distinction I not only admit, but upon occasion contend for: but granting $y = \epsilon_0{}^x$ to be fully represented by one continuous branch; $y = \epsilon_1{}^x$ by a system of isolated points having $x = \frac{1}{2} p$; $y = \epsilon_2{}^x$ by another system having $x = \frac{1}{4} p$, &c.; p being any integer, positive or negative—the curve $y = \epsilon_0{}^x\ (\cos 2\pi mx + \sin 2\pi mx . \sqrt{-1})$, or $y = \epsilon^x$ in the ordinary sense, collects all the cases, infinite in number, and adds the pointed branch to the continuous curve.

The discovery of equivocal value in the inverse functions of common algebra was accompanied by what is but the expression of the result in another way, the discovery of equivocal form in the direct functions. Thus the double square root of x implies the double form of x^2, $(+x)^2$ and $(-x)^2$. The triple distinction of +, 0, −, is connected with the similar distinction of expressions into logarithmic, common algebraical, and trigonometrical, in whichever order we read them. Each has its direct and inverse forms, as in ϵ^x and $\log x$, x^m and $x^{\frac{1}{m}}$, $\tan x$ and $\tan^{-1} x$. If univocal character be the proper attribute of a direct function, as opposed to an inverse one, then x^m (m integer), and $\tan x$, are direct. But both ϵ^x and $\log x$ are equivocal, and ϵ^x has an equivocal character resembling that of $x^{\frac{1}{m}}$, while $\log x$, in this respect, resembles $\tan^{-1} x$. Moreover, the nature of the equivocal values of $\log x$ and ϵ^x depends upon the value of x: in other respects, $\log x$ is analogous with the simple power of x, x^m, and ϵ^x with $x^{\frac{1}{m}}$; while $\log x$ is analogous with $\tan^{-1} x$, and ϵ^x with $\tan x$. Thus, x being the positive value of $\epsilon^{(2p+1):2q}$, $\log(-x) = \log x$, in its real value, while x and $-x$ are the two values of the exponential.

In order fully to trace the curve $F(x, \epsilon^{\phi x}, \epsilon^{\psi x}, \log \chi x, \ldots y) = 0$, we must first, without attending to conditions of punctuation, allow all the multiplicity of branches which would be given by $F(x, \pm\sqrt{P}, \pm\sqrt{Q}, R^2, \ldots y) = 0$, and then, selecting the branches in which one or both of $-\sqrt{P}$, $-\sqrt{Q}$, are used, *punctuate* them by only allowing y to exist when one or both of ϕx and ψx have the forms $(2p+1) : 2q$. If, by the general process, we desire to ascertain the singular points, then, rationalizing $F(x, \sqrt{P}, \sqrt{Q}, \ldots) = 0$ into $f(x, y, P, Q, R^2, \ldots y) = 0$, we must fully ascertain the singular points of $f(x, \epsilon^{2\phi x}, \epsilon^{2\psi x}, \log \chi x, \ldots y) = 0$, and then ascertain the pointed branches by reference to the original form, having previously taken only positive values of $\epsilon^{2\phi x}$ and $\epsilon^{2\psi x}$, and considered $\log(-a)$ as $\log a$. When χx is negative, the branch thence arising is punctuated by the condition $\chi x = \epsilon^{(2p+1):2q}$. The so-called *abrupt termination* is always the point of union of a continuous and pointed branch; and the *salient point* where two tangents can be drawn to one point, is but a double point at which two mixed branches meet.

The rejection of the pointed branch is an abandonment of geometrical algebra, and a return to algebraic geometry: but the stand is made too late, and, in fact, the place at which it shall be made is a matter of taste, unless it be made upon a definition. A system which admits an isolated point, any given number of them, any smallness of distance between isolated points, certainly draws upon algebra for its definitions, and must end with the pointed branch. An application of the Horatian pitchfork to any symbolic result will only be another fulfilment of the Horatian prophecy.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON, *April* 16, 1855.

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XXIX. *On the External Temperature of the Earth, and the other Planets of the Solar System. By* W. HOPKINS, ESQ., M.A., *of St Peter's College,* F.R.S., *and Vice-President of the Society.*

[Read *May* 21, 1855.]

1. GREAT interest has recently been excited respecting the physical state of the different planets which compose the Solar System, and more especially respecting those physical conditions by which their surfaces may severally be characterised, and to which the constitution of any organic beings existing on each particular planet must necessarily, as we conceive, bear a direct and intimate relation. Of these *external conditions*, as we may term them, such as heat, light, moisture, &c., I now propose to examine that of *temperature*. We have not, it is true, sufficient *data* to determine the superficial temperature of any planet besides our own. We know, however, that it must mainly depend on the temperature of the planetary space, and on the direct heat which the nearer planets, at least, must receive directly from the Sun, but modified, and possibly in a far greater degree than has been generally supposed, by the particular circumstances by which each planet may be characterised. The modifying circumstances to which I shall more particularly direct attention in this paper, are the existence of atmospheres surrounding the planets, the positions of their axes of rotation, and the conductivity and specific heat of the substances forming the outer crust of each planetary body of our system. No astronomer, I suppose, judging from the appearances which Mars and Jupiter present to us, would entertain any serious doubt as to the existence of atmospheres surrounding those planets, and the probability would seem to be almost equally strong of Saturn being likewise enveloped in a similar manner. The obliquity of the axis of rotation is known with considerable accuracy in the cases of Mars and Jupiter, and also in that of Saturn if, at least, it coincide with the axis of rotation of his ring. Venus presents great difficulties to the observer, on account of the dazzling quantity of light which she reflects to us, and the absence of well-defined permanent dark spots on her disc. This led formerly to considerable doubts as to the period of her revolution about her own axis, which appears now, however, to be pretty satisfactorily determined to be nearly the same as that of the Earth. The position of her axis presents a corresponding difficulty in its determination, but the most generally received opinion is, I believe, that the obliquity is large, amounting, in the opinion of some astronomers, to as much as about 75°. This must produce an extraordinary difference between the changes of *annual* temperature in that planet, and those which we experience. I have endeavoured, in this paper, to estimate numerically the effect of this anomalous obliquity. The indications of an atmosphere about Venus are not perhaps so distinct as those in Mars and Jupiter, but still the opinion of its existence is sanctioned by such observed facts as bear

on the question, nor am I aware of any practical astronomer having entertained an opposite opinion. Of Mercury we know too little by direct observation to form any opinion on these points founded on observed facts, and the same remark will apply to the remoter planets beyond Saturn; but in the absence of any such direct testimony respecting these planets, analogy would seem to throw the balance of probabilities far in favour of the opinion of the existence of atmospheres in these cases as well as in that of the Earth, or in the cases of those other planets which afford us more direct evidence on the subject. Most astronomers probably feel much the same conviction that Mercury, Uranus, and Neptune have atmospheres of greater or less extent, as that they revolve round their own axes with greater or less angular velocity.

It is not my object, however, to adjust the balance of probability for particular hypotheses in favour of planetary atmospheres, or against them; but, assuming their existence, to estimate their effects on the planetary temperatures. And, in like manner, I shall point out the influence which must be exercised by a greater or less conductivity and specific heat in the superficial matter of a planet, without professing to discuss the probability of such properties being materially different in the different planets. We are sure that there are powers in Nature by which the great Author of Nature has adapted the conditions of each planet to the purposes for which His wisdom has destined it; and my object is to shew that, however limited our knowledge of cosmical causes may be, we are still able to recognise certain natural agencies by which the effects of heat derived immediately from the Sun and from stellar space may be largely modified, and the resulting planetary temperatures adjusted to any purposes for which each planet may be intended.

2. Before I proceed to the particular points which the subject embraces, it may be well to premise a few general remarks on the manner in which heat may be transmitted through different media. This may take place (1) by *complete radiation*, (2) by *conduction*, (3) by *partial radiation* and *conduction*, or (4) by *convection*. The transmission through *vacuum*, or rather through space in which we conceive no other matter to exist than the thermaniferous medium itself, is by complete radiation. Heat also passes through perfectly *diathermanous* media in the same manner, precisely, as light passes through perfectly transparent media. In such cases no portion of the incident heat is absorbed by the constituent particles of the diathermanous medium, the temperature of which, consequently, remains entirely unaffected by the transmission. This may be regarded as one extreme case; the other is that in which heat is transmitted entirely by *conduction*. In such cases the whole of the incident heat is absorbed by the particles on which it first falls, and is then emitted by them to the next particles, and so on successively till emission takes place from the opposite boundary of the medium. Such media can only transmit heat when the temperature of the surface on which it is incident is higher than that from which it is finally emitted. Again, the incident heat may be only partly absorbed by the particles which form the external boundary of the medium, while other parts penetrate to a greater or less depth within it before they are completely absorbed, precisely as light penetrates to a certain depth in imperfectly transparent media. If this medium be of sufficiently small thickness, a portion of the incident light will be transmitted entirely through

it, and, under similar circumstances, a portion of the incident heat may pass instantaneously *by radiation* through an imperfectly diathermanous medium, while the remainder is absorbed by the particles of the medium.

It would appear improbable that the absorption of light should take place without producing some physical effect on the absorbing medium, though there are few cases in which we can form any distinct conception of what that effect may be; but we know that light once absorbed is finally extinguished. In like manner heat when absorbed may be destroyed as regards its power of affecting any thermometric instrument, while it produces other effects which the Dynamical Theory of heat has taught us how to appreciate; and, moreover, heat thus absorbed is reproducible as thermometric heat. When heat is first incident on any medium, a portion of it may be employed in producing motion in the particles of the medium or changes in its molecular state; but when the particles are again reduced to rest, or have assumed a state which, in reference either to their motion or the molecular condition of the medium, may be termed *steady*, all the heat incident on the medium will be propagated through it *as heat*, whatever may be the mode by which the transmission may take place. It is this conclusion which assures us that whatever may be the mean rate at which heat is supplied from the Earth's surface to the lower strata of the atmosphere, the same must be the mean rate at which the heat must be transmitted from the upper portions of the atmosphere into surrounding space; because the *mean state* of the general atmosphere, whatever may be the amount of local or periodical changes, must have been permanent for some indefinite period of time, and consequently no heat can ever be permanently absorbed in producing permanent change in the state of the atmosphere.

The transmission of heat by the last mode above-mentioned, *convection*, consists in the ascent of the hotter and descent of the colder particles of a medium in which the particles are easily moveable *inter se*. In the case of the atmosphere, this is one of the dynamical effects of the heat which enters it from the Earth's surface; but since the mean amount of the *vis viva* due to this motion, referred to a period of a few years, must be the same for consecutive periods, the remark at the end of the previous paragraph will be applicable to it equally as to other periodical atmospheric changes.

It must also be recollected that the particular manner in which heat is transmitted does not depend merely on the nature of the transmitting medium, but also on that of the heat transmitted. Glass and atmospheric air, for instance, are diathermanous for the direct solar rays of heat, but not so for the same heat after it has been absorbed and again emitted by any terrestrial body. It is this property of the atmosphere which renders it so important an agent with reference to the Earth's superficial temperature.

All the above-mentioned modes of transmitting heat are called into action in the atmosphere. It is almost entirely by direct radiation that the solar heat reaches the Earth's surface, when the atmosphere is pure, though a small quantity may even then be absorbed by the atmospheric particles. A considerably greater quantity will, of course, be absorbed when the atmosphere is laden with clouds and vapour. In the transference of the heat back again from the Earth's surface, all the other modes of transmission must be more or less effective, but we are altogether ignorant of their relative degrees of efficiency. For our present purpose, however, it is sufficient for us to know that none of them can be effective unless the temperature of the atmosphere decrease continually as we ascend from its lower to its upper regions.

Having made these general remarks, I will now proceed to the brief consideration of two or three hypothetical cases, which will lead us immediately to the questions which form the more especial objects of this communication.

3. Conceive a hollow spherical shell, or envelope, the interior of which contains nothing but the thermaniferous medium. It will be convenient with reference to our ultimate object to suppose its dimensions indefinitely large. If the temperature of the interior surface of the shell were maintained at a uniform temperature (t_0), a thermometer placed at any point within the envelope would indicate the temperature t_0, which would also be that finally assumed by any body placed within the shell, provided the body were not perfectly diathermanous for the kind of heat proceeding from the shell. It will be convenient to designate the particular kind of heat here supposed to radiate from the envelope by H_0. Again, let us conceive heat of a different kind (H_1) to radiate uniformly over the interior space, and in such quantity as to elevate the temperature at every point to $t_0 + t_1$; but suppose the body (which I will call A), placed within the envelope to be perfectly diathermanous for this particular kind of heat (H_1). Its particles would absorb none of this heat, and therefore would not have their own proper temperature (t_0) at all affected by it, but a thermometer placed within the body would not now indicate that temperature. For, its bulb not being diathermanous for either kind of heat, the temperature indicated by it would be raised above t_0 by the heat of the kind H_1 absorbed by its bulb, till this quantity should be equal to that emitted by the bulb to surrounding particles of the body. The indicated temperature would, therefore, be intermediate to t_0 and $t_0 + t_1$. A similar effect would be produced by the body A on the thermometer placed without it, but sufficiently near to be within the sphere of its cooling influence.

4. Taking another imaginary case: Suppose the general temperature t_0 of the envelope to be derived as in the first instance from H_0 alone; and conceive the body A to be a spherical shell, the center of which may be supposed, for greater distinctness, to coincide with that of the general envelope. Conceive also the center of A to be a source from which heat (H_2) emanates uniformly in all directions, and such that it can be transmitted through the matter of the shell *only by conduction*, in the same manner as H_0. The temperature (τ_1) of the internal surface of the shell will rise till its excess over the temperature (τ_2) of the outer surface of the shell shall be sufficient to enable the shell to transmit in a given time a quantity of heat equal to that which falls on its inner surface in the same time from the central source; while τ_2 must exceed the temperature (t_0) of the external space sufficiently to enable the same quantity of heat to radiate in the same time into surrounding space. In this state the temperature will be *steady*, decreasing from τ_1 to τ_2 at a rate depending on the quantity of heat transmitted in a given time, and on the conductivity of the matter of the shell; and since the rate of decrease would be independent of the thickness of the shell, it is manifest that the greater that thickness the greater must be the excess of τ_1 above τ_2.

Under these circumstances a thermometer placed beyond the sensible influence of the body (A) would indicate a temperature t_0, but would rise as it approached the outer surface of A, in consequence of radiation from that surface, supposing τ_2 sensibly greater than t_0. At that surface the instrument would indicate the temperature τ_2; and if τ denote the temperature at any point within the shell, τ would be the temperature indicated at that point, and would continually increase in approaching the lower surface of the shell, when it would become τ_1.

5. Let us now proceed a step further in our hypothetical case. Instead of supposing the interior of the shell (A) to be hollow, conceive it filled with homogeneous matter. Also suppose the envelope to send forth heat uniformly of the kind H_1 as previously supposed (Art. 3) in addition to the heat H_0, the heat H_1 having the property of permeating the matter of the shell (A) by radiation, and consequently without communicating any heat to the constituent particles of the shell. Suppose all the heat of this kind which falls upon, and thus passes through the shell, to be absorbed by the interior nucleus of the shell. The quantity thus absorbed by the nucleus in a given time, when the temperature has become *steady* throughout the nucleus and shell, must necessarily pass out of it in the same time; and if we assume the heat thus emitted from the nucleus to have lost its power of radiating back through the shell, it will return through it *by conduction;* and since the heat H_1, when incident from without, passes through A without affecting in any degree the thermal state of its particles, it will not affect the conduction of the heat (H_1) back from B through the shell into the surrounding space, and therefore τ_1, τ_2, and τ may be considered precisely the same in this case as in the case of the previous Article.

But here it must be carefully remarked that τ is the temperature of the constituent particles of the body, at any proposed point, but not that which would be indicated by a thermometer of which the bulb should be placed at that point. Let us again consider what would be the indications of a thermometer in different positions. If placed beyond the influence of A it would give the temperature $t_0 + t_1$, (derived from the radiating heat H_0 and H_1) as that of the general surrounding space. The temperature (τ_2) of the outer surface of the shell (A) would be the same as in the previous case, and under ordinary conditions would not much exceed t_0; we may therefore suppose $t_0 + t_1$ to exceed τ_2 as much as we please. If we suppose this excess to be considerable, the thermometer will be depressed below $t_0 + t_1$ as it approaches A by radiation from the surface of A, and this effect will increase till the instrument reaches that surface, the indicated temperature being always intermediate to τ_2 and $t_0 + t_1$. If the instrument be then placed within the shell (A) at a point where the temperature of its constituent particles $= \tau$, the instrument will still be equally affected by the radiating heat H_1, and the temperature indicated will be between $t_1 + t_0$ and τ; *i.e.* it will be higher than at the exterior surface, and will manifestly continue to rise till the instrument reaches the interior surface of the shell.

I have here supposed the density of the shell uniform, in which case the greatest depression of the thermometer would take place when it came in contact with the coldest particles, *i.e.* at the external surface of the shell, as above represented. If, however, the density of the shell decrease in ascending from the interior surface till it becomes sufficiently rare near the outer surface, the number of particles of the shell with which the thermometer will be in contact when placed there, will be smaller than at lower points within the shell, and consequently, their maximum effect in depressing the thermometer may be at some point within the shell, and not at its outer boundary. But whatever may be the point at which the instrument may indicate its lowest temperature, it will rise continually as it proceeds from that point till it reaches the interior surface of the shell, or external surface of the nucleus.

6. It has been already remarked that the excess of τ_1 above τ_2, and also that of τ_2 above t_0 must be such as to admit of the transmission of a certain quantity of heat from the inner to the outer surface of the shell in a given time, that heat being derived from the center of the shell in the hypothetical case of Art. 4, and by radiation from the external envelope and through the shell, in the present case. The difference, then, between τ_1 and t_0 will depend in the present case on the following circumstances:

(1) The quantity of heat (H_1) which radiates from the external envelope, and for which the shell (A) is perfectly diathermanous, and which is entirely absorbed and given out again by the internal nucleus.

(2) The conductivity of the matter of the shell, through which the heat given out by the nucleus is here supposed to pass entirely by conduction.

(3) The emissive power of the outer surface of the shell.

(4) The thickness of the shell.

The fourth condition is manifestly independent of the three preceding ones, and it follows that if the thickness of the shell be sufficiently great, the excess of τ_1 above t_0 may be increased to any amount, and may thus become greater than t_1, although this latter temperature is derived directly from the radiant heat (H_1) from which the heat of the nucleus, or τ_1, is derived. But this heat produces the temperature t_1 in the bulb of a thermometer from which the heat radiates again freely into surrounding space, whereas the temperature τ_1 is produced by heat radiating on the nucleus, from which, instead of again radiating freely, it can only pass away by the slower process of conduction, and consequently accumulates till τ_1 sufficiently exceeds t_0 to enable the heat to pass as rapidly through the shell by conduction as it first passes through it to the nucleus by radiation.

7. The general analogy between the case now considered and that of a planet, as a nucleus enveloped by its atmosphere, forming a spherical shell around it, is obvious; but to make the analogy complete with the case of the Earth, or that of any planet sufficiently near the Sun, we must consider the thermal condition of such planet independently of the heat received from a particular focus like the Sun, leaving the influence of that luminary for separate consideration. The heat received by the remoter planets from solar radiation must be too small to render such separate consideration essential. For the greater distinctness we may here limit ourselves to the case of the Earth, considering it independently of solar radiation.

Supposing, then, the Sun to be removed, the heat which the Earth would receive from external sources would be that which radiates from all surrounding bodies in space, of which we may regard as the most important the fixed stars.

The diathermanous property of our atmosphere with reference to direct solar heat has already been mentioned (Art. 2). We have no means of ascertaining by experiment whether it possess the same property with regard to heat which radiates from the fixed stars, but the analogy between the two cases is such that it would seem extremely difficult to suppose this property not to exist with respect to stellar as well as solar radiating heat. I shall here suppose it to do so equally. It is from this stellar radiation that the temperature of general stellar space is derived, but what may be the amount of the heat thus radiating to any particular spot, as our own planet, it is impossible to say, because we can form no idea of what would be the temperature of stellar space if stellar radiation were entirely to cease. The quantity of heat thus received by the earth may, for ought we know, be much greater than that which we receive from solar radiation. With respect to the uniformity of stellar radiation in different directions upon the Earth, we can only assert that it must be nearly uniform from those regions of space towards which the north and south poles of the Earth are almost permanently directed; for if it were otherwise there would necessarily be a greater difference between the temperatures of the northern and southern hemispheres of our globe than is shewn by observation to exist.

The same conclusion will not necessarily apply to the equatorial regions of the heavens, since nearly the whole of the Earth's surface is presented to them in turn. It will suffice, however, for our purpose, to suppose that stellar radiation upon the Earth, or any other point within the solar system, is the same in every direction, so as exactly to resemble the radiation which I have supposed in our hypothetical cases, to proceed from a uniformly heated spherical envelope of indefinite extent.

8. In our hypothetical case I have supposed the heat to return through the shell from the central nucleus entirely by conduction. There is no doubt, however, but that all the modes of transmission heretofore specified (Art. 2), except that by complete radiation, are efficient in the case of the Earth's atmosphere. The conclusion arrived at above will still be equally true—the excess of τ_1 above t_0 will increase *cæteris paribus*, with the extent of the atmosphere, though not in the same ratio as if the transmission were effected by the slower process of conduction alone. The temperature t_0 must now be considered as that to which if the atmosphere were reduced radiation from it into external space would entirely cease. We have no knowledge whatever of its value, except that it must doubtless be exceedingly low.

We may also remark that when radiation takes place between particles of the medium within a certain limited range, the radiation into external space will not take place altogether from those particles which form the extreme boundary, but from all those situated within a certain distance of that boundary; but this will not render a determinate relation the less necessary between τ_2 and t_0.

9. It has already been stated (Art. 5), that if the density of the shell (A) in our hypothetical case were sufficiently small at its outer boundary, the maximum effect in depressing a thermometer entering the upper region of the shell from the surrounding space would not necessarily be at the extreme boundary of the shell; and since the density of the atmosphere must be extremely small in its upper regions, it is very possible that the point of maximum depression of a thermometer moved gradually from stellar space to the Earth's surface, would be found at a considerable distance within the upper limit of the atmosphere, since the effect on the thermometer will depend on the number of atmospheric particles within a certain distance of the bulb, as well as on their proper temperature. Thus if the thickness of an atmosphere like the Earth's were smaller than the distance of the point of maximum thermometric depression from the external boundary of the atmosphere, an increase of the quantity of atmosphere might depress the temperature of the planet's surface. If, on the contrary, the thickness of the atmosphere be greater than the above-mentioned distance, an increase of atmosphere must necessarily produce an increase in the planet's superficial temperature. We are utterly ignorant respecting the lowest temperature which might thus be produced, or the amount of atmosphere which might be required for the purpose. We are certain, however, that the point of maximum thermometric depression in the Earth's atmosphere must be at a great distance above the Earth's surface, and, therefore, that the quantity of atmosphere which would produce the minimum temperature at that surface must be very much less than that which actually envelopes our own globe.

10. It appears then from this general reasoning that, assuming the absence of the Sun, the atmosphere in which a planet is enveloped may, according to its extent, either depress the superficial temperature of the planet below, or raise it above that of surrounding space. We have no means, however, of determining, even in the case of our own planet, which of these effects is produced, on account of our imperfect acquaintance with the conducting, radiating, and absorbing powers of the atmosphere, our ignorance of the state in which it exists in its upper regions, and of the temperature of space beyond. But whatever may be the actual effect of the existing atmosphere of our globe, we are sure that an increase or diminution of its quantity would produce a corresponding increase or diminution of the existing superficial terrestrial temperature, because we know that the temperature indicated by the thermometer descends as we ascend from the Earth's surface; and this is the important point as regards our object of ascertaining what may be the influence of an atmosphere enveloping any other planet according as that atmosphere may be greater or less than our own.

11. If the radiation of heat upon the Earth or other planets from different portions of space should be very different, the same general reasoning would be applicable, and would lead to similar conclusions. Whatever may be the quantity of heat which has to be transmitted back into surrounding space through the atmosphere, and whatever may be its original source, the greater the thickness of the atmosphere, the greater must be

its temperature at the planet's surface, that it may be enabled to transmit a given quantity of heat in a given time.

The effect of solar radiation is the same as if, instead of proceeding from a single focus, it proceeded generally from surrounding space, the intensity of radiation on any point of the Earth's surface, omitting daily variations, being a function of the latitude of the place and the time of the year. It will be my object, in the latter part of this paper, to examine the data which we possess for determining numerically what would be the effect of an increase or diminution of our atmosphere on terrestrial temperature, and also what is the actual effect of the sun's heat on that temperature. We shall then be able to judge what may be the probable influence of similar causes on the temperature of the other planets. But before I proceed to these points, I shall make some remarks on the mathematical treatment of the problem which has for its object the determination of the actual pressure, temperature, and density of the Earth's atmosphere.

12. Conceive a cone of which the angle at the apex is indefinitely small, the apex coinciding with the center of the Earth. If this cone be produced through the Earth's atmosphere, the same quantity of heat must pass upwards in the same time through every transverse section of the cone comprised within the atmosphere, the temperature being considered as *steady*. Let q denote the quantity of heat thus transmitted in a unit of time through any section at the height z above the Earth's surface; then must we have

$$\frac{dq}{dz} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

If the transmission take place by conduction, we have

$$q = k \cdot \frac{d\zeta}{dz},$$

where ζ is the temperature at the height z, and k the conductive power, probably some function of the density ρ. If the transmission take place by radiation within certain finite distances, or by convection, or by all three modes conjointly, it is manifest that q must depend on the difference of temperatures at the point whose height is z and at surrounding points, and the expression for it must therefore involve the differential coefficient $\frac{d\zeta}{dz}$, so that equation (1) will necessarily be a differential equation of the second order. Another equation must give us a relation between p, ρ, and z. If the atmosphere be at rest, this equation is

$$\frac{dp}{dz} = -g\frac{r^2}{(r+z)^2}\rho \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

where r = radius of the Earth. A third relation between p, ρ, and ζ depends merely on the property of air and not on the particular conditions under which it exists as an atmosphere surrounding a particular planet. We have as the result of experiment,

$$p = a^2\rho(1 + \alpha\zeta) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

where a^2 is constant. This law is known to hold within wide limits of the density, though it might not be safe to conclude that it would be true for the extremely small density and low temperature of the air, at the upper boundary of the atmosphere.

The last of the above equations combined with the integrals of the two former (or more strictly the equivalent exact equations, if they were known) would give the values of p, ρ, and ζ as functions of z, the required conditions being given for the determination of the constants. The integral of equation (1)

$$q = \text{const.} = c,$$

must involve $\frac{d\zeta}{dz}$, and, therefore, if the rate of decrease of temperature be given for any given height, or when $z = 0$, the value of c may be determined; or it will be immediately known if the quantity q be given. The other two constants may also be conceived to be determined from given values p_1 and ζ_1 of the pressure and temperature when $z = 0$.

Various other conditions may manifestly be assigned for the determination of the constants, but we may observe that if we suppose gravity constant throughout the height of the atmosphere (as we doubtless may without sensible error) that our three fundamental equations will not involve z explicitly, and consequently the results immediately deducible from them, while the constants introduced by integration remain arbitrary, must be independent of the position of the origin from which z is measured. Hence if q be given, and p_1 and ζ_1 be also absolutely given quantities, the constants determined by the above conditions will necessarily have the same values whether the origin of z be at the surface of the Earth or at any assigned height above it. Thus if there were two planets for which g was the same, and for one of which the pressure and temperature should be p_1 and ζ_1 at the surface, and should have the same values for the other at a height h above its surface, then the pressure, temperature, and density in the former at the height z would be the same as in the other at the height $h + z$, and the height of the atmosphere in the one case being H, that in the other would be $h + H$. In the second case the temperature would manifestly increase in descending below the point at the height h, according to the law which the final equations would determine. Hence the greater the value of h, i. e. the greater quantity of atmosphere by which a planet is surrounded, the greater *cæteris paribus* must be the temperature at its surface. This result is of such manifest importance in reference to the leading objects of this paper, that I have thought it worth while to establish it by considerations more detailed than the general reasoning of the previous articles.

If we suppose the heat to be transmitted upwards through the Earth's atmosphere solely by conduction we shall have for the integral of (1),

$$k\frac{d\zeta}{dz} + c = 0.$$

Poisson has worked out the problem of which the fundamental equations are (1), (2), and (3), assuming k to vary as $\frac{1}{\rho^2}$, which gives

$$\frac{1}{\rho^2} \cdot \frac{d\zeta}{dz} + c = 0,$$

and has obtained the complete expressions for p, ρ, and ζ in terms of z. The condition $p = 0$ then gives the height of the atmosphere and its density at its upper limit. The author does not attach any practical value to this solution, but merely gives it as an example of the manner in which the problem would have to be treated to obtain the actual values of p, ρ, and ζ, if we had more perfect knowledge of the law by which the atmosphere does actually transmit heat from the surface of the Earth. He obtains the result that the density of the atmosphere at its upper limit would, under the conditions assumed, be finite; and consequently that the condition $p = 0$ at that limit could only be satisfied by

$$1 + \alpha\zeta = 0,$$

$$\text{or} \quad \zeta = -\frac{1}{\alpha} = 273^\circ \text{ (C.)}$$

It should be remarked that this very low temperature would not be that of surrounding space, but that of the constituent particles of the atmosphere at its upper limit, where the air would consequently lose all its elasticity, and be reduced to the state of a *fluid*. This value of ζ would be that which I have designated by τ_2, and it would follow, according to this solution, that the temperature heretofore denoted by t_0 must be less than 273° (C.) But the result could give us no knowledge of what I have termed t_1, or consequently of $t_0 + t_1$ the proper temperature of surrounding space.

13. Another method of treating this problem consists in making use of an assumed expression for $1 + \alpha\zeta$ in terms of the density ρ, instead of equation (1) (Art. 12). Thus we may assume

$$1 + \alpha\zeta = M_0 + M_1\rho^{m_1} + M_2\rho^{m_2} + \&\text{c.};$$

and substituting in (3), we have

$$p = a^2\{M_0\rho + M_1\rho^{m_1+1} + \&\text{c.}\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and therefore

$$\frac{dp}{d\rho} = a^2\{M_0 + (m_1 + 1)\, M_1\rho^{m_1} + \&\text{c.}\}.$$

Equation (2) then becomes

$$a^2\left\{\frac{M_0}{\rho} + (m_1 + 1)\, M_1\rho^{m_1 - 1} + \&\text{c.}\right\}\frac{d\rho}{dz} = -g\,\frac{r^2}{(r + z)^2};$$

$$\therefore\ a^2\left\{M_0 \log\rho + \frac{m_1 + 1}{m_1}\, M_1\rho^{m}{}_1 + \ldots\right\} = C + gr^2\,\frac{1}{r + z};$$

and if $\rho = \rho_0$ at the Earth's surface,

$$a^2 \left\{ M_0 \log \rho_0 + \frac{m_1 + 1}{m_1} M_1 \rho_0^{m_1} \ldots \right\} = C + g r^2 \frac{1}{r};$$

$$\therefore\ a^2 \left\{ M_0 \log \frac{\rho_0}{\rho} + \frac{m_1 + 1}{m_1} M_1 (\rho_0^{m_1} - \rho^{m_1}) + \ldots \right\} = gr \frac{z}{r + z}.$$

Now since $p = 0$ at the upper limit of the atmosphere, we must also have by equation (4), $\rho = 0$ for that limit. But the first side of the last equation would then become infinite, whereas no value of z will render the second side infinite; and therefore the equation cannot be satisfied unless

$$M_0 = 0,$$

and we then have

$$a^2 \left\{ \frac{m_1 + 1}{m_1} M_1 (\rho_0^{m_1} - \rho^{m_1}) + \ldots \right\} = gr \frac{z}{r + z} \text{ .... } (5).$$

If z_1 be the limiting value of z, we thus obtain

$$a^2 \left\{ \frac{m_1 + 1}{m_1} M_1 \rho_0^{m_1} + \ldots \right\} = gr \frac{z_1}{r + z_1}$$

$$= g z_1 \text{ nearly};$$

$$\therefore\ z_1 = \frac{a^2}{g} \left\{ \frac{m_1 + 1}{m_1} M_1 \rho_0^{m_1} + \&c. \right\}$$

a finite quantity.

We have also (since $M_0 = 0$)

$$1 + \alpha\zeta = M_1 \rho^{m_1} + M_2 \rho^{m_2} + \&c. \ldots;$$

but this gives us for the determination of ζ at the upper atmospheric limit (or τ_2 of our previous notation)

$$1 + \alpha\tau_2 = 0,$$

$$\tau_2 = -\frac{1}{\alpha} = -273^0 \text{ (C.)},$$

the same as that determined by Poisson.

In this method the conductive power of the atmosphere is tacitly assumed to be such as is consistent with the upward transmission of a determinate quantity of heat, whether by conduction or limited radiation through the atmosphere at rest, the law of density being such as is expressed by equation (5). The condition of the emission of the same quantity of heat through the upper surface into the surrounding space is also assumed to be satisfied.

Another method consists in combining, with equations (2) and (3), an equation which expresses a relation between ζ and z. Thus we may assume

$$1 + a\zeta = c_0 - c_1 z + c_2 z^2 + \ldots + c_n z^n,$$

and since $\zeta = \tau_0$ when $z = 0$,

$$1 + a\tau_0 = c_0;$$

and therefore

$$1 + \alpha\zeta = 1 + \alpha\tau_0 - c_1 z + c_2 z^2 + c_n z^n.$$

Let β be a positive root of the equation

$$1 + \alpha\tau_0 - c_1 z + \ldots + c_n z^n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6);$$

then we may write

$$1 + \alpha\zeta = (\beta - z)\,\phi(z),$$

$\phi(z)$ being a rational function of z; and we shall have

$$\frac{1}{p}\frac{dp}{dz} = -\frac{gr^2}{a^2}\frac{1}{(\beta - z)(r + z)^2\,\phi(z)} \quad \text{(Art. 12)};$$

$$= -\frac{gr^2}{a^2}\left\{\frac{L}{\beta - z} + \frac{M}{(r + z)^2\,\phi(z)}\right\}$$

where L is independent of z, and M a function z, of lower dimensions in z than the denominator. Hence

$$\log\frac{p}{p_1} = \frac{gr^2}{a^2} L \log\frac{\beta - z}{\beta} - \frac{gr^2}{a^2}\int_0^z \frac{M\,dz}{(r + z)^2\,\phi(z)}.$$

Now $\dfrac{M}{(r + z)^2\,\phi(z)}$ can always be expressed in partial fractions of the following forms,

$$\frac{L_1}{\gamma - z},\quad \frac{L_2}{\delta + z},\quad \frac{L_3}{(r + z)^2},\quad \frac{L_4}{r + z},\quad \frac{P + Qz}{z^2 + \lambda z + \nu^2},$$

where the quantities denoted by the large letters are independent of z, γ being a positive root of equation (6), δ a negative root, and $z^2 + \lambda z + \nu^2$ a quadratic factor corresponding to a pair of impossible roots of the equation. Now it is easily seen that none of the integrals of the last four of the above fractions could become infinite for any finite positive value of z, or for such, therefore, as could be available for satisfying in the last equation the condition $p = 0$, which must hold at the upper atmospheric boundary. The only admissible value of z for that purpose must be a positive root of equation (6). And this, moreover, must be the *least* positive root of that equation; for β and γ being positive roots, if γ be taken for the height of the atmosphere and be greater than β, $1 + \alpha\zeta$, and therefore, from equation (3), $\dfrac{p}{\rho}$ must have different signs according as z is greater or less than β; *i. e.* either p or ρ must become negative at certain heights, which is absurd. Consequently if β be the least positive root of equation (6), it is the value of z admissible for satisfying the condition $p = 0$, or $\log\dfrac{p}{p_1} = -\infty$ in the last equation. Hence the least positive root of (6) will give the height of the atmosphere provided the increment of temperature at any height z can be expressed in a series proceeding according to positive and integral powers of z, and provided, also, the value of ρ does not become negative for any value of z less than β.

At the upper limit of the atmosphere, where $z = \beta$, $1 + \alpha\zeta$ is reduced to zero;

$$i.\,e. \quad 1 + \alpha\tau_z = 0,$$

$$\tau_2 = -\frac{1}{a} = -273^\circ \text{ (C.)}$$

as in the previous cases.

We may shew that ρ will not become negative for any value of z less than β. We have from equation (3)

$$\rho = \frac{p}{a^2(1 + a\zeta)},$$

which, at the upper boundary of the atmosphere, assumes the indeterminate form $\frac{0}{0}$. To find its real value we have, on substituting the assumed expression for $1 + a\zeta$, in the equation (3)

$$p = a^2\rho\left\{1 + a\tau_0 - c_1 z + c_2 z^2 + \ldots + c_n z^n\right\},$$

$$\therefore\ \frac{1}{p}\frac{dp}{dz} = \frac{1}{\rho}\frac{d\rho}{dz} - \frac{c_1 - 2c_2 z - \&c. - nc_n z^{n-1}}{1 + a\tau_0 - c_1 z + \ldots + c_n z^n}\Bigg\}.$$

Also from (1) and (2),

$$\frac{1}{p}\frac{dp}{dz} = -\frac{g}{a^2 c_1}\frac{r^2}{(r+z)^2}\frac{1}{\dfrac{1 + a\tau_0}{c_1} - z + \ldots},$$

or omitting $\frac{z}{r}$, which may be done without risk of error,

$$\frac{1}{p}\frac{dp}{dz} = -\frac{g}{a^2 c_1}\frac{1}{\dfrac{1 + a\tau_0}{c_1} - z + \ldots + \dfrac{c_n}{c_1}z^n}.$$

Hence $$\frac{1}{\rho}\frac{d\rho}{dz} = -\frac{\left(\dfrac{g}{a^2 c_1} - 1\right) + 2\dfrac{c_2}{c_1}z + \ldots\ldots + n\dfrac{c_n}{c_1}z^{n-1}}{\dfrac{1 + a\tau_0}{c_1} - z + \dfrac{c_2}{c_1}z^2 + \ldots\dfrac{c_n}{c_1}z^n}.$$

Since the numerator of this expression is of lower dimensions than the denominator, taking β as before, we shall have

$$\frac{1}{\rho}\frac{d\rho}{dz} = \frac{\left(\dfrac{g}{a^2 c_1} - 1\right) + 2\dfrac{c_2}{c_1}\beta + \ldots\ldots + n\dfrac{c_n}{c_1}\beta^{n-1}}{\beta - z} + \frac{M'}{\psi(z)};$$

$$\therefore\ \log\frac{\rho}{\rho_1} = \left\{\left(\frac{g}{a^2 c_1} - 1\right) + 2\frac{c_2}{c_1}\beta\ldots\ldots + n\frac{c_n}{c_1}\beta^{n-1}\right\}\log\frac{\beta - z}{\beta} + \int_0^z \frac{M'dz}{\psi(z)}.$$

The same reasoning will apply to the last term of this expression as in the previous case, and $z = \beta$ will give $\rho = 0$ provided

$$\left(\frac{g}{a^2} - c_1\right) + 2c_2\beta\ldots\ldots + nc_n\beta^{n-1}$$

be a positive quantity. If it were negative, $\log\frac{\rho}{\rho_1}$ would necessarily be positive when $\beta - z$ should be sufficiently small, and consequently ρ would be greater than ρ_1.

Hence we must have

$$c_1 < \frac{g}{a^2} + 2c_2\beta \ldots\ldots + nc_n\beta^{n-1},$$

If we neglect $c_2, c_3 \ldots c_n$, we obtain

$$\log\frac{\rho}{\rho_1} = \left(\frac{g}{a^2 c_1} - 1\right)\log\frac{\beta - z}{\beta},$$

$$\text{where } \beta = \frac{1 + \alpha\tau_0}{c_1};$$

and therefore $\frac{1 + \alpha\tau_0}{c_1}$ will be in this case the height of the atmosphere subject to the condition

$$c_1 < \frac{g}{a^2}.$$

Now we have

$$a = 916{,}118 \text{ feet*},$$

$$g = 32{,}172 \ \ldots.$$

Or if we take 100 feet for the unit of length

$$a = 9{,}16118$$

$$g = {,}32172.$$

Also $\frac{c_1}{\alpha}$ is the decrement of temperature in ascending through a unit of height, as appears from the assumed expression for $1 + \alpha\zeta$; and this decrement will be expressed in centigrade degrees if we make

$$\alpha = {,}0003665.$$

Hence $\quad \frac{c_1}{\alpha}$ must be $< \frac{g}{a^2\alpha}$

$$< 10^{\circ}{,}4. \text{ (C.)};$$

i. e. the above value of β, in the case we are considering, will give the height of the atmosphere, if the increment of temperature corresponding to the ascent of 100 feet be less than 10°,4 centigrade degrees. It is in fact only the fraction of a degree.

The limit for the values of $\frac{c_1}{\alpha}$ would manifestly be less than the above if $c_2, c_3, \ldots c_n$ were too large to be neglected and were negative.

At the limit of the atmosphere where $z = \beta$, we have still, as in the former cases.

$$1 + \alpha\tau_2 = 0,$$

$$\tau_2 = -\frac{1}{\alpha} = -273^{\circ} \text{ (C.)}$$

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* See Miller's *Hydrostatics*.

14. It appears then that in whichever of the above methods we solve this problem, we arrive at the same result respecting the temperature of the atmosphere at its upper limit. The only reasonable cause of doubt, I think, as to that temperature being so low as here determined, is to be found in the assumption that the equation

$$p = a^2\rho\,(1 + \alpha t),$$

where a and α are constant, holds true for extremely low values of ρ and t. But whether this be accurately so or not, there can be no sufficient reason to doubt the accuracy of the conclusion that the temperature τ_2 must at least approximate to some such low value as that determined above.

15. The best *data* we possess for the determination of the constant $\left(\frac{c_1}{a}\right)$ are furnished by Mr Welsh's observations made in four balloon ascents from Kew, of which two were made in the month of August 1852, one in October, and another in December. He reached the height of very nearly 23000 feet above the Earth's surface. In each ascent the temperature decreased in ascending a few thousand feet, and then became stationary for some three or four thousand feet, after which a regular decrease again commenced. This anomaly was manifestly due to peculiar atmospheric causes, and ought to be abstracted in determining the proper decrease of temperature due to general causes. This being done, Mr Welsh shews that the law of observed temperature is very nearly as well expressed by taking, in the series for $1 + \alpha\zeta$, only the term involving the first power of z, as by including also the term in z^2; and he obtains the following decrements of temperature for 1000 feet of ascent, in the four ascents respectively:

Aug. 17 ... 3°,434 (F.)
Aug. 26 ... 3°,440
Oct. 21 ... 3°,431
Nov. 10 ... 3°,205

Mean ... 3°,377 (F.)
or ... 1°,85 (C.)

Retaining the centigrade degrees, and taking 100 feet for our unit of length, we shall have

$$\frac{c_1}{a} = ,185.$$

Mr Welsh also gives the results of his observations, freed as far as possible from the effects of anomalous causes, in the following form, shewing the mean ascent corresponding to one degree of Fahrenheit:

Aug. 17 ... 292,0 feet.
Aug. 26 ... 290,7 ...
Oct. 21 ... 291,5 ...
Nov. 10 ... 312,0 ...

The corresponding heights for one degree centigrade are

Aug. 17 ... 525,6 feet.
Aug. 26 ... 523,2 ...
Oct. 31 ... 524,7 ...
Nov. 10 ... 561,6 ...

The first three of these results, it will be observed, are nearly identical, while the fourth is materially different. In these ascents the heights attained above the sea-level were, respectively, in feet

19510, 19100, 12640, 22930.

The height attained in the third being so much less than those in the other three ascents, it cannot be so well compared as the first two may be with the last. Restricting the comparisons, therefore, to those of the first two with the last, we may easily recognise a principal cause of the discrepancy in the results of the observations. The temperatures of the air, reduced to the sea-level, were, approximately,

Aug. 17 ... 71°,6, (F.)
Aug. 26 ... 67°,2,
Nov. 10 ... 50°,2;

the difference between the last and the mean of the first two being 19°,2 (F.), it might be expected that the decrement of temperature would be less in the last than in the first two ascents. Supposing this to be the true cause of the discrepancy in question, it would appear that a depression of temperature at the surface of the Earth of 20° (F.) would cause an increase of about 20 feet in the ascent corresponding to a decrement of 1°. The mean temperature at the pole is about 80° (F.) lower than that of our own region; and, therefore, according to this calculation, the increment of height corresponding to a decrement of 1° (F.) at the pole, would be about 80 feet more than in these latitudes.

It must not, however, be forgotten that there is one unavoidable source of error in the comparison of the results of observation and those of theory on this subject. In the fundamental equation

$$p = a^2\rho\,(1 + \alpha\zeta),$$

ζ denotes the proper temperature of the air itself, whereas the temperatures observed at different heights above the Earth's surface are those which result from the temperature of the air combined with the all-pervading influence of stellar radiation. The effect of direct solar radiation is, of course, supposed to be carefully obviated.

Observations on temperature, however, such as those above recorded, enable us, independently of the error just mentioned, to estimate the effect of an assigned increase or decrease in the Earth's atmosphere; for if that atmosphere were diminished by a quantity equal to that which occupies a spherical shell immediately surrounding the Earth of 1000 miles, for example, in thickness, the mean superficial temperature of the Earth would be diminished by rather more than 3° (F.), and to a proportionally greater amount if a greater quantity of atmosphere were abstracted. A like increase of temperature would result from a corresponding increase in the quantity of atmosphere.

16. Having thus considered the influence which an atmosphere like that of the Earth is capable of exerting on the superficial temperature of a planet, I shall proceed to consider the influence of solar heat on that temperature. It has been already stated that the Earth's atmosphere is nearly *diathermanous* for the heat radiating directly from the Sun. It is not probable, however, that it is entirely so, and if not, it must absorb some heat which must elevate in some degree the temperature of its constituent particles. This augmentation of temperature, however, must be too small to render it of any importance in a discussion like the present, and may therefore be omitted. We may regard, then, nearly the whole of the solar heat which radiates towards the Earth as penetrating by direct radiation to the lower region of the atmosphere, and, when not intercepted by clouds, to the surface of the Earth. That portion which reaches the surface is nearly all absorbed by the terrestrial objects on which it falls, which, after thus becoming heated to a temperature above that of the surrounding atmosphere, give it forth again, and in such degree that the mean quantity thus given back in the course of a year is equal to the mean annual supply derived from the Sun. This *terrestrial heat*, as already explained, not possessing the power of direct *solar heat* to permeate the atmosphere without loss of intensity, passes into it either by conduction or partial radiation, and, being absorbed, elevates the temperature of the atmosphere. But the atmosphere when thus heated gives back again to the Earth itself a portion of the heat which it has thus acquired; and this action and reaction is continued till the ultimate equilibrium of temperature is established. The direct solar heat, intercepted and absorbed by the clouds, is given forth again in all directions, and thus contributes also to elevate the temperature of the Earth's surface and of the stratum of the atmosphere reposing upon it.

We should expect that the mean annual temperature of that portion of the atmosphere which is constantly in contact with the Earth's surface, would be very nearly the same as that of the surface itself; and, as far as observation has extended, this has been verified; for it has been found, where the necessary observations have been made, that the mean annual temperature of the air, deduced from indications of a thermometer *in the shade*, and that of the surface of the Earth, deduced from a thermometer of which the bulb was placed just beneath the surface, are nearly identical. Hence if any assigned cause produce a certain effect on the mean superficial temperature of the Earth itself, we know that it must produce the same effect on the mean temperature of the adjacent atmosphere as a direct and immediate consequence. Thus Poisson has calculated the increase of the Earth's mean superficial temperature due to the quantity of heat which radiates directly upon it from the Sun. The mean atmospheric temperature receives directly on this account an equal augmentation. There then remains to be calculated the additional effect above mentioned arising from the mutual interchange of heat between the Earth and its atmosphere. Poisson has likewise given formulæ for this purpose; but experimental *data* are altogether wanting to render them of the slightest practical value. I shall endeavour in a different manner to estimate, in the sequel, this less direct effect of the solar heat in elevating the Earth's superficial temperature and the temperature of that portion of the atmosphere which determines the climatal conditions under which we live.

The statement now made, it will be observed, respecting the equality of the temperatures of the surface of the Earth itself and of the contiguous stratum of its atmosphere, applies only to the ***mean annual temperatures.*** We shall also have to examine what relations may exist between the periodical changes in these temperatures; an examination for which the researches of Professor Dove have furnished us with far better materials than we previously possessed. After thus estimating the entire effect of solar heat on the temperature of our own globe, we shall be prepared to make also some approximate estimate of the influence which it may exercise on the other planets according to the conditions under which they may be placed.

The *Théorie de la Chaleur*, the work of Poisson's in which the investigations alluded to are given, is as admirable for the physical as for the mathematical analysis of the problems there investigated; but it is so elaborate (in some cases perhaps uselessly so for any practical purpose) as to render it extremely difficult for reference to particular portions which may bear upon particular problems. I have therefore thought it worth while to give in regular sequence the formulæ which bear directly on the questions of this paper, though in doing so I am led to make such citations at greater length than I might otherwise deem admissible in a communication of this kind.

17. Poisson first determines the law of temperature within the Earth on the supposition of the temperature of surrounding space being everywhere equal to zero; and then extends the solution of the problem to the case in which the external temperature at any point of the Earth's surface is supposed to depend on the latitude of the place, and also to be a periodical function of the time, so that, denoting this temperature by ζ, we have

$$\zeta = B + A\cos(mt + \epsilon) + A'\cos(m't + \epsilon') + \&c.,$$

where the coefficients depend on the latitude*. If u be that part of the terrestrial temperature which depends on ζ, we then have

$$\left.\begin{aligned} u = B &+ \frac{b}{D} A e^{-\frac{x}{a}\sqrt{\frac{m}{2}}} \cos\left(mt + \epsilon - \frac{x}{a}\sqrt{\frac{m}{2}} - \delta\right) \\ &+ \frac{b}{D_1} A' e^{-\frac{x}{a}\sqrt{\frac{m'}{2}}} \cos\left(m't + \epsilon' - \frac{x}{a}\sqrt{\frac{m'}{2}} - \delta_,\right) \end{aligned}\right\} \text{................ (7),}$$

$$+ \&c.$$

where D and δ are determined by the equations,

$$\left.\begin{aligned} D\cos\delta &= b + \frac{\sqrt{\pi}}{a}, \\ D\sin\delta &= \frac{\sqrt{\pi}}{a}; \end{aligned}\right\} \text{.................................... (8),}$$

and generally D_i and δ_i are determined by

$$\left.\begin{aligned} D\cos\delta_i &= b + \frac{\sqrt{2i\pi}}{a}, \\ D_i\sin\delta_i &= \frac{\sqrt{2i\pi}}{a}. \end{aligned}\right\} \text{.... .... (9).}$$

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* *Théorie de la Chaleur*, p. 430.

The temperature ζ is that external temperature which, if it actually existed, would, by hypothesis, produce the same effect on the Earth's internal temperature as those external causes which really exist. The expression for ζ, therefore, at any point of the Earth's surface, depends on

(1) The exchange of radiating heat between the proposed point and any external bodies, such as the stars, or the component particles of the atmosphere;

(2) The heat communicated to or abstracted from the Earth at the proposed point by the atmospheric particles immediately in contact with it, such transference of heat taking place by conduction;

(3) The heat emanating from the Sun, and arriving at the proposed point after suffering a certain degree of absorption, whether in passing through the atmosphere, or through planetary space.

Thus the quantity of heat which passes through an element ω of the Earth's surface at the point proposed (P) in time δt, may be represented by

$$p \,.\, (u - \zeta)\, \omega \delta t,$$

where p is a constant which measures the emissive power of the Earth's surface at P. Also if ξ be the temperature of surrounding space which would produce the same effect as (1), this effect may be represented by

$$\lambda\,(u - \xi)\, \omega \delta t;$$

and similarly the effect of (2) may be expressed by

$$\lambda_{,}(u - \eta)\, \omega \delta t.$$

Again, let σ denote the quantity of heat which proceeds from the Sun, and would, independently of the absorption above mentioned, fall on a unit of the Earth's surface. Then $\kappa\sigma$ may represent that part of this quantity which actually reaches the Earth's surface and enters it at P: and if θ be the angle of incidence at P, the quantity of solar heat which passes through ω in time δt will be equal to

$$\kappa\sigma \cos\theta \,.\, \omega \delta t.$$

Hence, we obtain

$$p\,(u - \zeta) = \lambda\,(u - \xi) + \lambda_{,}(u - \eta) - \kappa\sigma \cos\theta,$$

and since we must here have

$$p = \lambda + \lambda_{,},$$

we have

$$\zeta = \frac{\lambda}{\lambda + \lambda_{,}}\,\xi + \frac{\lambda_{,}}{\lambda + \lambda_{,}}\,\eta + \frac{\kappa\sigma \cos\theta}{\lambda + \lambda_{,}}.$$

In the sequel, Poisson assumes that the effect of the atmospheric particles in immediate contact with the Earth is much less than that produced by the inter-radiation between the

Earth's surface and the atmospheric particles within the sphere of this mutual radiation. On this supposition, which may be readily admitted, $\frac{\lambda_{\prime}}{\lambda}$ will be small, and we shall have approximately

$$\zeta = \xi + \frac{\kappa\sigma \cos\theta}{\lambda}.$$

Poisson shews how the complete value of ξ might be determined, but, as already intimated, experiment altogether fails to give us the *data* which are required to render this theoretical determination practically available. We have much better means, however, of determining the approximate value of the second term in the above expression for ζ.

If r denote the instantaneous distance of the Earth from the Sun, we shall have

$$\sigma = \frac{S}{r^2},$$

where S is a measure of the intensity of the Sun's radiation. Also θ is the Sun's zenith distance, and, therefore,

$$\cos\theta = \sin\mu \sin\gamma \sin\upsilon + \cos\mu \cos\psi \sqrt{1 - \sin^2\gamma \sin^2\upsilon},$$

where μ = latitude of the place,
γ = obliquity of the ecliptic,
υ = Sun's longitude,
ψ = Sun's angular distance from the meridian, υ and ψ being consequently functions of the time t. If then we substitute for $\cos\theta$, we have

$$\frac{\kappa\sigma \cos\theta}{\lambda} = \frac{\kappa S}{\lambda r^2} \{\sin\mu \sin\gamma \sin\upsilon + \cos\mu \cos\psi \sqrt{1 - \sin^2\gamma \sin^2\upsilon}\},$$

in which expression, neglecting the eccentricity of the Earth's orbit, r may be taken as the mean distance of the Earth from the Sun. The expression will only be true when the Sun is above the horizon of the place, $\sigma \cos\theta$ being zero when the Sun is below the horizon. Hence, $\frac{\kappa\sigma \cos\theta}{\lambda}$ is a discontinuous function of ψ, and can be expressed by a well-known formula in a series of terms containing respectively the cosines of ψ and of its multiples. Effecting this transformation, our author obtains

$$\frac{\kappa\sigma \cos\theta}{\lambda} = \frac{\kappa S}{\pi\lambda r^2} \{V + V_1 \cos\psi + V_2 \cos 2\psi + \&c\},$$

where

$$V = \frac{\pi}{2} \sin\mu \sin\gamma \sin\upsilon + Q + Q_1 \cos 2\upsilon + Q_2 \cos 4\upsilon, \&c.,$$

Q, Q_1, &c. being numerical quantities the values of which are immediately expressible by means of elliptic integrals. V_1, V_2, &c. may be expressed in a similar manner. Thus we have

$$\zeta = \xi + \frac{\kappa S}{\pi\lambda r^2} (V + V_1 \cos\psi + \&c.),$$

which gives the value of ζ at any place whose latitude is μ, on the day when the Sun's longitude is υ, and at the time when his angular distance from the meridian is ψ.

The terms involving ψ will have for their period a day or a submultiple of a day; and, therefore, if we omit these daily inequalities, and restrict ourselves to the consideration of the mean value of ζ and its annual and semi-annual inequalities, we shall have, substituting for V, and putting $\frac{\kappa S}{\pi \lambda r^2} = h$,

$$\zeta = \xi + h\left\{\frac{\pi}{2} \sin\mu \sin\gamma \sin\upsilon + Q + Q_1 \cos 2\upsilon + \&c.\right\},$$

and, taking a year for the unit of time, we have

$$\upsilon = 2\pi t,$$

and, therefore,

$$\zeta = \xi + h\left\{\frac{\pi}{2} \sin\mu \sin\gamma \sin 2\pi t + Q + Q_1 \cos 4\pi t + \&c.\right\},$$

and therefore, the above equation may be written,

$$\zeta = \xi + hQ + h \,.\, \frac{\pi}{2} \sin\mu \sin\gamma \cos\left(2\pi t - \frac{\pi}{2}\right) + hQ_1 \cos 4\pi t + \&c.$$

This expression for ζ will coincide with that given above (p. 646), if we put

$$B = \xi + hQ,$$

$$A = h \,.\, \frac{\pi}{2} \sin\mu \sin\gamma,$$

$$A' = hQ,$$

$$\&c. = \&c.$$

$$m = 2\pi, \quad \epsilon = -\frac{\pi}{2},$$

$$m' = 4\pi, \quad \epsilon' = 0,$$

$$\&c. \quad \&c.$$

Hence the equation (7) becomes

$$\begin{aligned} u = {} & (\xi + hQ) \\ & + \frac{b}{D} h \frac{\pi}{2} \sin\mu \sin\gamma e^{-\frac{x}{a}\sqrt{\pi}} \cos\left(2\pi t - \frac{\pi}{2} - \frac{x}{a}\sqrt{\pi} - \delta\right) \\ & + \frac{b}{D_1} h Q_1 e^{-\frac{x}{a}\sqrt{2\pi}} \cos\left(4\pi t - \frac{x}{a}\sqrt{2\pi} - \delta_1\right) \\ & + \&c. \end{aligned}$$

18. This expression contains three unknown quantities a, b, and h. Poisson* thus determines their values. The third term, containing $Q_{,}$, and the subsequent terms con-

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* p. 499.

taining $Q_{,,}$, $Q_{,,,}$, &c., are very small on account of the smallness of those factors; so that the *maximum* and *minimum* values of u for any assigned value of x will be very approximately determined by those of the second term, provided μ be not too small. Let H be the difference between these values; then

$$H = \frac{b}{D} \cdot h \cdot \pi \sin \mu \sin \gamma \epsilon^{-\frac{x}{a}\sqrt{\pi}}.$$

Again, let H' be the value of H at a depth x', at the same place. Then

$$H' = \frac{b}{D} \cdot h \cdot \pi \sin \mu \sin \gamma \epsilon^{-\frac{x'}{a}\sqrt{\pi}}.$$

Therefore,

$$\frac{H'}{H} = \epsilon^{-\frac{x'-x}{a}\sqrt{\pi}},$$

whence a is determined by means of observed values of H, H', x and x'.

The value of b is determined by means of observed values of the times at which the above *maximum* and *minimum* values of u occur; and a and b being thus determined, the value of h is known by the preceding expression for H.

The values of these quantities are

$a = 5{,}11655$,
$b = 1{,}05719$,
$h = 35^0{,}925$ (C.).

These determinations were obtained by means of careful observations made for many years at Paris, on the temperature of the Earth at different depths. It must not, however, be supposed that these values are as accurately determined as the number of decimal places might seem to imply. The probable errors of observation render such accuracy unattainable. The exact times of the *maxima* and *minima* values of u are difficult to obtain, and hence arises a somewhat greater uncertainty in the value of b than in those of a and h. The above value is the mean of two determinations which differed from each other by nearly ,1. There can be no reason, however, to doubt the approximate accuracy of the above values as determined from the observations at Paris. Poisson remarks that h would probably be somewhat greater in lower, and somewhat less in higher latitudes, on account of greater general purity of the atmosphere in the former, and less purity in the latter case. Its value also depends on the ratio of the absorbing and radiating powers of the Earth's surface; and this ratio may possibly differ materially at different points not remote from each other. Its mean value, however, throughout a whole district, will probably differ but little from its mean value in another district; and this is the principal consideration in reference to its influence, which it is our object to ascertain, on the *mean temperature of the air* in any proposed latitude.

19. If u_0 denote the value of u at the Earth's surface, we have (putting $x = 0$ in the preceding expression for u)

$$u_0 = \xi + Q \cdot h$$
$$+ h \left\{ \frac{b}{D} \cdot \frac{\pi}{2} \sin \mu \sin \gamma \cos \left(2\pi t - \frac{\pi}{2} - \delta\right) + \frac{b}{D_1} Q_1 \cos (4\pi t - \delta_1) + \&c. \right\} \ldots\ldots\ldots (10).$$

Thus, in any latitude μ, the mean annual temperature of the Earth's surface will be equal to

$$\xi + Qh,$$

where Q depends on the latitude μ.

The temperature denoted by ξ depends upon the heat which the Earth receives from stellar space, and on that which it receives by radiation from the atmosphere*. Supposing the former source to be constant, the second will vary with all the causes which may locally affect the mean temperature of the air in the shade, at any proposed place. If the condition of the Earth were the same at every point of its surface, the mean annual temperature of its surface would be a function only of the latitude of the place. The principal deviations from this simple law arise from local causes, and are comprised in ξ. The value of h also may vary, as above explained, though probably in a very much less degree, with local influences.

Poisson shews that the value of Q is

$,95910$ when $\mu = 0$,
$,66662$ ... $\mu = 48^\circ 50'$ (at Paris),
$,398$ ... $\mu = 90^\circ$.

Also, $h = 35^\circ,925$ (C.).

Consequently, the direct effect of the solar heat is to increase the mean temperature of the Earth's surface by

$34^\circ,45$ (C.) at the equator,
$23^\circ,948$ at Paris (lat. $= 48^\circ 50'$),
$14^\circ,996$ at the pole.

On account of the general purer state of the atmosphere, however, at the equator, and its more clouded state at the pole, the first of the above numbers ought probably to be somewhat increased, and the last somewhat diminished, as above mentioned.

20. The second term in the expression for u_0 has a year for its period, and is the principal annual inequality in u_0, unless the latitude (μ) be that of a place near the equator.

The difference between its greatest and least values

$$= h\frac{b}{D} \cdot \pi \sin\gamma \sin\mu.$$

D is determined from the expressions for $D\sin\delta$ and $D\cos\delta$ (Art. 17); whence

$$D = 1,44573,$$

$$\therefore \frac{b}{D} = ,731.$$

Also $\gamma = 23^\circ 28'$,

$$\therefore \sin\gamma = ,398.$$

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* That which it receives from the air by actual contact has been omitted, in omitting the constant λ_1 as compared with λ (Art. 17).

Hence the above expression

$$= 32^\circ{,}88 \sin \mu.$$

At places sufficiently near the equator μ becomes so small that this term in the expression for u_0 becomes of no more importance than the next term, which is small on account of Q_1. Its period is half a year.

21. The whole effect of the solar heat on the mean temperature of the Earth's surface is not expressed by hQ, for there is also, as above explained, the effect of the higher temperature of the atmosphere superinduced by the higher temperature of the Earth's surface, and by the direct absorption of a certain portion of the solar heat in passing through the atmosphere. Though we have no sufficient *data* for calculating mathematically the amount of the effect thus produced, we may still form an estimate of it.

It has been above explained that ξ depends on the interchange of heat between the Earth on the one hand, and the atmosphere and stellar space on the other. Let ψ denote the value which ξ would have if it depended only on radiation from stellar space and the atmosphere in the state in which it would exist if the solar heat were absent; and let s denote that part of ξ which depends on the higher temperature of the atmosphere arising from the existing solar heat. We shall then have

$$\xi = \psi + s.$$

If we suppose the radiation of heat to the Earth from stellar space to be the same in all directions, the temperature of the atmosphere, in the absence of the Sun, would be the same in all latitudes. Consequently ψ would be constant, as I shall here suppose it to be, while s will be a function of the latitude, and will also be affected by any local causes brought into action by the difference of temperature in different latitudes. Ocean currents, for instance, which would produce no effect on local temperature if the general temperature were the same for all latitudes, do actually produce very considerable effects. We shall, however, eliminate in some degree these local irregularities in the values of s if we take its mean value for any proposed parallel of latitude, instead of its value for any particular place situated upon that parallel. In such case the values of s will only be affected by local causes so far as those causes may affect the mean of the temperatures for a whole parallel of latitude, and not merely the temperature of one particular part of it. Taking only that part of the expression for u_0 which gives its mean value, we have

$$u_0 = \psi + s + hQ.$$

The value of u_0 might be ascertained by observing the mean annual temperature of the surface of the Earth at each particular place, but very few observations of this kind have hitherto been made. We can only take, therefore, instead of them, observations on the mean temperature of the atmosphere. If we take for u the mean annual temperature for a whole parallel of latitude, the corresponding value of s will be referred likewise to the whole of the same parallel, for which also the corresponding value of Q can be calculated. Adopting Dove's results*, and calculating Q by the formulæ given by Poisson†, we have

$$\left.\begin{aligned} u_0 &= 26^\circ{,}5\ (\mathrm{C}) \\ Q &= {,}95910 \end{aligned}\right\}\ \text{at the Equator.}$$

$$\left.\begin{aligned} u &= 21^\circ \\ Q &= {,}77397 \end{aligned}\right\}\ \text{in latitude } 30^\circ\ \text{N.}$$

$$\left.\begin{aligned} u_0 &= 6^\circ{,}35 \\ Q &= {,}66662 \end{aligned}\right\}\ \ldots\ldots\ldots\ 48^\circ.50'\ \text{N.}$$

$$\left.\begin{aligned} u_0 &= -\ 7^\circ{,}5 \\ Q &= \ \ {,}47729 \end{aligned}\right\}\ \text{at the Arctic Circle.}$$

$$\left.\begin{aligned} u^0 &= -\ 16^\circ{,}5 \\ Q &= \ \ {,}398 \end{aligned}\right\}\ \text{at the Pole.}$$

Hence we obtain

$$\begin{aligned} \psi + s_0 &= u_0 - hQ \\ &= 26^\circ{,}5 - 34^\circ{,}5 \\ &= -\ 8^\circ\ (\mathrm{C})\ \text{at the Equator.} \\ \psi + s_1 &= 21^\circ - 27^\circ{,}72 \\ &= -\ 6^\circ{,}72\ \text{in lat. } 30^\circ. \\ \psi + s_2 &= 6^\circ{,}35 - 23^\circ{,}76 \\ &= -\ 17^\circ{,}41\ \text{in lat. } 48^\circ.\ 50'. \\ \psi + s_3 &= -\ 7^\circ{,}5 - 17^\circ{,}3 \\ &= -\ 24{,}^\circ 8\ \text{at the Arctic Circle.} \\ \psi + s_4 &= \\ &= -\ 16^\circ{,}5 - 14^\circ{,}4 \\ &= -\ 30^\circ{,}9\ \text{at the Pole.} \end{aligned}$$

From these equations we can obtain the differences between any two of the quantities s_0, s_1, &c. Thus (omitting for the present s_1 which would give an anomalous result) we have

$$\begin{aligned} s_0 - s_2 &= 9^\circ{,}41, \\ s_0 - s_3 &= 16^\circ{,}8, \\ s_0 - s_4 &= 22^\circ{,}9. \end{aligned}$$

These numbers express the differences between the effects of the Sun in different latitudes on the Earth's superficial temperature (u_0), arising from the increased temperature of the atmosphere, in addition to the direct effect of the heat which radiates through the atmosphere and is absorbed by the Earth's surface. Thus this effect on u_0 at the equator exceeds the similar effect in the latitude of Paris by $9^\circ{,}41$ (C), at the Arctic Circle by $16^\circ{,}8$, and at the Pole by $22^\circ{,}9$. Let us compare these differences with the differences in the quantities of heat which radiates directly from the Sun to the Earth's surface in the above latitudes respectively. If we denote hQ by H, and adopt a notation analogous to the above, we shall have

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* *Distribution of Heat over the Surface of the Globe*, p. 14.

† *Théorie de la Chaleur*, p. 487.

$$H_0 - H_2 = 10^\circ{,}84 \text{ (C)},$$

$$H^0 - H_3 = 17^\circ{,}2,$$

$$H_0 - H_4 = 20^\circ{,}1.$$

Hence
$$\frac{s_0 - s_2}{H_0 - H_2} = \frac{9{,}41}{10{,}84} = {,}87,$$

$$\frac{s_0 - s_3}{H^0 - H_3} = \frac{16{,}8}{17{,}2} = {,}97,$$

$$\frac{s_0 - s_4}{H_0 - H_4} = \frac{22^\circ{,}9}{20{,}1} = 1{,}14.$$

22. It will be observed that in the *data* from which these results have been deduced, I have omitted the value of $s_0 - s_1$, the difference between the values of s at the equator and in the latitude of 30° N. It leads, as above remarked, to an anomalous result. In fact we have

$$s_0 - s_1 = -1^\circ{,}18,$$

which would give the particular effect of the solar heat, expressed in s, greater in the latitude of 30° than at the equator. The anomaly of s_1 being greater than s_0 is due to the circumstance, that the effect of currents and other local causes has been only partially eliminated in the lower latitudes by taking for u_0 the mean value for a whole parallel. In fact the influence of these causes is such that the parallel of maximum annual temperature, instead of coinciding with the equator, lies in about 10° N. lat. The anomaly would be in some degree corrected by taking the mean of the values of u_0 for 30° N. and 30° S. latitude; but the difference between s_0 and s_1 would still be much smaller than that which, I conceive, must necessarily result from the difference of position with reference to the Sun, of the equator, and either parallel of 30° of latitude. On referring to the values above given of the temperatures (u_0) of each parallel, it will be observed that the difference of these values for the equator and the latitude of 30° is only 5°,5, while the difference for the latter parallel and that of 48°. 50′ amounts to 14°,65, though the difference of latitude in the latter case little exceeds half that in the former. This is due, I conceive, to the peculiar position of the region immediately about the equator as the region of maximum temperature. If the surface of the Earth were perfectly uniform, and there were no atmospheric currents producing a *horizontal* transference of heat from one parallel of latitude to another, the influence of the Sun expressed in s would be some determinate function of the latitude, equally with u_0. But the values which u_0, and therefore also s, would have on this hypothesis in any assigned latitude, would manifestly be changed by the difference between the heat brought to the proposed region by horizontal transference, and that carried away from it in a similar manner, in a north and south direction. Now this compensation afforded by the heat which is brought, for that which is carried away must generally exist in a more or less perfect degree in those latitudes which are intermediate to the equatorial and polar regions, while in those regions themselves no such compensation can exist; and this must be a reason for a more approximate equability of temperature in those regions than if the transference of heat were entirely vertical. Thus in the equatorial region the heat must constantly be carried by atmospheric currents and other causes both to the north and to the south, and the mean temperature of that region must be materially diminished, while that of the regions immediately contiguous must receive a corresponding increase. This is probably the principal cause of the anomaly which has given rise to these remarks. Let us examine how far it will be removed by an assumed amount of the change in the values of the mean temperature at the equator and on the parallel of 30° N. For this purpose let us take the following values of u_0, instead of those above given (Art. 21), as assumed approximate values of the mean annual temperatures of different parallels, supposing that no horizontal transmission of heat should take place by atmospheric or local causes, so that the corresponding values of s may give the real effect of the Sun on the temperature of the air in different latitudes, in addition to the effect produced upon it by the heating of the surface of the Earth, arising from the solar heat which radiates directly upon it.

$u_0 = 30^0$ (C) at the Equator,
$= 16^0,50$ in lat. 30,
$= 6^0,35$ …… $48^0. 50'$,
$= -7^0$ at the Arctic Circle,
$= -15^0$ at the Pole.

I have here added $3^0,5$ to the observed mean temperature of the equator, and deducted $4^0,5$ from that of the parallel of 30^0 N., which is rather higher than the mean of the north and south parallel of 30^0. I have increased the temperature at the pole by $1^0,5$: the other temperatures are not altered. We should thus have a depression of temperature of $13^0,5$ (C) in passing from the equator to the parallel of 30^0; while it would equal $20^0,5$ between that parallel and that of 60^0, and 11^0 between the latter parallel and the pole.

23. We may also here make another correction of which we know the *sign*, but can only assume the amount. Poisson has calculated the numerical value of h by observations made at Paris; but since it depends on the intensity of solar radiation after the heat has passed through the atmosphere, he suggests, as before stated, that its value must be somewhat greater at the equator and somewhat less at the pole than at Paris. Instead of using $34^0,4$ for the value of hQ at the equator, I shall now use 36^0; and instead of $14^0,4$ at the pole, I shall substitute $12^0,4$. This is equivalent to increasing the value of h about $1^0,7$ at the equator, and diminishing it about 5^0 at the pole. Changing the values of h, also, for different latitudes in a proportionate degree, we shall have

$hQ = H_0 = 36^0$ (C) at the Equator,
$H_1 = 29^0$ in lat. 30^0,
$H_2 = 23^0,76$ in lat. $48^0. 50'$,
$H_3 = 16^0,5$ at the Arctic Circle,
$H_4 = 12^0,4$ …… Pole.

Hence we deduce as before the following corrected results:

$$s_0 - s_1 = 6^\circ{,}5 \qquad H_0 - H_1 = 7^\circ$$
$$s_0 - s_2 = 11^\circ{,}4 \qquad H_0 - H_2 = 12^\circ{,}2$$
$$s_0 - s_3 = 17^\circ{,}8 \qquad H_0 - H_3 = 19^\circ{,}5$$
$$s_0 - s_4 = 22^\circ{,}4. \qquad H_0 - H_4 = 23^\circ{,}6.$$

Therefore
$$\frac{s_0 - s_1}{H_0 - H_1} = {,}93$$
$$\frac{s_0 - s_2}{H_0 - H_2} = {,}92$$
$$\frac{s_0 - s_3}{H_0 - H_3} = {,}91$$
$$\frac{s_0 - s_4}{H_0 - H_4} = {,}95.$$

These results are free from the anomaly presented by those previously obtained. Taking the mean of them, we may assume generally, where s and H correspond to any assigned latitude, that

$$\frac{s_0 - s}{H_0 - H} = {,}93,$$
$$\text{or} \quad s_0 - s = {,}93\,(H_0 - H).$$

We may also take the condition that s and H would vanish together, which gives

$$s_0 = {,}93H_0,$$

and therefore

$$s = {,}93H,$$
$$= {,}93hQ,$$

as the general relation between s and H.

24. The whole effect, it will be recollected, of the solar heat on the mean annual temperature of the Earth's surface in any proposed latitude consists of the direct effect (hQ) of solar heat radiating upon the Earth, and the indirect effect (s) of that heat arising from the consequent elevation of the temperature of the atmosphere. The above result shews *that the latter of these is very nearly equal to the former*, or rather that it would be so if there were no causes for a lateral transference of heat perpendicular to the parallels of latitude, and no local causes of disturbance of mean annual temperature. We may assert, therefore, in general terms, that the mean annual temperature of the surface of the Earth at any point is higher by nearly $2hQ$ than it would be if the solar heat did not exist. And the same proposition may be enunciated equally with reference to the lower stratum of the atmosphere, which, as already stated, may be considered to have the same mean temperature as the surface of the Earth itself

25. Let us further consider what the superficial temperature of our globe would be independently of the Sun's heat. Since h and s would then vanish, the corresponding value of the temperature u_0 would become equal to ψ. Now we have, generally,

$$\begin{aligned}\psi &= u_0 - hQ - s \\ &= u_0 - 1{,}93\,hQ,\end{aligned}$$

an equation which gives ψ when u_0 and hQ are given for any latitude. Since ψ is by hypothesis constant, the same value would be obtained for every latitude. Thus taking any of the above cases for which u_0 and hQ are given, we find

$$\psi = -39^\circ{,}5\ (\mathrm{C}),$$

which is also the required value of u_0.

It must be borne in mind that this value of ψ is not the temperature of that portion of stellar space in which the solar system is placed (considered independently of the heat radiating from the Sun), but that uniform temperature which, in the absence of solar heat, would exist at every point of the Earth's surface. It would differ from the temperature of surrounding space, either in excess or defect, in consequence of the Earth's atmosphere, as already explained. The above value of ψ expresses the limit towards which the mean superficial temperature of our globe would ultimately approximate if the solar heat were gradually to diminish and become finally extinct, supposing always, that all other external conditions should remain unaltered.

We have here been considering *mean temperatures*. Let us now proceed to the consideration of *annual variations* of terrestrial temperature.

26. Though the mean annual temperature of the lowest portion of the atmosphere as given by a thermometer sheltered from the direct rays of the Sun, at any place, may be considered the same as the mean annual temperature of the Earth's surface at that place, it is manifest that there can be no such accordance between their respective variations of temperature, where the periods of such variations are sufficiently short. The daily changes of temperature in the atmosphere are frequently far greater than the corresponding changes in that of the surface of the Earth at the same place. To make some approximate comparison between the *annual variations* of the temperature of the surface of the Earth, and those of its atmosphere, I have extracted the following table from M. Dove's Work, already referred to. It gives the difference between the temperature of those two months in the year, for one of which the temperature is greater, and for the other less, than for any other month, under different parallels of latitude respectively, the monthly temperatures of each parallel being taken as the mean of the monthly temperatures of different places situated upon that parallel. Calling these differences $D_1 D_2$. &c., we have

$D_1 = 2^\circ$ (C) in latitude 10°, N.
$D_2 = 6^\circ{,}5$ .... 20°,
$D_3 = 12^\circ{,}22$ .... 30°,
$D_4 = 17^\circ{,}90$ .... 40°,
$D_5 = 23^\circ{,}75$ .... 50°,
$D_6 = 29^\circ{,}35$ .... 60°,

$D_7 = 31^\circ,65$ .... 70°,
$D_8 = 30^\circ,30$ .... 80°,
$D_9 = 31^\circ,75$ .... 90°.

In the expression for u_0, (Art. 19) the first periodical term gives nearly the whole amount of the annual variation when the latitude (μ) is not too small, the value of Q_1 being small. At the equator it is 0,04152, and in the latitude of Paris only 0,00253, and it decreases with the increase of latitude. Restricting ourselves, then, to the principal inequality in u_0, and denoting by Δ the difference between the greatest and least values of its coefficients, we have

$$\Delta = 32^\circ,88 \ \sin \mu, \text{ (Art. 20)},$$

and hence

$\Delta_1 = 5^\circ,69$ (C) in latitude 10°, N.
$\Delta_2 = 11^\circ,24$ .... 20°,
$\Delta_3 = 16^\circ,44$ .... 30°,
$\Delta_4 = 21^\circ,11$ .... 40°,
$\Delta_5 = 25^\circ,18$ .... 50°,
$\Delta_6 = 28^\circ,47$ .... 60°,
$\Delta_7 = 30^\circ,90$ .... 70°,
$\Delta_8 = 32^\circ,31$ .... 80°,
$\Delta_9 = 32^\circ,88$ .... 90°.

Comparing the corresponding values of D and Δ, it will be observed that, in the higher latitudes, they are nearly the same, but that the former are considerably less than the latter in the lower latitudes. This will necessarily be the case wherever there are causes which tend especially to an equalization of temperature. Such is the case, as I have shewn, near the equator. It seems probable that, in the absence of such causes, there would be little difference between the annual variations of temperature in the surface of the Earth, and those of the contiguous portion of the atmosphere. The latter probably approximate to the former as their limit.

27. It will be observed, that the annual inequality in the expression for u_0 is multiplied by the factor $\frac{b}{D}$, and the semi-annual inequality by the factor $\frac{b}{D_1}$; and from the equations (Art. 17), we have

$$D^2 = b^2 + 2\sqrt{\pi}\ \frac{b}{a} + \frac{2\pi}{a^2},$$

$$D_1^2 = b^2 + 2\sqrt{2\pi}\ \frac{b}{a} + \frac{4\pi}{a^2};$$

$$\frac{D^2}{b^2} = 1 + 2\sqrt{\pi}\ \frac{1}{ab} + \frac{2\pi}{a^2b^2},$$

$$\frac{D_1^2}{b^2} = 1 + 2\sqrt{2\pi}\ \frac{1}{ab} + \frac{4\pi}{a^2b^2}.$$

I have hitherto spoken of a and b as two constants to be determined by observation (Art. 18), but it is important to consider their physical meaning. Now

$$a^2 = \frac{k}{c}, \quad \text{and } b = \frac{p}{k},$$

where k = the conductivity of the matter of the Earth's crust near to its surface;

c = its specific heat;

p = the transmitting power of the Earth's surface.

Hence,

$$\frac{1}{a^2b^2} = \frac{ck}{p^2};$$

and, therefore, if the ratio $\frac{c}{p}$ of the specific heat to the transmitting power of the surface, or the ratio $\frac{k}{p}$ of the conductivity to the same power, be increased, $\frac{1}{ab}$ will be increased, as will also, therefore, $\frac{D}{b}$ and $\frac{D_1}{b}$. Consequently the coefficients of the two periodical inequalities above mentioned will be diminished. This ought manifestly to be the case; for if a given quantity of heat pass in a given time into a planet, for example, through a given area of its surface, the effect on the temperature of the matter into which it passes will be the less, the greater the specific heat of that matter; and the greater its conductivity, the more rapidly will the incident heat be carried away downwards, and the less will be its effect in raising the superficial temperature. In this latter case, the *amplitude* of the inequality, or the depth to which it would sensibly extend, would be increased, but its magnitude at the surface would be diminished. It is obvious how much periodical inequalities in the superficial temperature of a body like the Earth, and therefore also in the temperature of the lowest stratum of its atmosphere, might be modified by an adaptation of the specific heat and conductivity of the matter composing it, and the transmissive power of its surface, to the conditions under which it may be placed.

28. It may be well to give a brief summary of the previous portions of this paper before I proceed to apply the results obtained with reference to the Earth to the other planets of the solar system.

(1) I have endeavoured to explain, as fully as possible, the manner in which the atmosphere produces its influence on the temperature of the Earth's surface; and to shew that the temperature of the constituent particles of the atmosphere at its upper limit must be exceedingly low.

(2) If as much atmosphere were removed from the Earth as now forms an inferior stratum of about 525 feet in thickness, the mean annual temperature of each point on the Earth's surface would be diminished by 1^0 (C). In different latitudes at the same period of the year, and in different seasons at the same place the thickness of the stratum of atmosphere corresponding to 1^0 (C) would be somewhat different, being probably somewhat

greater in high than in low latitudes, and in summer than in winter. If the thickness of the stratum removed were greater the diminution of temperature would be greater in the same ratio, as proved by Mr Welsh's observations to the height of 23,000 feet, and as may be thence inferred, for much greater heights. An increase in the quantity of atmosphere would produce an exactly similar but opposite effect on the temperature at the Earth's surface.

(3) After explaining Poisson's method of treating the problem of terrestrial temperature, and his mode of estimating the direct effect of the solar heat radiating to the Earth, on the temperature of its surface and of the atmosphere in contact with it, I have shewn that the *whole effect* of the solar heat on the mean value of this superficial temperature is equal to nearly twice the immediate effect which was the object of Poisson's calculation, except in cases in which there may be a considerable horizontal transference of heat from local causes.

(4) Assuming that the temperature of every part of the Earth's surface would be the same in the entire absence of the Sun's influence, I have estimated that temperature at $-39^{\circ},5$ (C). This must not be confounded with the temperature of that portion of stellar space in which the solar system now exists, a temperature which we have not the means of determining.

(5) I have also shewn that *annual inequalities* in the temperature of the lower stratum of the atmosphere are nearly equal in amount to those in the Earth itself at its surface, with exceptions similar to those mentioned in (3).

With these results we are prepared to estimate the external temperature of the planets under certain assumed conditions.

29. I commence with Jupiter. The intensity of solar radiation at the distance of Jupiter, compared with that at the distance of the Earth is nearly as 1 : 27. Consequently the value of the quantity which has been heretofore denoted by h will, in the case of Jupiter, be only $\frac{1}{27}$th of its value in the case of the Earth, so far as it depends only on the intensity of radiation. Conceive the Earth itself to be placed at the same distance from the Sun as Jupiter. If there were no sensible effect from solar radiation, the temperature of every part of the surface would become the same, supposing equilibrium of internal and external temperature to have become established in the same degree as it is in our globe at present.

This temperature I have estimated at $-39^{\circ},5$ (C) (Art. 25), assuming the intensity of stellar radiation towards any point of the solar system to be the same in all directions. To this temperature we must add that due to solar radiation at the distance of Jupiter. The whole effect of this radiation, according to our calculation, would be nearly $2hQ$; and since hQ will, in the present case, be $\frac{1}{27}$th part of its actual value on the Earth, we shall have

$$2hQ = 2\frac{36^{\circ}}{27} = 2^{\circ},6 \text{ (C)}$$

at the equator. Consequently, if the Earth were placed at the distance of Jupiter, the *mean annual temperature* at the equator would be $-36^{\circ},9$ (C), or rather more than 20 (C) lower than the actual mean temperature of the N. pole. The mean temperature of the pole, at the distance of Jupiter, would manifestly very little exceed $-39^0,5$ (C).

The annual inequality in the temperature in any latitude (μ) would be about

$$\pm \frac{16^0,44}{27} \sin \mu$$

$$= \pm 0^0,6 \sin \mu,$$

so that the temperature at each place would be sensibly constant during the year; and in passing from the equator to the poles the whole change of temperature would never exceed about 3^0 (C).

Such would be the temperature of the Earth revolving about the Sun in Jupiter's orbit, and having her axis of rotation inclined at the same angle as at present to the plane of her orbit. I omit all consideration of the effect which a reduced general temperature of the lower portion of her atmosphere would have upon its height, density, transmissive power, &c., as being unnecessary in the general estimate which only we can here pretend to make, of what would be the thermal condition of the Earth under the circumstances supposed. But if now we suppose her atmosphere to be increased, there would be a corresponding increase of temperature which would be sensibly the same at every point of her surface. It has been shewn that the temperature decreases, according to Mr Welsh's observations, at the rate of $3^0,37$ (F), or $1^0,85$ (C), for each 1000 feet of ascent; and since this rate may be regarded as independent of the density (as far as observation has extended), we conclude that if the thickness of the atmosphere were increased by 1000 feet, the mean temperature of the surface of the Earth would, in her actual orbit, be increased by the above amount. If she moved in Jupiter's orbit, there is reason to believe that the increase of temperature under the same circumstances would be somewhat less, on account of the lower temperature to which the surface of the Earth would be reduced in this case, while at the upper limit of the atmosphere it would be the same as at present under our assumed condition of the equality of temperature in surrounding space. I have already pointed out the probable influence of this cause in the result of Mr Welsh's observations (Art. 15), and have given a rough estimate of the effect which it might produce at the pole where the mean temperature is about 80^0 (F) lower than in our own latitude. Our estimated temperature of the Earth's equator, if she moved in Jupiter's orbit, is still lower by about 37^0 (F). In this case, therefore, we may conclude that the decrement of temperature in ascending would be somewhat less than it is actually now at the N. or S. pole. Thus, suppose the present decrement at the pole to be 1^0 (F) for about 400 feet of ascent, instead of about 300 feet as in this latitude; then the height corresponding to the same decrement of temperature if the Earth moved in Jupiter's orbit might be estimated at about 450 feet. Other causes with which we are but imperfectly acquainted might possibly have some effect also in diminishing the rate of decrement of temperature in ascending; but taking the most unfavourable case that can be supposed, admitting always that stellar heat has the same power as solar heat in radiating through the Earth's atmosphere, the general conclusion that if the Earth moved in Jupiter's orbit, the decrement of temperature in ascending would be something like what I have here stated, is unquestionable, and the necessary consequence would be that the addition of 35,000 or 40,000 feet to the thickness of her atmosphere would elevate her equatorial temperature to about the mean temperature of our temperate zone at present.

It is easy to pass from this hypothetical case of the Earth moving in Jupiter's orbit, to the case of Jupiter himself, supposing him to have an atmosphere of the same nature as to the transmission of heat as that of the Earth. The principal difference would be that an atmosphere of given height would have a greater density about Jupiter than about the Earth, on account of his greater attraction. But the very approximately uniform rate of decrement of temperature in our actual atmosphere up to a height at which the density is less than half that at the Earth's surface, implies that the transmissive power of the atmosphere is, in a great degree, independent of its density; so that the greater density of Jupiter's atmosphere, *cæteris paribus*, would probably have no material influence on the rate at which the temperature would decrease in ascending from his surface. There seems reason to believe that Jupiter has an atmosphere much greater than that of the Earth, and it may be of such a thickness as entirely to counterbalance the defect of heat derived from the Sun by direct radiation. His climate, however, must differ from that of the planets nearer to the Sun, in the comparative absence of periodical variations, which can scarcely be sensible in his equatorial regions, and can never amount to more than a few degrees in his polar regions.

30. The cases of the planets more remote than Jupiter are still more simple than that of Jupiter himself, since solar radiation can have no sensible effect whatever upon them. If the Earth were placed in the position of any one of these planets, and equilibrium of temperatures due to internal and external causes, respectively, were so far established as it is under existing conditions, her superficial temperature would, according to our previous calculations, be about $-39^{0},5$ (C); but if her atmosphere were increased to the requisite amount, this temperature might be increased, as we have seen, in any proposed degree. And so it must be with these remote planets themselves. Their superficial temperatures must be entirely uniform independently of variations due to local or temporary causes, and must depend on the quantity of atmosphere by which they may be respectively surrounded.

31. It seems to be an opinion very generally received that a superficial temperature may perhaps be preserved in these remoter planets, higher than that which may be due to external causes, by the remains of the primitive heat which they may originally have possessed. This notion, however, is contrary to all analogy with the case of our own globe; for it is distinctly proved that the superficial temperature of the Earth has already approximated within the very small fraction of a degree to that ultimate temperature below which it can never descend so long as its atmosphere and all other external conditions remain unaltered. This close approximation to its final state can be due only to the enormous length of time during which the Earth must have been exposed to the cooling influences of surrounding space; and these influences must have been equally effective with respect to the other planets as with respect to the Earth, supposing them to have existed for the same period of time, for the magnitudes of the planets would have little effect on their superficial temperature after a sufficient lapse of

time, though it must produce a great effect on their internal temperatures at points sufficiently remote from their surfaces. It is easy to imagine that these remoter planets may be older than the Earth, as they must be, for instance, according to Laplace's theory or any analogous one; but no hypothesis can be more perfectly arbitrary, or less sanctioned by all theory, than that of the more recent formation of the remoter than of the nearer planets. For these reasons I consider that the cause now spoken of cannot be admitted among those which can be regarded as exercising any probable influence on the present superficial temperatures of the planets.

32. The planet Mars is usually recognised as having more in common with the Earth than any of the other planets. It receives a sufficient quantity of heat from the Sun to mark the different seasons of its year, and the inclination of the axis of rotation to a perpendicular to the plane of its orbit, which is about 28°. 42′, does not differ much from the corresponding inclination in the case of the Earth. Its mean distance from the Sun is to that of the Earth in the ratio of 1,52 : 1, so that the intensity of solar radiation at the distance of Mars to that at the distance of the Earth from the Sun, is nearly as the ratio ,44 : 1. Hence taking the value of h at the equator and pole of the Earth as 36° (C) and 12°,4 respectively, we may take the corresponding values for Mars as 16° and 5°,5. The value Q at the equator of Mars will be rather less than ,9591, its value at the Earth's equator, on account of the greater obliquity of Mars. Without farther calculation we may assume it to be ,9. At the pole it is equal to the sine of the obliquity = ,48. Hence we have

$$hQ = 14^\circ{,}4 \text{ (C) at the equator of Mars,}$$
$$\text{and} = 2^\circ{,}6 \text{ .... pole.}$$

Now in the entire absence of the Sun's influence, the uniform superficial temperature of the planet, with an atmosphere equal to that of the Earth, would $= -39^\circ{,}5$ (C) (Art. 25). Hence if the Earth with her present atmosphere had been placed in the orbit of Mars, with an obliquity equal to that of Mars, the mean temperature of the equator would have been

$$= -39^\circ{,}5 + 2hQ,$$
$$= -10^\circ{,}7 \text{ (C)},$$
$$= -12^\circ{,}7 \text{ (F)};$$

and the mean temperature of the pole would have been

$$= -39^\circ{,}5 + 5^\circ{,}2,$$
$$= -34^\circ{,}3 \text{ (C)},$$
$$= -29^\circ{,}7 \text{ (F)}.$$

If, however, we allow to Mars an atmosphere similar to that of the Earth but of greater thickness, to the amount of some 15000 or 20000 feet, the mean temperature may be 60° (F), or 15°,5 (C) at the equator, and about 14° or 15° (F) at the pole, supposing the additional atmosphere to have rather less effect at the pole than at the equator. Any greater increase of temperature would, of course, result from a still greater increase of atmosphere.

The coefficient of the annual inequality for any latitude, supposing $\frac{b}{D}$ to be constant, depends upon the value of $h \sin\gamma$ (Art. 20), (γ being the obliquity). Now in the case of the Earth

$$h \sin\gamma = 14^0{,}33 \text{ (C)};$$

and in the case of Mars

$$h \sin\gamma = 7^0{,}68;$$

and hence the coefficient of the inequality, which in the case of the Earth

$$= \pm 16^0{,}44 \sin\mu,$$

becomes in the case of Mars,

$$= \pm 8^0{,}71 \sin\mu.$$

Consequently the variations of temperature in different seasons of Mars would little exceed half the corresponding variations on the Earth, assuming $\frac{b}{D}$, or the ratio $\frac{ck}{p^2}$, to be the same in both planets (Art. 27). If it be different, the values of this inequality in Mars and the Earth must bear a different ratio to each other. We may remark, however, that the least value of $\frac{D}{b}$ and, therefore, the greatest value of $\frac{b}{D}$ is unity, to which it approximates as $\frac{1}{a^2b^2}$ or $\frac{ck}{p^2}$ approximates to zero. Now the value of $\frac{b}{D}$ for the Earth, as above given (Art. 25), is ,731. Consequently, the annual inequality of temperature, so far as it depends on the conductivity and specific heat of the superficial matter of the planet, and the transmissive power of its surface, cannot be much greater in any other planet than in our own globe. On the other hand, it may be diminished in any ratio by the increase of the quantity $\frac{ck}{p^2}$.

33. Let us now take the case of Venus. Her distance from the Sun is such that the intensity of the solar radiation must be very nearly twice that at the distance of the Earth. Hence we may take generally,

$$h = 70^0 \text{ (C)}.$$

And first let us suppose the Earth, with its existing atmosphere, to occupy the place of Venus, with the inclination of her axis of rotation to the plane of her orbit the same as at present. Then should we have

$$hQ = 67^0{,}2 \text{ (C) at the equator,}$$

$$\text{and} \quad = 28^0 \quad \ldots\ldots \text{ pole;}$$

and if we suppose the entire effect of solar heat on the superficial temperature of the planet to be $2hQ$, in the absence of all horizontal transference of heat, we should have the mean temperature of the equator

$$= -39^0{,}5 + 134^0{,}4$$

$$= 94^0{,}9 \text{ (C)};$$

and the mean temperature of the pole

$$= 16^\circ,5 \text{ (C)}.$$

The former of these temperatures would doubtless be much reduced by a lateral transference of heat both to the north and south from the equatorial regions, by which we may conceive the mean temperature of those regions to be reduced to between 70° and 80° (C). The mean temperature of the poles would probably be somewhat greater than the above.

The greatest value of the annual inequality at the poles would be nearly twice as great as at present, or about

$$\pm 32^\circ \text{ (C)}.$$

Such would be the temperature of Venus in her equatorial and polar regions, with intermediate temperatures for the intermediate regions, supposing her to have the same atmosphere as the Earth, and her superficial crust to have the same properties with respect to heat as that of our own globe. By a diminution of the atmosphere and an increase of the value of $\frac{ck}{p^2}$ (Art. 27), both the mean temperature and the annual inequality might be diminished in any assigned degree.

34. I have here supposed the obliquity in Venus to be the same as that of the Earth, but the observations on this planet are so difficult to make as to leave the determination of this element in great uncertainty. It has been estimated by some astronomers at as much as 70° or 75°, and more recently M. de Vico has estimated the inclination of the equatorial plane of Venus to the ecliptic at 53°. 11'. 26"[*]. Allowing this amount of obliquity, the seasons of Venus must manifestly be utterly different from the terrestrial seasons. In this uncertainty as to the real obliquity in this planet, I will take the extreme case of 75°. The temperatures being calculated for this and the case already given, it will not be difficult to form a general estimate for any intermediate case.

I have calculated the values of Q by Poisson's formulæ[†], with an obliquity of 75°, for the equator, the arctic circle (latitude of 15°), and for the pole, cases for which the general expression for Q becomes much simplified. I thus find the values of Q to be

,685 at the Equator,
,701 in lat. 15°,
,966 at the Pole.

The semi-annual inequality of temperature becomes, in this case, of importance in the region corresponding to our tropical region, extended in Venus to 75° of latitude. Its value depends on Q_1 the value of which I have calculated for the equator, where this inequality has its greatest relative importance. I find

$$Q_1 = ,355.$$

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[*] See *The Planetary Worlds*, by Mr Breen, of the Cambridge Observatory, p. 154.

[†] *Théorie de la Chaleur*, Art. 215.

Hence we have

$$hQ = 48^\circ \text{ (C) at the Equator,}$$
$$= 49^\circ \text{ in lat. } 15^\circ,$$
$$= 67^\circ{,}6 \text{ at the Pole;}$$

and consequently, taking, as before, the mean temperature

$$= -39^\circ{,}5 + 2hQ,$$

we have the mean annual temperature of Venus

$$= 56^\circ{,}5 \text{ (C) at the Equator,}$$
$$= 58^\circ{,}5 \text{ in lat. } 15^\circ,$$
$$= 95^\circ{,}7 \text{ at the Pole.}$$

These would be the temperatures supposing the atmosphere to be exactly like that of the Earth, and assuming the absence of all horizontal transference of heat. The greater heat, however, would necessarily produce a greater evaporation, and the atmosphere would consequently be far more laden with moisture, so that the absorption of the solar heat in its passage to the surface of Venus would probably be considerably greater than in the case of the Earth. The value of h would be thus reduced. Let it become $70^\circ - t$. Then will each of the values of the above mean temperatures be reduced by $2Qt$. Again, on account of the horizontal transference of heat from the pole, now become the point of maximum temperature, the mean temperature there would undoubtedly become considerably lowered. Let it be thus diminished by t'; then the mean temperature at the pole would become

$$= 95^\circ{,}7 - 2Qt - t'$$
$$= 95^\circ{,}7 - 2t - t' \text{ very nearly.}$$

The mean temperature at the equator would also be depressed $2Qt$ by the diminution of h, and would probably be somewhat raised by the tendency to equalization of temperature. It would therefore be

$$56^\circ{,}5 - 2Qt + t'',$$

where $Q = {,}685$.

These results would further be subjected to the modification due to an increase or diminution of atmosphere as compared with that of the Earth. If the atmosphere of Venus be equal to that of the Earth diminished by a quantity equal to that which forms a stratum resting on the Earth's surface, and whose thickness $= a$, the mean temperature would be reduced generally by about $1^\circ{,}85a$, the unit of length being = 1000 feet (Art. 15). Hence the mean temperature at the pole would

$$= 95^\circ{,}7 - 2t - t' - 1^\circ{,}85a,$$

and that at the equator would

$$= 56^\circ{,}5 - 1{,}37t + t'' - 1^\circ{,}85a.$$

If the atmosphere of Venus be supposed equal to that of the Earth, the value of t would, I conceive, be necessarily positive; but if we suppose that atmosphere to be diminished, as implied by the last term in the above expressions, its value would be reduced and might become negative, but it would seem extremely improbable that the transparency of Venus's atmosphere should so far exceed that of the Earth as to render any negative value of t otherwise than very small. On the contrary, a diminished transparency of the atmosphere of that planet compared with that of the Earth (such as the appearances of the planet would seem to render probable) might render a positive value of t of considerable magnitude. The quantities t' and t'' must manifestly (unless especially affected by any local causes) be positive, and may possibly amount to several degrees; but the most important term in each of the above expressions is the last. Thus suppose the height denoted by a to be 25000 feet, a little more only than Gay Lussac and Mr Welsh ascended in their balloons; the consequent reduction of the mean superficial temperature for any latitude would amount to about 46° (C), so that the mean temperature of the equinoctial regions of Venus would be actually reduced to that of the temperate regions of the Earth, while that of her polar regions would equal about 49° (C) reduced by a quantity depending on the values of t and t'. If we suppose this reduced value to be about 40° (C), it will exceed the mean temperature of our own tropical region by 12° or 13° (C).

It must be recollected that the temperatures here spoken of are mean annual temperatures, such as would be indicated by a thermometer *shaded from the direct rays of the Sun*. An inhabitant of Venus would also experience during the day the heat of *sunshine*, like ourselves, but of about double the intensity, supposing her atmosphere to be as diathermanous to the Sun's rays as that of the Earth. But this supposition seems, as above intimated, to be contradicted by the general appearance of this planet. The Moon, without any sensible atmosphere, exhibits to us her mountains in minute detail, and Mars indicates the outlines of apparent continents; but Venus, when examined even through the best telescopes, exhibits very little more than a dazzling and uniform brightness, which has generally been attributed to the reflexion from an atmosphere of small comparative transparency, and therefore hiding the real surface of the planet. This does not imply a more extended atmosphere than that of the Earth, but rather one containing that larger amount of aqueous vapour which would necessarily result in an atmosphere like our own from increased heat and evaporation. It would seem probable therefore that the glare of sunshine on this planet may be so far modified as not to be materially greater than on the Earth.

We have still to consider the periodical inequalities of temperature, which, with the above assumed obliquity, become much more important than on our own globe. The most important will be the annual inequality, the maximum value of which will

$$= \pm h \frac{b}{D} \frac{\pi}{2} \sin 75^\circ \sin \mu, \ (\mu = \text{lat.}).$$

Taking the value of $\frac{b}{D} = {,}731$, the same as for the Earth, and $h = 70^\circ$ (C), this becomes

$$\pm 77^\circ{,}5 \sin \mu.$$

The maximum value of the semi-annual inequality

$$= \pm h \frac{b}{D_1} Q_1.$$

Taking $\frac{b}{D_1}$ the same as for the Earth, its value is ,66; and taking the value above given of Q_1, we have the greatest value of the semi-annual inequality at the equator of Venus

$$= \pm 16^\circ{,}4 \text{ (C)}.$$

The annual inequality at the pole would doubtless be subject to a very considerable reduction in an atmosphere resembling the Earth's, on account of the rapid degradation of temperature in passing from the pole to the equator, at the time when this inequality should be near the above, *i.e.* its greatest value, and when also the semi-annual inequality at the equator would reduce the temperature there to nearly its lowest value. Judging by the tendency to equalization of temperature in our own tropical regions, it seems probable that the temperature would be lowered at the pole of Venus by many degrees, while that at her equator would be increased. It must also be recollected that, as already explained, both these inequalities in the temperature might be lessened in any degree by a change in the value of the quantity $\frac{ck}{p^2}$. As the atmosphere would appear to be the principal agent with which we are acquainted for regulating the mean temperatures of the planets, so the conductivity, specific heat of their outer crusts, and transmissive powers of their surfaces, appear to afford the most effective agency by which we can understand the greatest modifications to be made in the periodical variations of temperature to which the nearer planets may be subjected. The densities of Jupiter and Saturn, combined with the enormous pressure to which their central portions must be subjected, shew that the matter of which those planets are composed must be of different kinds, while each must differ still more from the Earth; and though the densities of Venus and Mars do not afford similar indications of difference, it would at least seem contrary to the analogy afforded by the larger planets to assume the matter of which the two former are constituted, to have exactly the same properties as the crust of our own globe.

To take a numerical illustration, suppose that the tendency to equalize the temperature of the atmosphere at the poles should reduce the coefficient of the annual inequality there from 75° (C) to 60°. Also suppose the matter composing the superficial crust of Venus to be such that the fraction $\frac{ck}{p^2}$ should be five or six times as great as for the Earth, which is entirely admissible as a supposition, considering the differences in the values of c, k, and p, which we find in different terrestrial substances. This would reduce the coefficient of the inequality by nearly 20°. Thus the reduction of the coefficient by both these causes would amount to nearly 35°, leaving it equal to rather more than 40° (C); and if we take the value of the mean temperature as above estimated (about 40°) with the diminished atmosphere, the extreme temperature on the surface of Venus (which would be at the poles) would not much exceed 80° (C). At the same time, taking the mean temperature of the equator at 12° (C) and the coefficient of the semi-annual inequality as modified by the causes above mentioned, at 15°, the equatorial temperature would oscillate between -3° and $+27^\circ$ (C). If we further suppose the *sunshine* to be modified as above intimated, a large portion of the surface of Venus would be habitable by beings little better calculated than the human race to bear the extremities of heat and cold.

If we should take a case intermediate, as regards the obliquity, we should obtain intermediate results. The mean temperatures at the equator and pole would be more nearly equal, and the oscillations of temperature due to the annual and semi-annual inequalities would diminish with the obliquity. The general high temperature of this planet would in such case be more equably diffused than in either of those cases which we have here examined in more detail.

35. We know too little of Mercury to form any judgment as to the possible effects of any known causes in modifying the intense heat of radiation to which he must be subjected in consequence of his proximity to the Sun.

36. In considering the case of the Moon it will be sufficient for my purpose to suppose her axis of rotation to be perpendicular to the plane of her orbit, and that plane to pass through the Sun. I shall moreover restrict myself to the effect of the solar heat on a place situated on the Moon's equator. The problem may be solved by means of the general formulæ given by Poisson, but so far as regards the numeral quantity Q, it may be more simply solved as a particular case.

Let θ = the zenith distance of the Sun at any proposed place on the Moon's equator, at any time t; and let $\frac{S}{a^2}$ denote the intensity of solar radiation at the distance a. It will be sufficient to consider the Sun's distance from the Moon as constant, and equal to the Earth's mean distance from the Sun. Then shall we have

$$\zeta = \xi + \frac{kS}{\lambda a^2} \cos\theta,$$

$$= \xi + \pi h \cos\theta \text{ (Art. 17)};$$

ζ being the temperature of an imaginary medium surrounding the Moon (Art. 17), and ξ now denoting the temperature derived from stellar radiation alone, the Moon being assumed to be without sensible atmosphere. Now the last term will have the value $\pi h \cos\theta$ only so long as the Sun is above the lunar horizon of the proposed place, *i. e.* from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$, and will be equal to zero from $\theta = \frac{\pi}{2}$ to $\theta = \pi$. It will therefore be a periodical discontinuous function. But any function, $f(\psi)$, of this kind, can be expressed by the well-known formula

$$f(\psi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi')\,d\psi' + \frac{1}{\pi}\,\Sigma\left\{\int_{-\pi}^{\pi} \cos i\,(\psi - \psi')\,f\psi'\,d\psi'\right\};$$

and hence putting $f(\psi) = \cos\theta$, and taking the integrals only between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, since the other portions vanish, we have

$$\pi h \cos\theta = \frac{h}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta'\,d\theta' + h\Sigma\left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos i\,(\theta - \theta')\cos\theta'\,d\theta'\right\},$$

the last term including all values of i from unity to infinity. This expression gives the value of $h\cos\theta$ subject to the condition that it shall equal zero while the Sun is below the horizon of the proposed place. Performing the operations indicated, and taking all values of i from unity to infinity inclusive, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta'\,d\theta' = 2,$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos i\,(\theta - \theta')\cos\theta'\,d\theta' = \left(\frac{\sin\,(i+1)\frac{\pi}{2}}{i+1} + \frac{\sin\,(i+1)\frac{\pi}{2}}{i-1}\right)\cos i\theta;$$

and, therefore,

$$\pi h \cos\theta = h\left\{1 + \frac{\pi}{2}\cos\theta + \frac{2}{3}\,.\cos 2\theta - \frac{2}{3\,.\,5}\cos 4\theta + \frac{2}{5\,.\,7}\cos 6\theta - \&c.\right\}.$$

Neglecting the differences between the true and mean values of the angular motion of the Earth round the Sun, and that of the Moon round the Earth, we shall have, at any time t,

$$\theta = n(t - \tau),$$

where τ denotes the time of the Sun's crossing the lunar meridian of the proposed place at the Moon's equator, and n denotes the mean angular velocity with which the Sun separates from that meridian. Preserving the unit of time, one year, which Poisson has taken, we shall have for the length of the mean solar day at the proposed place on the Moon (since it equals the Moon's mean synodic period)

$$= \frac{\text{Moon's mean synodic period}}{\text{Earth's mean period}} = \frac{1}{12{,}37};$$

and consequently we have

$$2\pi = \frac{n}{12\,.\,37},$$

$$n = 2\pi\,.\,(12{,}37).$$

Hence if we put

$$\omega = \frac{\sqrt{\pi(12{,}37)}}{a}*,$$

the expression for the temperature of the Moon at her equator, and at a depth x beneath her surface, will be

$$\begin{aligned} u &= \xi + h, \\ &+ h\,.\frac{b}{D'}\frac{\pi}{2}\,\epsilon^{-x\omega}\cos\left\{n(t-\tau) - x\omega - \delta'\right\}, \\ &+ h\,.\frac{b}{D'}\frac{2}{3}\,\epsilon^{-x\omega\sqrt{2}}\cos\left\{2n(t-\tau) - x\omega\sqrt{2} - \delta''\right\}, \\ &- h\,.\frac{b}{D''''}\frac{2}{3\,.\,5}\,\epsilon^{-x\omega\sqrt{4}}\cos\left\{4n(t-\tau) - x\omega\sqrt{4} - \delta''''\right\}, \\ &+ \&c. \;\ldots\ldots, \end{aligned}$$

where

$$b + \omega\sqrt{i} = D^{(i)}\cos\delta^{(i)},$$

$$\omega\sqrt{i} = D^{(i)}\sin\delta^{(i)},$$

and, therefore,

$$D^{(i)} = \sqrt{b^2 + 2b\omega\sqrt{i} + 2i\omega^2}.$$

It will be observed that the temperature here denoted by ξ is what is properly termed the temperature of stellar space, or, more accurately, that temperature which the portion of space occupied by the solar system would derive from the stars alone. It must not be confounded with the temperature heretofore denoted by ψ (Art. 21), and which I have estimated at $-39^0,5$ (C). We have no means, as already stated, of determining the value of ξ. The above expression for u only tells us that the mean equatorial temperature of the Moon at points not far beneath her surface, must be augmented by the temperature h, derived from solar radiation; *i. e.*, by the same quantity as that by which the mean equatorial temperature of the Earth would be augmented supposing the equator and ecliptic to coincide, and the intensity of solar radiation to be the same in the two cases. If we assume it to be somewhat greater for the Moon, and estimate h, for example, at 40^0 (C), then the Moon's mean equatorial temperature, in the case supposed respecting the position of her axis and the plane of her orbit, would be greater or less than zero (Centigrade), according as ξ should be greater or less than -40^0 (C). At the pole the value of h would be reduced to zero, and the temperature to ξ. With the actual positions of her orbit and axis of rotation, the equatorial temperature would be somewhat less and the polar temperature rather greater.

---

* *Théorie de la Chaleur*, Art. 222.

The first periodical term expresses a monthly inequality. To find its greatest value we must know the value of $\frac{b}{D'}$.

Now,

$$\frac{D'^2}{b^2} = 1 + \frac{2\sqrt{\pi(12{,}37)}}{ab} + \frac{2\pi(12{,}37)}{a^2b^2},$$

so that the value of $\frac{b}{D'}$ depends on that of $\frac{1}{ab}$ (or $\frac{ck}{p^2}$ as before explained Art. 27), and can only be determined, therefore, with some assumed value of this fraction. If we suppose a and b to have the same values as for the Earth, we shall have

$$\frac{b}{D'} = {,}40,$$

and, therefore,

$$\frac{b}{D'} \cdot \frac{\pi}{2} = {,}628.$$

Hence the term in question becomes at an equatorial point of the Moon's surface,

$$h \,.\, ({,}628) \cos\{n(t - \tau) - \delta\};$$

and consequently its greatest value with our assumed value of h, will

$$= 25^0{,}12 \text{ (C)}.$$

The coefficient $\left(h\frac{b}{D''}\frac{2}{3}\right)$ of the next term is nearly

$$= 8^0 \text{ (C)}.$$

The values of δ and δ' are respectively $26\,.34'$ and $32^0.13'$ nearly, so that, though these two terms cannot attain their respective *maxima* at the same time, it is easily seen that the greatest value of their sum will amount to more than 30^0 (C). The succeeding terms are much smaller, and may be neglected in a rough approximation like the present.

Hence, then, it appears that the superficial temperature of the Moon's mass, in the immediate vicinity of her equator, must have a mean value exceeding that of surrounding stellar space, by an amount which may probably be estimated at nearly 40° (C), and will be subject to an enormous monthly inequality, which may possibly amount to upwards of 30° (C) in excess or defect, and therefore producing in the extreme monthly temperatures a difference of some 60° (C). It must, however, be recollected that this conclusion involves the hypothesis that the values of c, k, and p, the specific heat, conductivity and radiating power, are the same for the Moon as for the Earth. At the poles the mean temperature can little exceed that of stellar space, and the inequalities must be very small.

The Moon's *external temperature*, or that which would be indicated by a thermometer placed at any point not remote from her surface and external to it, must depend, in the absence of a sensible atmosphere in contact with the bulb of the instrument, entirely on radiation from the Moon herself, the Sun, and all other bodies in stellar space. If it received heat from the stellar bodies alone, it would indicate the temperature above denoted by ξ; and if it were then exposed also to the radiation from the Moon, but sheltered from that of the Sun, the instrument would rise till the heat received by the bulb in consequence of inter-radiation between the bulb and the Moon should equal that emitted in consequence of the inter-radiation between the bulb and the stellar bodies. The indicated temperature, therefore, would be less or greater than the superficial temperature of the lunar mass, according as the latter should be higher or lower than that of stellar space; it would, probably, be much less, when the superficial temperature should acquire its greatest monthly value. This defect, however, of temperature might possibly be more than counterbalanced by the direct heat from the Sun, when the instrument should be exposed to the solar as well as to the lunar and stellar radiation.

The monthly inequality of temperature above noticed is manifestly analogous to the daily inequality in the terrestrial temperature. Its magnitude depends on the slow angular motion of the Moon about her axis, and shews the remarkable degree in which a more or less rapid rotation may affect the variations of temperature on any planet or satellite sufficiently near to the central source of heat of our system.

W. HOPKINS.

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